Concepts of Potential Energy, Conservative and Non-Conservative Forces

LECTURE-III

B. Tech 1st Semester

(PH103: Physics-I)

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Curriculum Delivery in the E-Mode

PATH INTEGRAL OF FORCE:

We have a curved line S.

Force $\vec{F}$ acts on a particle which moves the particle from M to N on the curve.

X and Y are two nearby points on the curve.

$\vec{r}$ and $\vec{r} + \delta\vec{r}$ are the position vectors of the points X and Y respectively.

O is the origin. Hence, $XY = \delta r$.

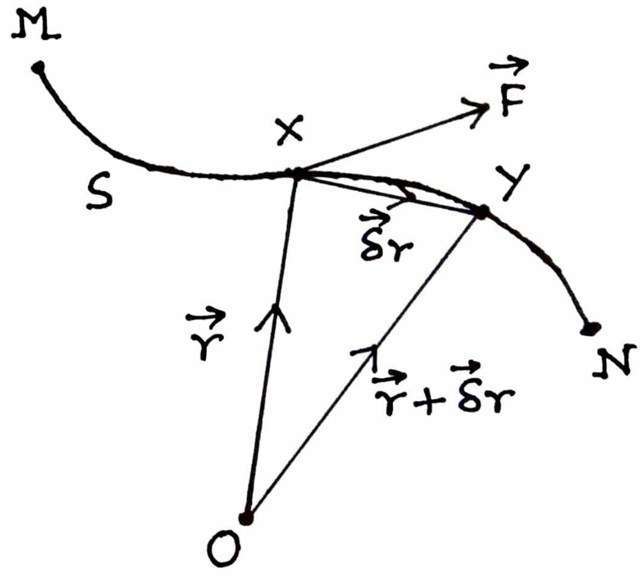


Fig. 1

Work done by the force in displacing the particle from X to Y is $\vec{F} \cdot \delta\vec{r}$ (scalar product)

The path integral of the force $\vec{F}$ along the curve S is defined as $\sum \vec{F} \cdot \delta\vec{r}$

Total work done by the force in displacing the particle from point M to point N on the curve S is,

$$W = \underset{\delta r \to 0}{Lt} \sum_{M}^{N} \vec{F} \cdot \delta\vec{r} = \int_{M}^{N} \vec{F} \cdot d\vec{r} \qquad \text{—— (1)}$$

$\int_{M}^{N} \vec{F} \cdot d\vec{r}$ ← Path integral (or line integral) of force $\vec{F}$ between N and M along curve S ⟵ total work done by $\vec{F}$

Suppose, instead of only a single force, a number of forces $\vec{F}_a, \vec{F}_b, \vec{F}_c, \ldots$ act simultaneously on the particle.

The resultant of the forces is given by,

$$\vec{F}_r = \vec{F}_a + \vec{F}_b + \vec{F}_c + \cdots$$

Then the work done on the particle is given by,

$$W = \int_M^N \vec{F}_r \cdot d\vec{r}$$

$$\text{or, } W = \int_M^N (\vec{F}_a + \vec{F}_b + \vec{F}_c + \cdots) \cdot d\vec{r}$$

$$\text{or, } W = \int_M^N \vec{F}_a \cdot d\vec{r} + \int_M^N \vec{F}_b \cdot d\vec{r} + \int_M^N \vec{F}_c \cdot d\vec{r} + \cdots \qquad (2)$$

So, Work done on particle $= \sum$ work done by each individual force.

Work–Energy Theorem

If $\vec{F}$ denotes the force (or the resultant of the forces) acting on a particle, then we have

$$W = \int_M^N \vec{F} . d\vec{r}$$

Now, $\vec{F} = m \frac{d\vec{v}}{dt}$. So, $W = m \int_M^N \frac{d\vec{v}}{dt} . d\vec{r}$

we know, $\vec{v} = \frac{d\vec{r}}{dt} \Rightarrow d\vec{r} = \vec{v} dt$

So, $W = m \int_M^N \frac{d\vec{v}}{\cancel{dt}} . (\vec{v} \cancel{dt}) = m \int_M^N \vec{v} . d\vec{v} = \frac{m}{2} \int_M^N d(\vec{v} . \vec{v})$

Work-Energy Theorem

or, $W = \frac{m}{2}\int_M^N d(v^2) = \frac{m}{2}\left[v^2\right]_M^N$

or $W = \frac{1}{2}mv_N^2 - \frac{1}{2}mv_M^2$

= change in kinetic energy

So, the work done on the particle equals the change in kinetic energy of the particle. This is known as the Work-Energy theorem.

CATEGORIES OF FORCES

In general, forces can be divided into the following two broad categories:

(i) Conservative force:

If the work done by a force in displacing a particle from one point to another does not depend on the path joining the two points and is solely dependent on the position of the two given points, then the force is called conservative.

Examples: Force of gravity, Coulomb force, force of spring.

(ii) Non-conservative force:

If the work done by a force in displacing a particle from one point to another depends upon the actual path taken, then the force is called non-conservative.

Examples: Force of friction, Air resistance force.

THE CURL

- Multiplication involving the del (nabla) operator, $\vec{\nabla}$, is extremely useful in physics.
- If $\vec{F}$ is a vector function, the product denoted by $\vec{\nabla}\times\vec{F} \equiv$ curl $\vec{F}$ is called the curl of $\vec{F}$.
- The curl of $\vec{F}$ is given by

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

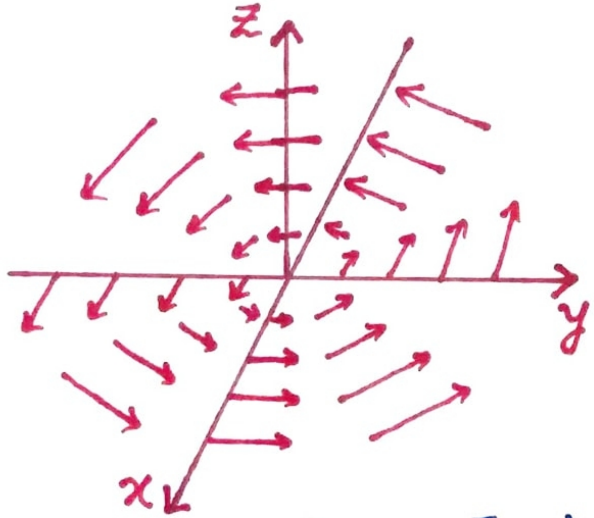


Figure I(a)

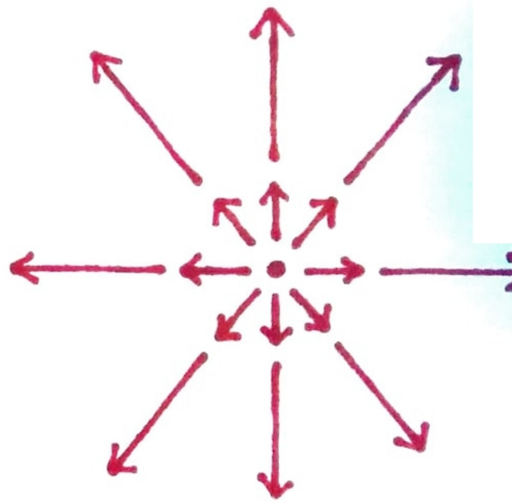


Figure I(b)

Geometrical Interpretation:

- $\vec{\nabla}\times\vec{F}$ is a measure of how much the vector $\vec{F}$ "curls around" the point in question.
- The function in Figure I(a) has a substantial curl, pointing in the z-direction as suggested by the right-hand rule.
- However, the function in Figure I(b) has zero curl.

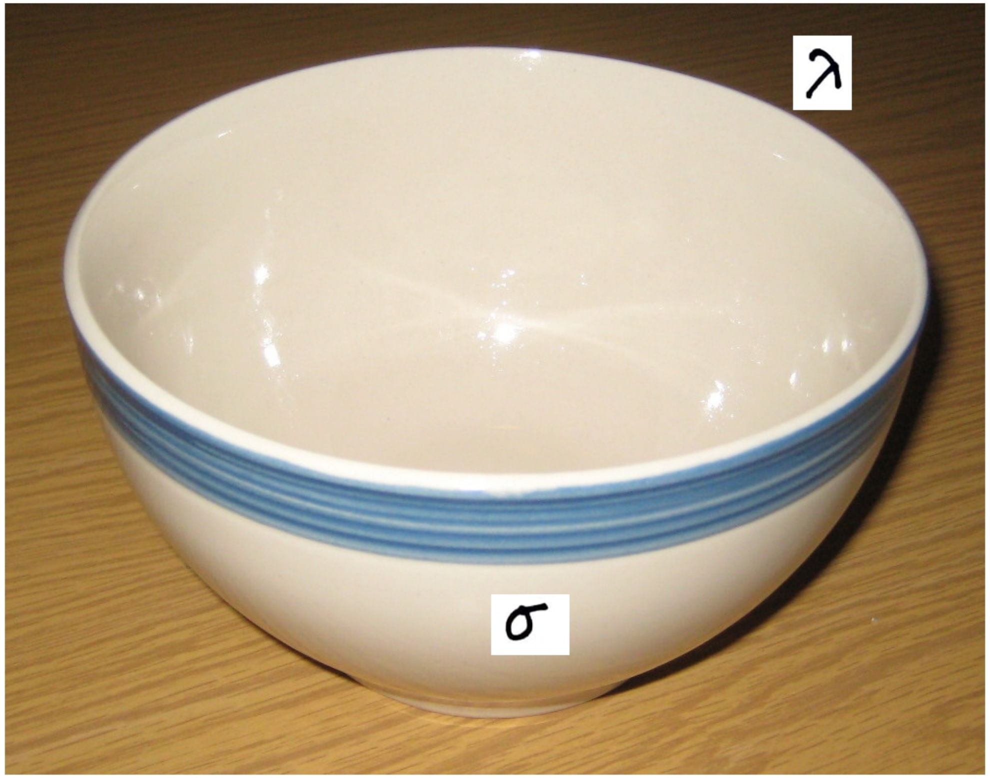
STOKES' THEOREM

- The curl theorem due to Stokes states that: If the vector function $\vec{F}$ and its first derivatives are continuous, the line integral of $\vec{F}$ around a closed curve λ is equal to the normal surface integral of curl $\vec{F}$ over an open surface bounded by λ.

- In other words, the surface integral of curl $\vec{F}$ taken over any open surface σ is equal to the line integral of $\vec{F}$ around the periphery λ of the surface.

- In equation form, we write

$$\boxed{\int_{\sigma} (\vec{\nabla} \times \vec{F}) \cdot d\vec{\sigma} = \oint_{\lambda} \vec{F} \cdot d\vec{\lambda}}$$



CONCEPT OF FORCE FIELD AND CONSERVATIVE FORCE

A force field is a region of space in which at every point a particle is acted upon by a force that varies regularly from position to position in that region of space.

- The field of a conservative force is called conservative force field.
- Suppose in such a conservative field, a particle moves from point M to point N along path S_1 and returns to point M from N along another path S_2.
- The total work done during the round trip MS_1NS_2M will be zero.
- So, a force may be called conservative if the work done by it round a closed curve is zero.

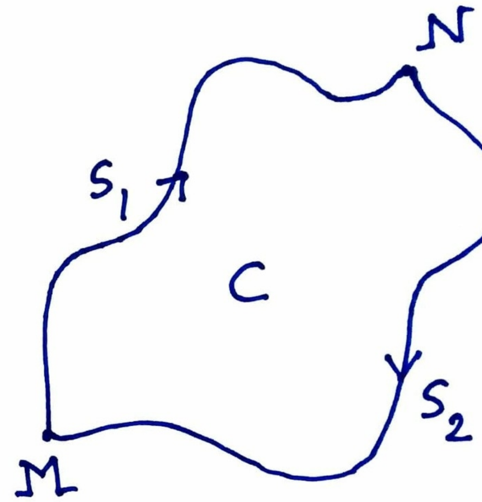


Fig. 2

CONSERVATIVE FORCE

- Mathematically, we represent it by $\oint \vec{F}\cdot d\vec{r} = 0$, where the symbol $\oint$ indicates that the integration must be performed over a closed path.

- Let us now consider the closed curve MS_1NS_2M. We have,

$$\oint_{MS_1NS_2M} \vec{F}\cdot d\vec{r} = \int_{MS_1N} \vec{F}\cdot d\vec{r} + \int_{NS_2M} \vec{F}\cdot d\vec{r} \quad \text{——(4)}$$

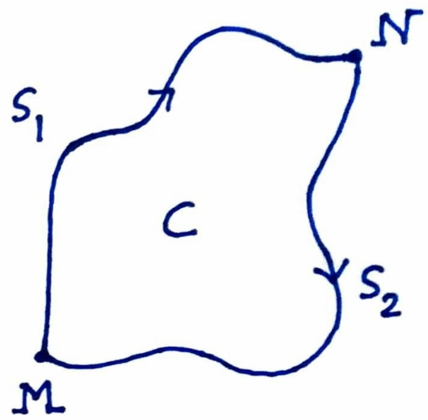


Fig. 2

For a conservative field, $\oint_{MS_1NS_2M} \vec{F}\cdot d\vec{r} = 0$. Thus,

$$\int_{MS_1N} \vec{F}\cdot d\vec{r} + \int_{NS_2M} \vec{F}\cdot d\vec{r} = 0 \quad \text{——(5)}$$

or

$$\boxed{\int_{MS_1N} \vec{F}\cdot d\vec{r} = -\int_{NS_2M} \vec{F}\cdot d\vec{r} = \int_{MS_2N} \vec{F}\cdot d\vec{r}} \quad \text{——(6)}$$

From Equation (6), we find that the work done by the conservative force does not depend on the path S_1 or S_2 but depends only on the coordinates of the end points M and N.

From Stoke's theorem, we know,

$$\int_{\sigma} (\vec{\nabla} \times \vec{F}) \cdot d\vec{\sigma} = \oint_{C} \vec{F} \cdot d\vec{r} \qquad \text{— (7)}$$

Since for a conservative force field $\oint_C \vec{F} \cdot d\vec{r} = 0$, for such a field we have $\int_{\sigma} (\vec{\nabla} \times \vec{F}) \cdot d\vec{\sigma} = 0$, where the surface bounded by the curve is denoted by σ.

Thus, a force $\vec{F}$ acting on a particle is conservative if the curl of the force is zero, i.e.

$$\vec{\nabla} \times \vec{F} = 0 \qquad \text{— (8)}$$

If U is a scalar function of coordinates of the particle, then curl of gradient of scalar function U equals zero.

Thus, $$\vec{\nabla} \times (-\vec{\nabla} U) = 0 \quad \text{—(9)}$$

From (8) and (9), we have

$$\vec{F} = -\vec{\nabla} U \quad \text{—(10)}$$

- So, if a force can be expressed as the negative gradient of a scalar function U, then it is called a <u>conservative force</u>.

- U is known as the <u>potential energy</u> of the particle.

CONCEPT OF POTENTIAL ENERGY

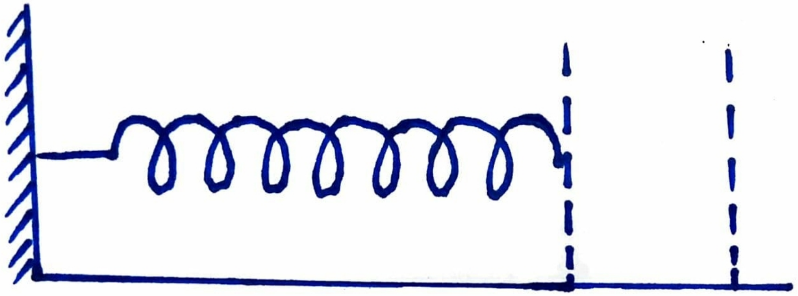
The work done by a conservative force $\vec{F}$ on a system when the system goes from the initial configuration i to final configuration f is expressed as

$$W = \int_i^f \vec{F} \cdot d\vec{r} = -(U_f - U_i) \qquad \text{—}(11)$$

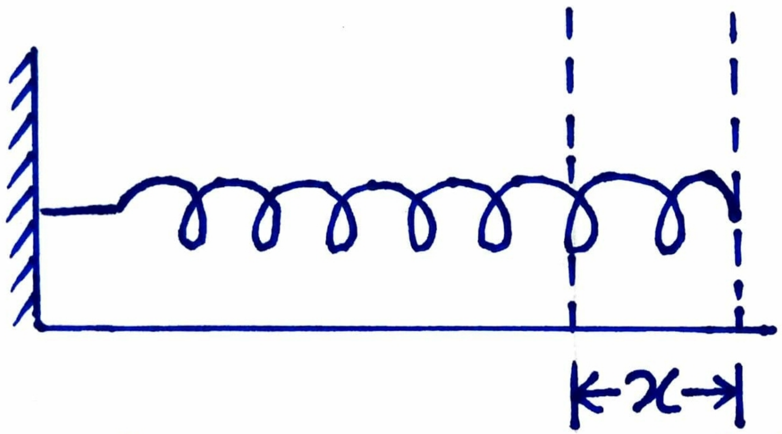
Here, the quantity U depends on the configuration of the system and is called the potential energy of the system.

$(U_f - U_i)$ is the change in potential energy of the system corresponding to the conservative force $\vec{F}$.

Idea of change in configuration of a system:



(a) Unstretched spring ($x=0$)



(b) Spring in stretched position ($x>0$)

The quantity x measures the displacement of the free end of the spring from its unstretched position.

When x changes, the configuration of the spring system changes.

- The energy possessed by a body by virtue of its configuration is called the potential energy.

Measurement of Potential Energy:

In order to measure the potential energy, the conservative force which gives rise to the potential energy needs to be specified.

If we consider $U_i = 0$ in Equation (11), we get

$$\int_i^f \vec{F}.d\vec{r} = -U_f$$

$$\text{or} \quad U_f = -\int_i^f \vec{F}.d\vec{r} = \int_f^i \vec{F}.d\vec{r} \qquad \text{—— (12)}$$

Hence, the potential energy possessed by a system in a given configuration is measured by the work done by the relevant conservative force in taking the system from the given configuration to some standard configuration.

By putting $U_i = 0$ in Equation (11), we have chosen i as the standard configuration.

Thus Equation (12) gives us the potential energy of the system in configuration f with respect to the standard configuration i.

Example (I)

Let us consider a special spring which is governed by a force law: $\vec{F} = -k x^2 \hat{i}$ where k is a constant.

We consider the unstretched position of the spring ($x = 0$) as the standard where the potential energy $U = 0$.

At any other position x, let us calculate the potential energy of the spring. We use Equation (12).

Here f corresponds to the position having x-coordinate equal to x and i corresponds to the position $x=0$.

We have, $\vec{F} = -kx^2\hat{i}$ and $d\vec{r} = dx\hat{i}$

The potential energy is:

$$U = -\int_0^x (-kx^2\hat{i}) \cdot (dx\,\hat{i}) = \int_0^x kx^2\,dx = \frac{1}{3}kx^3$$

- Thus, we cannot measure the potential energy of a system absolutely. We are free to choose any configuration of a system to correspond to zero potential energy, which is the standard.

- The potential energy of the system will then be measured w.r.t this standard.

- Moreover, we also need to know the concerned force in order to determine the potential energy.

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Thank You