

**Topic: Subbasis for product topology & Subspace Topology**  
**MI226 : Introductory Topology**

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Before we begin a discussion on subspace of a topological let us quickly discuss about a subbasis for the product topology.

### 1. SUBBASIS FOR PRODUCT TOPOLOGY

First let us talk about projection map.

**Definition 1.** Let  $\pi_1 : X \times Y \longrightarrow X$  be the map defined as  $\pi_1(x, y) = x$ .

Let  $\pi_2 : X \times Y \longrightarrow Y$  be the map defined as  $\pi_2(x, y) = y$ . These maps are called a projection maps.

Let  $U$  be a subset of  $X$ . Then What is  $\pi_1^{-1}(U)$ ? Can you see that  $\pi_1^{-1}(U) = U \times Y$ . It is easy to check. So it is left as an exercise.

Similarly, if  $V$  is a subset of  $Y$ , then  $\pi_2^{-1}(V) = X \times V$ .

**Lemma 2.** *The collection  $\mathcal{B} = \{\pi_1^{-1}(U), \pi_2^{-1}(V) : U \text{ is open in } X, V \text{ is open in } Y\}$  is a subbasis for the product topology on  $X \times Y$ .*

*Proof.* We know that

$$(U \times Y) \cap (X \times V) = (U \cap X) \times (V \cap Y) = U \times V.$$

Thus  $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times V$ .

We also know that basis elements for the product topology on  $X \times Y$  are  $U \times V$ , where  $U$  is open in  $X$ ,  $V$  is open in  $Y$ . Thus for  $U$  open in  $X$ ,  $V$  open in  $Y$ , we have

$$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V).$$

Thus each basis element of the product topology on  $X \times Y$  is a finite intersection of elements of  $\mathcal{B}$ . Hence  $\mathcal{B}$  is a subbasis for the product topology on  $X \times Y$ . □

### 2. SUBSPACE TOPOLOGY

We are now going to define subspace topology.

**Definition 3.** Let  $(X, \mathcal{T})$  be a topological space, and let  $A$  be a subset of  $X$ . Consider the collection  $\mathcal{T}_A = \{A \cap U : U \in \mathcal{T}\}$ . Clearly,  $\mathcal{T}_A$  is a collection of subsets of  $A$ . Can we see that  $\mathcal{T}_A$  is a topology on  $A$ ? Yes. How? (we will answer this shortly). This topology  $\mathcal{T}_A$  is called a subspace topology on  $A$ .  $(A, \mathcal{T}_A)$  is called a subspace of  $X$ .

We now answer that  $\mathcal{T}_A$  is a topology on  $A$ .

(1) Since  $\emptyset \in \mathcal{T}$ , and  $\emptyset = A \cap \emptyset$ , we see that  $\emptyset \in \mathcal{T}_A$ . Since  $X \in \mathcal{T}$ , and  $A = A \cap X$ , we see that  $A \in \mathcal{T}_A$ .

(2) Let  $\{A \cap U_\alpha : \alpha \in \Delta\}$  be a subcollection of  $\mathcal{T}_A$ . Then

$$\bigcup_{\alpha \in \Delta} A \cap U_\alpha = A \cap \bigcup_{\alpha \in \Delta} U_\alpha \in \mathcal{T}_A, \text{ because } \bigcup_{\alpha \in \Delta} U_\alpha \in \mathcal{T}.$$

(3) Let  $A \cap U_1, A \cap U_2, \dots, A \cap U_n \in \mathcal{T}_A$ . Then

$$\bigcap_{i=1}^n (A \cap U_i) = A \cap \left( \bigcap_{i=1}^n U_i \right) \in \mathcal{T}_A, \text{ because } \bigcap_{i=1}^n U_i \in \mathcal{T}$$

Hence  $\mathcal{T}_A$  is a topology on  $A$ .

**Remark 4.** Let  $(X, \mathcal{T})$  be a topology space and  $(A, \mathcal{T}_A)$  be a subspace of  $(X, \mathcal{T})$ . Then a subset  $V$  of  $A$  is said to be "open in  $A$ " means  $V \in \mathcal{T}_A$ , that is  $V = A \cap U$  for some  $U \in \mathcal{T}$ .  $V$  is open in  $A$  does not necessarily imply that  $V$  is open in  $X$ . Can you think of an example?

We recall the following lemma.

**Lemma 5.** Let  $(X, \mathcal{T})$  be a topological space. Suppose that  $\mathcal{C}$  is a collection of open sets of  $X$  such that for each open subset  $U$  of  $X$  and each  $x \in U$ , there is an element  $C$  of  $\mathcal{C}$  such that  $x \in C \subseteq U$ . Then  $\mathcal{C}$  is a basis for the topology  $\mathcal{T}$  on  $X$ .

**Lemma 6.** Let  $\mathcal{B}$  be a basis for a topology  $\mathcal{T}$  on  $X$ . Then the collection

$$\mathcal{B}_A = \{A \cap B : B \in \mathcal{B}\}$$

is a basis for the subspace topology  $\mathcal{T}_A$  on  $A$ .

*Proof.* Let  $x \in A$  and  $V$  be open in  $A$  such that  $x \in V$ .  $V$  is open in  $A$  means  $V \in \mathcal{T}_A$ , that is,  $V = A \cap U$  for some  $U \in \mathcal{T}$ . Now  $x \in V = A \cap U$  implies that  $x \in U$ . Thus  $x \in U$ ,  $U$  is open in  $X$ . Since  $\mathcal{B}$  is a basis, there exists  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ . Take  $B' = A \cap B$ . Then  $B' \in \mathcal{B}_A$  and  $B' = A \cap B \subseteq A \cap U = V$ . Observe that  $x \in B'$ . Thus  $x \in B' \subseteq V$ . Hence  $\mathcal{B}_A$  is a basis for  $\mathcal{T}_A$ , by Lemma 5.  $\square$

**Example 7.** Let  $(\mathbb{R}, \mathcal{T})$  be the usual topological space. We know that  $\mathbb{N} \in \mathbb{R}$ . How the subspace topology  $\mathcal{T}_{\mathbb{N}}$  on  $\mathbb{N}$  look like? Note that  $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}\}$  is a basis for  $\mathcal{T}$ . Thus the the set  $\mathcal{B}_{\mathbb{N}} = \{\mathbb{N} \cap (a, b) : a, b \in \mathbb{R}\}$  is a basis for  $\mathcal{T}_{\mathbb{N}}$ , by lemma 6. Thus  $\mathbb{N} \cap (n - 1/2, n + 1/2) = \{n\} \in \mathcal{T}_{\mathbb{N}}$  for each  $n \in \mathbb{N}$ . Hence  $\mathcal{T}_{\mathbb{N}}$  is the discrete topology on  $\mathbb{N}$ .

**Example 8.** Let  $(\mathbb{R}, \mathcal{T})$  be the usual topology. Then  $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}\}$  is a basis for  $\mathcal{T}$ . Consider the subset  $A = [0, 1)$  of  $\mathbb{R}$ . Then by Lemma 6,

$$\{[0, 1) \cap (a, b) : a, b \in \mathbb{R}\} = \{[0, b), (a, b) : 0 < a < b \leq 1\}$$

is a basis for the subspace topology on  $A$ . Observe that  $[0, 1/3)$  is open in  $A$ , because,  $[0, 1/3) = [0, 1) \cap (-1/2, 1/3)$ . Note that the set  $[0, 1/3)$  is not open in  $\mathbb{R}$ . why?

Let  $(X, \mathcal{T})$  be a topology. Let  $A$  be a subspace of  $X$ . Let  $V \subseteq A$ . Since  $A \subseteq X$ , we see that  $V$  is a subset of  $X$  also. Now if  $V$  is open in  $A$  (open in the subspace topology) can we say that  $V$  is open in  $X$ ? The answer is not always. Look at the previous example.

**Definition 9.** Let  $(X, \mathcal{T})$  be a topology. Let  $A$  be a subspace of  $X$ .  $A$  is said to be an **open subspace** of  $X$  if the set  $A$  is an open subset of  $X$ .

**Lemma 10.** *Let  $A$  be an open subspace of  $X$ . Let  $V \subseteq A$ . Then the subset  $V$  is open in  $A$  if and only if  $V$  is open in  $X$ .*

*Proof.* First suppose that  $V$  is open in the subspace topology on  $A$ . Then  $V = A \cap U$  for some open set  $U$  in  $X$ . Since  $A$  is open in  $X$  and  $U$  is open in  $X$ , we see that  $V = A \cap U$  is open in  $X$ .

Conversely, assume that  $V$  is open in  $X$ . Note that  $V = A \cap V$ . So  $V$  is open in the subspace topology on  $A$ . □

**Exercise 11.** *Let  $(\mathbb{R}^2, \mathcal{T})$  be the usual topological space. Consider  $X$ -axis, that is, the set  $A = \{(x, y) \in \mathbb{R}^2 : y = 0\}$ . Can you find a basis for the subspace topology  $\mathcal{T}_A$  on  $A$ ? Do you see any connection between the open sets in  $\mathcal{T}_A$  and in the usual topology on  $\mathbb{R}$ .*

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<sup>1</sup>NB: For more details the students are advised to go through *Chapter 2* of the Book *Topology: A First Course* by James R. Munkres.