Topic: Topological spaces: Basis
MI226 : Introductory Topology

A topology $\mathcal{T}$ on a set X can be a complicated collection of subsets of a set, and it can be very difficult to describe the entire collection $\mathcal{T}$ of open sets. In most cases one specifies a smaller collection of subsets of X that generates the topology $\mathcal{T}$. One such collection is a basis and another is a subbasis.

Definition 0.1. Let X be a set and $\mathcal{T}$ be a topology on X. A *basis* for $\mathcal{T}$ is a collection $\mathcal{B}$ of subsets of X such that

(1) For each $x \in X$, there is at least one element $B \in \mathcal{B}$ containing x.
(2) If $x \in B_1 \cap B_2$, where $B_1, B_2 \in \mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

The elements of $\mathcal{B}$ are called *basis elements*.

Definition 0.2. If $\mathcal{B}$ is a basis for a topology on set X, then the topology $\mathcal{T}$ generated by $\mathcal{B}$ is described as follows:
A subset U of X is said to be open in X (that is, $U \in \mathcal{T}$), if for each $u \in U$ there is a basis element $B \in \mathcal{B}$ such that $u \in B$ and $B \subset U$.

Remark 0.3. *If $\mathcal{B}$ is a basis for $\mathcal{T}$, then $\mathcal{B} \subseteq \mathcal{T}$.*

Example 0.4. The collection $\mathcal{B}$ of all open intervals is a basis for the usual topology on $\mathbb{R}$. How? For any $x \in \mathbb{R}$, we have $(x-1, x+1) \in \mathcal{B}$ such that $x \in (x-1, x+1)$.

Next let $x_0 \in (a,b) \cap (c,d)$. Then $a < x_0 < b, c < x_0 < d$. Consider

$$r_0 = \min\{|x_0 - a|, |x_0 - c|\}, s_0 = \min\{|b - x_0|, |d - x_0|\}.$$

Then $x_0 \in (x_0 - r_0, x_0 + s_0) \in (a,b) \cap (c,d)$.

(3,1.5) (0,.5) []¡-¿(0,0)(15,0) (5,-0.25)(5,0.25) (2,-0.25)(2,0.25) (11,-0.25)(11,0.25) (10,-0.25)(10,0.25)
[]¡-(5,0.6)(5.8,0.6) []-¿(6.25,0.6)(7,0.6)
[]¡-(7,-0.3)(8.3,-0.3) []-¿(8.7,-0.3)(10,-0.3)
(7,0.2)x_0(7,0) (6,0.6)r_0 (8.5,-0.3)s_0

Example 0.5. The collection $\mathcal{B}$ of all open disks, that is $\mathcal{B} = \{B_r(a,b) : r > 0, (a,b) \in \mathbb{R}^2\}$ is a basis for the usual topology on $\mathbb{R}^2$. How? The following picture will give you the answer. Can you now argue mathematically? Try.

(3,6) (0,4) (5.5,1)1.2 (7,0)1.6 (6.2,0.8).35 (7,0)(5.5,1) [linestyle=dashed](5.5,1)(6.7,.72) [linestyle=dashed](7,0)(5.9,1.2) []¡-¿(0,-2)(12,-2) []¡-¿(4,-4)(4,2) [linecolor=red](6.22,.8) (6.12,0.63)$_{x_0}$

Example 0.6. Let X be a set and $\mathcal{B} = \{\{x\} : x \in X\}$. Then $\mathcal{B}$ is a basis for the discrete topology on X. Why? Recall that the discrete topology on X contains all the subsets of X.

For each $x_0 \in X$, we have $\{x_0\} \in \mathcal{B}$ such that $x_0 \in \{x_0\}$.

Also if $x_0 \in B_1 \cap B_2$, where $B_1, B_2 \in \mathcal{B}$, then $B_1 = B_2 = \{x_0\}$, which implies that $x_0 \in \{x_0\} \subseteq B_1 \cap B_2$.

Example 0.7. Let $\mathcal{B} = \{[a, b) : a, b \in \mathbb{R}, a < b\}$. Then the topology $\mathcal{T}$ generated by $\mathcal{B}$ is called the *lower limit* topology.

Example 0.8. Let $X = \{1, 2, 3\}$ and let $\mathcal{B} = \{\{1, 2\}, \{2, 3\}, X\}$. Then $\mathcal{B}$ is not a basis for any topology on X. To the contrary, assume that $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on X. Note that $2 \in \{1, 2\} \cap \{2, 3\}$. But there exists no $B \in \mathcal{B}$ such that $2 \in B \subseteq \{1, 2\} \cap \{2, 3\}$. Thus we arrive at a contradiction.

The following lemma says that each member of $\mathcal{T}$ is a union of some members of $\mathcal{B}$.

Lemma 0.9. *Let X be a set and let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all possible unions of elements of $\mathcal{B}$.*

Proof. Let $\mathcal{F}$ be an arbitrary collection of elements of $\mathcal{B}$. Since $\mathcal{B} \subseteq \mathcal{T}$, it follows that $\mathcal{F} \subseteq \mathcal{T}$. Thus all the elements of $\mathcal{F}$ belongs to $\mathcal{T}$. By property of topology, we see that union of elements of $\mathcal{F}$ belongs to $\mathcal{T}$.

Next let $U \in \mathcal{T}$. For each $u \in U$, there exists a $B_u \in \mathcal{B}$ such that $u \in B_u \subseteq U$. Thus $\bigcup_{u \in U} B_u = U$. So U equals to union of elements of $\mathcal{B}$.

Hence $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$. □

Recall the following definition.

Definition 0.10. Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on a given set X. If $\mathcal{T}' \subseteq \mathcal{T}$, then $\mathcal{T}$ is called *finer* than $\mathcal{T}'$ or $\mathcal{T}'$ is *coarser* than $\mathcal{T}$.

If $\mathcal{T}'$ properly contains $\mathcal{T}$, then $\mathcal{T}'$ is called *strictly finer* than $\mathcal{T}$.

The following lemma says that: in terms of the bases one can determine whether one topology is finer than another.

Lemma 0.11. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively on a set X. Then the following are equivalent:*

(1) $\mathcal{T}'$ *is finer than* $\mathcal{T}$

(2) For each $x \in X$ *and each basis element* $B \in \mathcal{B}$ *containing* x*, there is a basis element* $B' \in \mathcal{B}'$ *such that* $x \in B' \subseteq B$*.*

Proof. (2) $\implies$ (1) Assume that (2) holds. To show $\mathcal{T} \subseteq \mathcal{T}'$. Let $U \in \mathcal{T}$. If $U = \emptyset$, then clearly $U \in \mathcal{T}'$. Suppose that $U \neq \emptyset$. Let $x \in U$. Since $\mathcal{B}$ generates $\mathcal{T}$, we see that there exists a basis element $B \in \mathcal{B}$ such that $x \in B \subseteq U$. By our assumption, it follows that, there is a basis element $B'_x \in \mathcal{B}'$ such that $x \in B'_x \subset B$. Thus $x \in B'_x \subseteq U$. So $U = \bigcup\limits_{x \in U} B'_x \in \mathcal{T}'$.

(1) $\implies$ (2) Assume that (1) holds. Suppose that $x \in X$ and $B \in \mathcal{B}$ such that $x \in B$. Note that $B \in \mathcal{T}$. Since $\mathcal{T} \subseteq \mathcal{T}'$, we see that $B \in \mathcal{T}'$. Since $\mathcal{B}'$ generates $\mathcal{T}'$, we see that there exists an element $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

Example 0.12. Let $\mathcal{B}$ be the collection of all circular regions in the $x - y$ plane. Let $\mathcal{B}'$ be the collection of all rectangular regions (sides of the rectangles are parallel to co-ordinate axes) in the $x - y$ plane. Let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies generated by $\mathcal{B}$ and $\mathcal{B}'$, respectively. Observe that, inside a rectangular region containing a point x_0 we can always have a circular region containing x_0 (see the following picture). So by Lemma 0.11, $\mathcal{T} \subseteq \mathcal{T}'$.

(3,5) (0,2) (6.22,.8).4 [](7,1.5)(4.5,1.5)(4.5,0)(7,0)(7,1.5) []¡-¿(4,-2)(4,2)[]¡-¿(0,-1)(10,-1) [linecolor=red](6.22,.8) (6.12,0.63)x_0

Next observe that, inside a circular region containing a point x_0 we can always have a rectangular region containing x_0 (see the following picture). So by Lemma 0.11, $\mathcal{T}' \subseteq \mathcal{T}$.

(3,5) (0,2.5) (6,0)1.4 [](5.8,1)(6.5,1)(6.5,0.5)(5.8,0.5)(5.8,1) []¡-¿(4,-2)(4,2)[]¡-¿(0,-1)(10,-1) [linecolor=red](6.22,.8) (6.12,0.63)x_0 Thus $\mathcal{T} = \mathcal{T}'$. Hence the topologies generated by $\mathcal{B}$ and $\mathcal{B}'$ are same.

□

The following lemma provides a way of obtaining a basis for a given topology.

Lemma 0.13. *Let* $(X, \mathcal{T})$ *be a topological space. Suppose that* $\mathcal{C}$ *is a collection of open sets of* X *such that for each open set* U *of* X *and each* $x \in U$*, there is an element* C *of* $\mathcal{C}$ *such that* $x \in C \subseteq U$*. Then* $\mathcal{C}$ *is a basis for the topology* $\mathcal{T}$ *on* X*.*

Proof. We need to show that $\mathcal{C}$ is a basis for $\mathcal{T}$.

Let $x \in X$. Since X itself is an open set, we see (by hypothesis) that there is an element C of $\mathcal{C}$ such that such that $x \in C \subseteq X$.

Next let $x \in C_1 \cap C_2$, where $C_1, C_2 \in \mathcal{C}$. Since C_1 and C_2 are open sets, we see that $C_1 \cap C_2$ is an open set. So, by hypothesis, there is an element C of $\mathcal{C}$ such that $x \in C \subseteq C_1 \cap C_2$. Thus $\mathcal{C}$ is a basis for some topology, say $\mathcal{T}'$ on X.

We now need to show that $\mathcal{T} = \mathcal{T}'$. By lemma 0.11, it follows that $\mathcal{T}'$ is finer than T, that is, $\mathcal{T} \subseteq \mathcal{T}'$. Next let $U' \in \mathcal{T}'$. Then U' is the union of elements of a sub-collection $\mathcal{C}'$ of $\mathcal{C}$, by Lemma0.9. Note that each element of $\mathcal{C}$ is an element of $\mathcal{T}$. So union of elements of the sub-collection $\mathcal{C}'$ is an element of $\mathcal{T}$ (by definition of topology), that is, $U' \in \mathcal{T}$. Thus $\mathcal{T}' \subseteq \mathcal{T}$. hence $\mathcal{T} = \mathcal{T}'$.

□

Example 0.14. Let $\mathcal{B}$ be the collection of all half-open intervals of the form $[a, b)$, where $a < b$. The topology $\mathcal{T}$ generated of $\mathcal{B}$ is called the *lower limit* topology on $\mathbb{R}$.

We have already seen that the collection of all open intervals of the form (a, b), where $a < b$ forms a basis for the usual toplogy on $\mathbb{R}$. So the natural question arises here, is whether the lower limit topology on $\mathbb{R}$ is same as the usual topology on $\mathbb{R}$. The answer is no. Are they comparable? If so, which one is bigger?

Lemma 0.15. *The lower limit topology $\mathcal{T}$ on $\mathbb{R}$ is strictly finer than the usual topology $\mathcal{T}'$ on $\mathbb{R}$.*

Proof. Let (a, b) be a basis element of $\mathcal{T}'$, and let $x \in (a, b)$. Then $[x, b)$ is a basis element of $\mathcal{T}$ and $x \in [x, b) \subseteq (a, b)$. Thus by lemma 0.11, it follows that $\mathcal{T}$ is finer than $\mathcal{T}'$.

Next let $[x, d)$ be a basis element for $\mathcal{T}$. Then there is no basis element (a, b) of $\mathcal{T}'$ such that $x \in (a, b) \subseteq [x, d)$. Thus T' is not finer than $\mathcal{T}$, by lemma 0.11. Thus $\mathcal{T}' \subset \mathcal{T}$. □

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[1]NB: For more details the students are advised to go through *Chapter 2* of the Book *Topology: A First Course* by James R. Munkres.