Topic: Topological spaces: Product topology
MI226 : Introductory Topology

Definition 1. Let X and Y be two topological spaces. The product topology on $X \times Y$ is the topology generated by the basis

$$\mathcal{B} = \{U \times V : U \text{ is open in } X, V \text{ is open in } Y\}.$$

Lemma 2. *The collection $\mathcal{B} = \{U \times V : U$ is open in X, V is open in $Y\}$ is a basis for a topology on $X \times Y$.*

Proof. Note that $X \times Y \in \mathcal{B}$. So for any (x, y), we have a basis element, namely, $X \times Y$ such that $(x, y) \in X \times Y$.

Next let $B_1 = U_1 \times V_1$ and $B_2 = U_2 \times V_2$ and $(x, y) \in B_1 \cap B_2$. We know that

$$B_1 \cap B_2 = (U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$$

Take $U = U_1 \cap U_2, V = V_1 \cap V_2$. Since intersections of two open sets is open, it follows that both U and V are open sets in X and Y, respectively. Thus $B = U \times V \in \mathcal{B}$. Hence $(x, y) \in B = U \times V \subseteq B_1 \cap B_2$.

Hence $\mathcal{B}$ is a topology for $X \times Y$. □

Remark 3. *Then open sets in the product topology on $X \times Y$ are the unions $\bigcup_{\alpha \in \Delta} U_\alpha \times V_\alpha$, of arbitrary collections $\{U_\alpha \times V_\alpha : \alpha \in \Delta\}$ of basis elements.*

Recall the following lemma:

Lemma 4. *Let $(X, \mathcal{T})$ be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open subset U of X and each $x \in U$, there is an element C of $\mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology $\mathcal{T}$ on X.*

Theorem 5. *Let X have the topology generated by $\mathcal{B}$ and Y have the topology generated by a basis $\mathcal{C}$. Then the collection*

$$\mathcal{D} = \{B \times C : B \in \mathcal{B}, C \in \mathcal{C}\}$$

is a basis for the product topology on $X \times Y$.

Proof. Since $B \in \mathcal{B}$ is open in X, $C \in \mathcal{C}$ is open in Y, it follows that the elements $B \times C$ of the collection $\mathcal{D}$ are open in the product topology on $X \times Y$.

Let W be an open subset in the product topology on $X \times Y$ and $(x, y) \in W$. Since product topology on $X \times Y$ is the topology generated by the basis $\{U \times V : U \text{ is open in } X, V \text{ is open in } Y\}$, it follows that there exists a basis element $U \times V$, where U is open in X, V is open in Y such that $(x, y) \in U \times V \subseteq W$.

Now since $x \in U$, U is open in X and $\mathcal{B}$ is a basis for the topology on X, we see that there exists a basis element $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Now since $y \in V$, V is open in Y and $\mathcal{C}$ is a basis for the topology on Y, we see that there exists a basis element $C \in \mathcal{B}$ such that $x \in C \subseteq V$. Thus we have $B \times C \in \mathcal{D}$ such that $(x, y) \in B \times C \subseteq U \times V \subseteq W$. By Lemma4, $\mathcal{D}$ is a basis for the product topology on $X \times Y$. $\square$

Example 6. Since $\{(a, b) : a, b \in \mathbb{R}, a < b\}$ is a basis for the usual topology on $\mathbb{R}$, we see that the collection $\mathcal{B}$ of all open rectangles of the form $(a, b) \times (c, d)$, where $a, b, c, d \in \mathbb{R}, a < b, c < d$ is a basis for the product topology on $\mathbb{R} \times \mathbb{R}$.

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[1]NB: For more details the students are advised to go through *Chapter 2* of the Book *Topology: A First Course* by James R. Munkres.