Concepts of Potential Energy, Conservative and Non-Conservative Forces

LECTURE-IV

B. Tech 1st Semester

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Dr Biplob Sarkar

Asst. Prof. (Deptt. of Applied Sciences)

Tezpur University, Napaam-784028

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CENTRAL FORCE – A CONSERVATIVE FORCE

- A force that is always directed to, or away from, a fixed centre is called a <u>central force</u>.

- A central force is a position-dependent force which solely depends upon the instantaneous position of the particle with respect to the fixed centre.

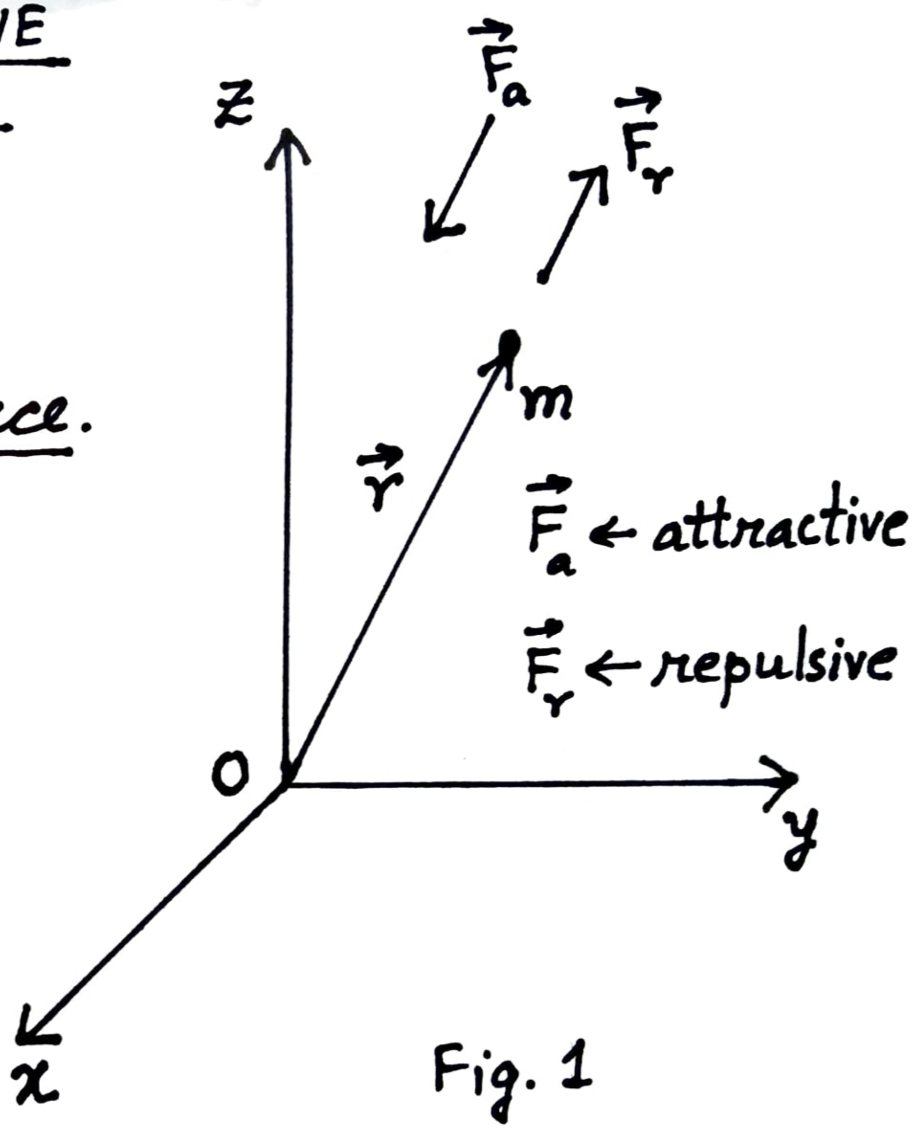

Fig. 1

<u>Examples</u>: gravitational force, electrostatic force, magnetostatic force.

- A central force can be expressed as $\boxed{\vec{F} = f(r)\,\hat{r}}$ where $f(r)$ is a scalar function of r only and $\hat{r}$ is a unit vector along $\vec{r}$.

A central force is a conservative force and the proof is as follows:

- Suppose a central force $\vec{F}$ acts on a particle which displaces the particle through a small distance $d\vec{r}$. Then the work done by the central force is

$$dW = \vec{F} \cdot d\vec{r} \qquad \text{—(1)}$$

If the force $\vec{F}$ displaces the particle from position $A = r_1$ to another position $B = r_2$, then the work done is given by

$$W = \int_{A=r_1}^{B=r_2} \vec{F} \cdot d\vec{r} = \int_{r_1}^{r_2} f(r)\,\hat{r} \cdot \hat{r}\, dr \quad [\because d\vec{r} = \hat{r}\, dr]$$

$$\text{or } W = \int_{r_1}^{r_2} f(r)\, dr = [V]_{r_1}^{r_2}$$

Since $f(r)$ is a function of position r only, the integral of $f(r)$ is also a function of r only. Thus,

$$W = [V]_{r_1}^{r_2} = V_{r_2} - V_{r_1} \qquad \text{——(2)}$$

$V_{r_1} \leftarrow$ value of integral V at $r = r_1$ at position A

$V_{r_2} \leftarrow$ value of integral V at $r = r_2$ at position B.

From Eq^n (2), it is clear that work done W depends only on the position of the end points $r = r_1$ and $r = r_2$.

W is independent of the actual path followed between points A and B.

Hence, a central force is a conservative force.

Conservative Force as Negative Gradient of Scalar Potential

- Suppose a conservative force $\vec{F}$ does work W on a system as the system passes from the initial configuration i to the final configuration f.
- If U is the potential energy corresponding to the conservative force $\vec{F}$, then according to the definition of potential energy

$$W = \int_i^f \vec{F} \cdot d\vec{r} = U_i - U_f = [-U]_i^f = -\int_i^f dU$$

$$\therefore -dU = \vec{F} \cdot d\vec{r} \quad \text{and} \quad -U = \int \vec{F} \cdot d\vec{r} \quad —(3)$$

The negative sign indicates that the work done by a conservative force on a system decreases the potential energy of the system.

In Cartesian co-ordinates

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

and $\vec{F} = F_x\,\hat{i} + F_y\,\hat{j} + F_z\,\hat{k}$

According to Equation (3)

$$-U = \int \vec{F} \cdot d\vec{r}$$

or $-U = \int (F_x\,\hat{i} + F_y\,\hat{j} + F_z\,\hat{k}) \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$

or $-U = \int F_x\,dx + \int F_y\,dy + \int F_z\,dz \quad \text{--- (4)}$

Partially differentiating U with respect to x, y and z, we get

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz$$

or $-U = \int \left(-\frac{\partial U}{\partial x}\right) dx + \int \left(-\frac{\partial U}{\partial y}\right) dy + \int \left(-\frac{\partial U}{\partial z}\right) dz \quad \text{--- (5)}$

Comparing (4) and (5), we have

$$F_x = -\frac{\partial U}{\partial x} \; ; \; F_y = -\frac{\partial U}{\partial y} \; ; \; F_z = -\frac{\partial U}{\partial z}$$

$$\therefore \vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$

$$= -\left[\frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}\right]$$

$$= -\left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]U$$

$$= -\vec{\nabla} U = -\text{grad } U \qquad \text{—(6)}$$

Hence, a conservative force can be expressed as the negative gradient of scalar potential U.

LAW OF CONSERVATION OF MECHANICAL ENERGY

we know,

$$\vec{F}\cdot d\vec{r} = F_x\,dx + F_y\,dy + F_z\,dz$$

or $$\vec{F}\cdot d\vec{r} = -\left(\frac{\partial U}{\partial x}\right)dx - \left(\frac{\partial U}{\partial y}\right)dy - \left(\frac{\partial U}{\partial z}\right)dz = -dU$$

$\therefore$ Work done in displacing a particle from M to N is given by

$$W = \int_M^N \vec{F}\cdot d\vec{r} = -\int_M^N dU = U_M - U_N$$

But from Work-energy theorem,

$$W = \int_M^N \vec{F}\cdot d\vec{r} = \frac{1}{2}mv_N^2 - \frac{1}{2}mv_M^2$$

Evidently, thus, $U_M - U_N = \frac{1}{2}mv_N^2 - \frac{1}{2}mv_M^2$

This indicates that the loss in potential energy of a particle leads to the corresponding gain in kinetic energy of the particle.

So that, $$\frac{1}{2} m v_M^2 + U_M = \frac{1}{2} m v_N^2 + U_N$$

Or, putting T_M and T_N respectively for $\frac{1}{2} m v_M^2$ and $\frac{1}{2} m v_N^2$, we have

$$T_M + U_M = T_N + U_N, \text{ or, } T + U = E, \text{ a constant} \quad \text{— (7)}$$

- The sum total (E) of the kinetic energy and the potential energy of a particle is called the total mechanical energy.

- Equation (7) shows that the total mechanical energy of a particle remains constant as the particle moves from one point M to another point N in a conservative field.

- This is called as the law of conservation of mechanical energy for conservative forces.

- In presence of non-conservative forces, such as friction, the total mechanical energy E (= T+U) is not constant and we cannot apply the law of conservation of mechanical energy.

- However, the work-energy theorem holds even in the presence of non-conservative forces.

Non - conservative force:

A force $\vec{F}$ which acts on a particle and moves it from point A to point B will be non-conservative if:

(i) The curl of the force is not zero i.e. $\vec{\nabla} \times \vec{F} \neq 0$.

or (ii) It is not possible to express the force $\vec{F}$ as a gradient of a scalar function, say U.

or (iii) If the work done by the force $\vec{F}$ depends on the path taken between the initial and final states.

or (iv) If the work done by the force $\vec{F}$ during the round trip of a system is not zero.

* Velocity-dependent forces like frictional and viscous force, are, in general, non-conservative.

* The value of forces like frictional and viscous ones, depends upon the magnitude and direction of velocity.

Example: 1. Determine whether the force field $\vec{F} = (y^2 + 2zx)\hat{i} + (2xz)\hat{j} + (y^2)\hat{k}$ is conservative or non-conservative.

Answer: If the force field $\vec{F}$ is conservative, then $\vec{\nabla} \times \vec{F} = 0$.

Now, $$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2 + 2zx) & (2xz) & (y^2) \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(y^2) - \frac{\partial}{\partial z}(2xz)\right] + \hat{j}\left[\frac{\partial}{\partial z}(y^2 + 2zx) - \frac{\partial}{\partial x}(y^2)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(2xz) - \frac{\partial}{\partial y}(y^2 + 2zx)\right]$$

$$= \hat{i}[2y - 2x] + \hat{j}[2x] + \hat{k}[2z - 2y] \neq 0.$$

$\therefore$ The force field is non-conservative.

Example: 2. Show that the force $\vec{F} = (4yz - 6xy)\hat{i} + (4xz + 2z^2 - 3x^2)\hat{j} + (4xy + 4zy)\hat{k}$ is conservative.

Answer: If the force $\vec{F}$ is conservative curl $\vec{F} = 0$ i.e., $\vec{\nabla} \times \vec{F} = 0$

$$\therefore \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (4yz - 6xy) & (4xz + 2z^2 - 3x^2) & (4xy + 4zy) \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(4xy + 4zy) - \frac{\partial}{\partial z}(4xz + 2z^2 - 3x^2)\right] + \hat{j}\left[\frac{\partial}{\partial z}(4yz - 6xy) - \frac{\partial}{\partial x}(4xy + 4zy)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(4xz + 2z^2 - 3x^2) - \frac{\partial}{\partial y}(4yz - 6xy)\right]$$

$$= \hat{i}\left[4x + 4z - 4x - 4z\right] + \hat{j}\left[4y - 4y\right] + \hat{k}\left[4z - 6x - 4z + 6x\right]$$

$$= \hat{i}[0] + \hat{j}[0] + \hat{k}[0]$$

$$= 0$$

$\therefore$ the force $\vec{F}$ is conservative.

Example: 3. Find the potential at the point (x, y, z) in the field associated with the force $\vec{F} = \hat{i}(y^2 + 2zx) + \hat{j}(2xy) + \hat{k}(x^2)$

Answer:

Given, $\vec{F} = \hat{i}(y^2 + 2zx) + \hat{j}(2xy) + \hat{k}(x^2)$

$= \hat{i} F_x + \hat{j} F_y + \hat{k} F_z$ (say)

We know that a conservative force $\vec{F}$ is equal to the negative gradient of a scalar function U called the potential. That is,

$$\vec{F} = -\vec{\nabla} U$$

When the conservative force $\vec{F}$ is given, then potential associated with the force can be found out as below:

Potential $U = -\int \vec{F} \cdot d\vec{r}$

or $U = -\int (\hat{i} F_x + \hat{j} F_y + \hat{k} F_z) \cdot (\hat{i}\, dx + \hat{j}\, dy + \hat{k}\, dz)$

or $U = -\int (F_x\, dx + F_y\, dy + F_z\, dz)$

or $U = -\int [(y^2 + 2zx)\, dx + 2xy\, dy + x^2\, dz]$

or $U = -\int [(y^2\, dx + 2xy\, dy) + (2zx\, dx + x^2\, dz)]$

or $U = -\int [d(xy^2) + d(zx^2)]$

or $U = -\int d(xy^2 + zx^2)$

or $U = -(xy^2 + zx^2)$

So, U at $(x, y, z) = -(xy^2 + zx^2)$.

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Thank You