Collaborative activities with other institutions/ research establishment/industry for research and academic development of faculty and students

Sl No	Year	Name of the Faculty/Student	Collaborating Institutes	Supported by
1	2016	Basanta Saikia	Durham University, UK	NP Visiting Research - Chemistry YAF130
2	2016	Prof. Ashwini K. Phukan	Department of Science and Technology under Indo-German (DST-DAAD) collaborative research project	DST-DAAD
3	2017	Dr. Utpal Bora Prof A. J. Thakur	Institute of Materials and Environmental Sciences, Research Centre for Natural Sciences, Hungarian Academy of Sciences, Budapest, Hungary	DST collaborative project
4	2017	Dr. Kusum Kumar Bania	RMIT University, Australia	INSA
5	2019	Dr. Pankaj Bharali	Boreskov Institute of Catalysis, Novosibirsk, Russia	DST Collaborative Project
6	2019	Shivanee Borpatra Gohain	University of Nottigham, UK, under Commonwealth Split fellowship	Commonwealth Split fellowship
7	2019	Prof. Ashwini K. Phukan	Department of Science and Technology under Indo-German (DST-DAAD) collaborative research project	DST-DAAD
8	2019	Dr. Utpal Bora	Institute of Catalysis, Bulgarian Academy of Sciences, Sofia, Bulgaria	DST Indo-Bulgaria joint research project
9	2020	Dr. Utpal Bora Prof A. J. Thakur Dr. Pankaj Bharali	CSIR-NEIST Jorhat	Collaboration
10	2020	Dr. Utpal Bora	Gifu Pharmaceutical University, Japan	Collaboration
11	2020	Gunanga J Gogoi	Queen’s University, Belfast	Joint PhD programme

Durham University

Shaped by the past, creating the future

Ref: Student ID Number: 000701685

11 November 2016

Basanta Saikia
Department of Chemical Sciences
Tezpur University
Napaam, Tezpur
Assam
784028
India

Dear Mr Saikia

I am pleased to offer you a place on a postgraduate programme at Durham University. This letter contains very important information concerning your offer; please read it carefully. You must also read the additional information relating to your offer and preparing to study at Durham University which can be found on our website at www.durham.ac.uk/postgraduate/offerinformation.

The summary details of your offer are as follows:

Programme	NP Visiting Research - Chemistry YAF130
Field Of Study	Supramolecular chemistry of gels
Supervisors	Professor Jonathan Steed and Dr Katharina Edkins
JACS Code	F100 Chemistry
ATAS Statement	The project will involve metastable crystal form identification using organic functional self-assembled monolayers and functional polymers. The project will also involve pharmaceutical crystallization using supramolecular. Work will also be undertaken studying cocrystal formation to improve drug properties with particular emphasis on the and nucleation of variable stoichiometry cocrystals.
Department / School	Chemistry
Faculty	Science
Plan Of Study	Full Time
Start Date	01 October 2017
Duration	1 year(s)
End Date	30 September 2018 (end of supervised study)
College	Not yet allocated
Conditions	• Achieve an overall band score of at least 6.5 in the IELTS test (with no element below 6.0). In all cases the relevant certificate should have been issued within two years of the programme start date. Details of alternative accepted English language proficiency tests can be found at: www.dur.ac.uk/learningandteaching.handbook/1/3/3/1/
Fee Status	Overseas
Fees	£400 for academic year 2017/18.

Durham University

Shaped by the past, creating the future

For any conditions detailed above, please supply your department with documentary evidence (original or certified copies) that you have met these conditions, as soon as it is available. This information can be submitted via the online application form.

Departments may charge additional fees as appropriate, for example, to cover specific relevant costs, eg bench costs or to recognise other costs in supporting students. Further details of any additional fees will be provided by your department.

Applicants who do not need to attend an English Language pre-sessional course to meet the English language condition of their offer may be interested in attending a one month pre-sessional course, although this is not a requirement. The one month Preparatory Academic Research Skills & English (PARSE) course is recommended for all applicants wishing to develop their academic English writing and speaking skills. Further details are available at the English Language Centre website: www.dur.ac.uk/englishlanguage.centre.

Overseas applicants will receive a separate communication regarding the information that will be required for their visa application. Please do not try to use this offer letter to apply for a visa as it will result in your visa application being refused. If you require a visa to study in the UK and ATAS clearance is required for your programme, please note that you will need to apply online for an ATAS clearance certificate at the Foreign and Commonwealth Office (FCO) website. Please follow the official guidance at: www.fco.gov.uk/atas. Please be advised that you will need to apply for your ATAS certificate before applying for your visa, allowing at least 4 weeks to obtain it.

As a commitment to supporting students, Durham University offers a range of financial support opportunities in the form of Scholarships, Bursaries, Studentships and from the start of the 2016/17 academic year, a tuition fee discount for Durham University Alumni returning to study at postgraduate level at Durham. Further information can be found at www.dur.ac.uk/scholarships and www.dur.ac.uk/postgraduate/finance/pgscholarships/alumni/. Students who are in receipt of financial awards from Durham University will receive a separate communication confirming the terms and conditions of the award.

Once you have accepted your offer, you will be contacted by the Collegiate Office regarding arrangements for your college membership and accommodation.

Please note that the personal information provided by you to Durham University will be used only for the purposes of processing your application. Following acceptance of your offer, the University will add further data at registration and then during the course of your education at Durham University, in line with the business purposes outlined in our Data Protection Notification and Data Protection Policy, available at www.dur.ac.uk/data.protection/policy/.

We hope you will accept this offer and would be grateful if you could reply to us to let us know your decision as soon as possible. Acceptance of this offer indicates acceptance of Durham University's terms and conditions of study, full details of which are given in Volume II of the University's Calendar at www.dur.ac.uk/university.calendar/. Please send your reply to pg.research@durham.ac.uk quoting your name and Student ID number (shown at the top of this letter). You can also contact us on this email address or telephone us on +44 (0) 191 334 6492 if you have any questions regarding your offer; we will be happy to help.

Yours sincerely



Mrs Elizabeth Dodds BA (Hons)
Head of University Admissions

The Palatine Centre Durham University Stockton Road Durham DH1 3LE

Telephone +44 (0) 191 334 2000
www.durham.ac.uk

सत्यमेव जयते

Telegram : Scientech
दूरभाष/Tel. : 26962819, 26567373,
26562134, 26562122 (EPBAX)
फैक्स/Fax : 26569908, 26864570,
26863847, 26862418
वेबसाईट/website : www.dst.gov.in

भारत सरकार
विज्ञान और प्रौद्योगिकी मंत्रालय,
विज्ञान और प्रौद्योगिकी विभाग,
टेक्नोलॉजी भवन, नया महरौली मार्ग,
नई दिल्ली-110 016

GOVERNMENT OF INDIA
MINISTRY OF SCIENCE AND TECHNOLOGY,
DEPARTMENT OF SCIENCE AND TECHNOLOGY,
TECHNOLOGY BHAVAN, NEW MEHRAULI ROAD,
NEW DELHI-110 016

No. INT/FRG/DAAD/P-9/2016

Dated: 1-6-2016

ORDER

Sanction of the President is hereby accorded for implementation of the Indo-German (DST-DAAD) joint research project entitled "Homocoupling, Heterocoupling and Liberation of Borylene Ligands" at a total cost of **Rs. 10,15,000/- (Rupees Ten Lac Fifteen Thousand Only).** The duration of the project shall be 2 years from date of issue of this sanction order.

2. The composition of the project team is as below:

Indian side	German side
i.) Dr. Ashwini Kumar Phukan, Associate Professor, Deptt. of Chemical Sciences, Tezpur University, Napam – 784 028, Assam (India). ii)Bitupon Borthakur, Graduate doctoral candidate.	i.) Dr. Holger Braunschweig, Professor Institute of Inorganic Chemistry, Lehrstuhl II, Am Hubland, Julius-Maximilians-Universitat Wurzburg 97074 Wurzburg, Germany ii)Dr. Dewhurst, Rian(Senior Researcher) iii)Nutz, Macro(M.Sc)

3. The change/ replacement of project participants is not allowed. The participating scientists should be working on the project and affiliated with the Institute/University holding the indicated position at the time of start of the project as well as at the time of exchange visits under the project. The expenditure related to approve visits of participating scientists, as mentioned above, only can be met from the project grant.

4. The break-up of estimated expenditure is as indicated below and no re-appropriation shall be allowed under this project:-

Headings	Estimated expenditure(s)		
	1st year	2nd year	Total
1. Expenses for Indian scientists to visit Germany under the project International air-fare (by IATA excursion) (2 persons) 1) Air fare: 65,000*2 2) Surface travel in india: 32,000*2 3) Surface travel in germany:17,000*2 4) Cost of overseas insurance cover for Indian visitors in RS: (3,500+3,000) = 6,500	Rs2,34,500/-	Rs2,34,500/-	Rs4,69,000/-
2.Expenses for German scientists to visit India under the project **(i)** Accommodation charges @ Rs. 4000/- per day.(duration 21days, 2persons) **(ii)** Per-diem charges @ Rs. 2500/- per day (2 persons)	Rs2,73,000/-	Rs2,73,000/-	Rs5,46,000/-
3. Total	**5,07,500/-**	**5,07,500/-**	**10,15,000/-**

(Rupees Ten Lac Fifteen Thousand Only)

Contd. p/2

mta ttk

DEPARTMENT OF ENVIRONMENTAL CHEMISTRY AND CATALYSIS
INSTITUTE OF MATERIALS AND ENVIRONMENTAL CHEMISTRY
RESEARCH CENTRE FOR NATURAL SCIENCES
HUNGARIAN ACADEMY OF SCIENCES
H-1117 BUDAPEST, MAGYAT TUDÓSOK KÖRÚTJA 2.

ADDRESS: H-1519 BUDAPEST, P.O.BOX 286.
PHONE: +36 1 382 6860
E-MAIL: VALYON.JOZSEF@TTK.MTA.HU
WWW.TTK.MTA.HU

Professor Dhanapati Deka
Department of Energy
Tezpur University
Tezpur, Assam, India

April 27, 2017

Re: Invitation letter to Prof. Dhanapati Deka and Professor Utpal Bora.

Dear Professor Deka,

The program-based bilateral R&D cooperation project of India and Hungary, titled "Biochemicals, and biofuels from lignocellulosic biomass by green catalytic processes", supported in Hungary by the National Research, Development and Innovation Office, Hungary (Project No: TÉT_15_IN-1-2016-034) reached the stage when personal meeting of cooperating scientists seems to be necessary to move the work on. Therefore, it is my pleasure to invite you, as the Indian principal investigator (PI) of the project, and Assistant Professor Dr. Utpal Bora, who is the Co-PI of the project, to visit our institute during June 12 - June 20, 2017.

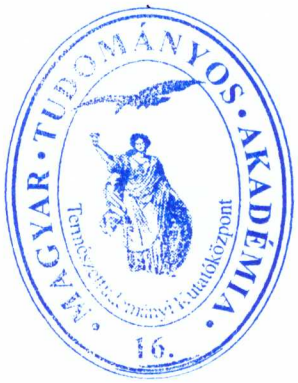
We booked two rooms for you for 8 nights at Hotel Ibis (Ráday utca 6., Budapest, Hungary 1092).

Your visit in our Institute was a great opportunity to change views about current scientific questions. I would be glad if you presented a lecture that covers an interesting topic on heterogeneous catalysis or nanostructured materials.

Sincerely yours

Valyon József

Dr.Valyon, József
PI of the Hungarian Project Partner



एस.पी. मिश्र
उप कार्यकारी निदेशक-I
S. P. MISHRA
Deputy Executive Director-I

स्पीड पोस्ट
SPEED POST

भारतीय राष्ट्रीय विज्ञान अकादमी
बहादुर शाह ज़फर मार्ग, नई दिल्ली-110 002
INDIAN NATIONAL SCIENCE ACADEMY
Bahadur Shah Zafar Marg, New Delhi-110 002

No. IA/INDO-AUST/F-13/2017/1881
9 January 2017

10 JAN 2017

Dr Kusum Kumar Bania
Assistant Professor
Department of Chemical Sciences,
Tezpur University, Napaam
Tezpur-784028
Assam
kusum@tezu.ernet.in

Sub: Indo-Australia Early & Mid-Career Researchers Fellowship Programme 2016-17

Dear Dr. Bania,

The Indian National Science Academy (INSA) New Delhi is pleased to award you the **Indo-Australia Early & Mid-Career Researchers Fellowship Programme 2016-2017** for conducting advance research on the project titled **"Zeolite-Y Encapsulated Chiral Catalyst for Asymmetric Synthesis"** for a duration of **7 months at the RMIT University Australia.**

The Academy will provide funds for your international air travel, fellowship amount, contingency amount and fund for travel within the host country as per the guidelines governing the fellowship programme **(Guidelines-Annexure-I is enclosed)**.

Early & Mid-Career Researchers Fellowship:

Sl. No.	Details	Amount
1	Fellowship amount including accommodation charges in Australia (as per actual)	**USD 3000/- per month**
2	International Air fare by economy class (Air India) including domestic and airport transportation (as per actual)	**Rs. 1,00,000/-**
3	Visa charges, medical insurance and contingency	**Rs. 25,000/-**
4	Grant for attending workshop/conference in Australia	**USD 1200/-**

The Fellowship should be commenced between January 2017 and June 2017 and must be completed by 31 March 2018 and no extension is possible.

The advance payment for your visit will be released to your institute and the balance will be settled after completion of fellowship and return to India on the receipt of the detailed report, statement of expenditure and other required documents as per INSA formats dully forwarded by the Head of your Institute/University. **Kindly provide Institute Bank details for release of money.**

Continue....2/-

BORESKOV INSTITUTE OF CATALYSIS

Letter of Invitation

Dear Dr. Bharali,

It is pleasure for me to invite you to visit our institute in the framework of our joint scientific program that is supported by the RFBR and DST (RFBR project number – RFBR# 18-53-45012). According to our agreement, you are invited to visit and work in our laboratory for a time span of 14 days between June, 5th and June, 30th 2019. We will arrange accommodation for you and we will be glad to offer you assistance in all arrangements during your stay in Russia. You will know that you will require a health insurance valid in Russia to get your visum – a travel health insurance, which is certainly also offered in India, will be sufficient if it covers the length of your stay. If there any problems we will advise and support you.

I hope that you will be able to obtain your visum for Russia in time. Please let us know what we can do to support you in finding the smoothest way to come to Novosibirsk.

Looking forward to see you in Novosibirsk soon!

Sincerely Yours, 11.03.2019

(Dr. Kerzhentsev Mikhail Anatolievich)

Boreskov Institute of Catalysis
5 Lavrentiev Ave.
630090, Novosibirsk, Russia
http://catalysis.ru
tel.: +7 383 330 67 71
fax: +7 383 330 80 56
bic@catalysis.ru

BORESKOV INSTITUTE OF CATALYSIS

Letter of Invitation

Dear Mr. Borah,

It is pleasure for me to invite you to visit our institute in the framework of our joint scientific program that is supported by the RFBR and DST (RFBR project number – RFBR# 18-53-45012). According to our agreement with your supervisor, Dr. Pankaj Bharali, you are invited to visit and work in our laboratory for a time span of 14 days between June, 5th and June, 30th 2019. We will arrange accommodation for you and we will be glad to offer you assistance in all arrangements during your stay in Russia. You will know that you will require a health insurance valid in Russia to get your visum – a travel health insurance, which is certainly also offered in India, will be sufficient if it covers the length of your stay. If there any problems we will advise and support you.

I hope that you will be able to obtain your visum for Russia in time. Please let us know what we can do to support you in finding the smoothest way to come to Novosibirsk.

Looking forward to see you in Novosibirsk soon!

Sincerely Yours, 11.03.2019

(Dr. Kerzhentsev Mikhail Anatolievich)

Boreskov Institute of Catalysis
5 Lavrentiev Ave.
630090, Novosibirsk, Russia
http://catalysis.ru
tel.: +7 383 330 67 71
fax: +7 383 330 80 56
bic@catalysis.ru

Commonwealth Scholarship Commission in the UK

Woburn House
20-24 Tavistock Square
London WC1H 9HF
United Kingdom

T +44 (0)20 7380 6700
F +44 (0)20 7387 2655

www.dfid.gov.uk/cscuk

Ms. Shivanee Borpatra Gohain
Room 404 Pobitora Madam Curie Women's Hostel
Tezpur University
Napaam
Tezpur, Assam
784028
India

08/08/2019
INCN-2019-102

Dear Ms. Borpatra Gohain

Confirmation of Award

Thank you for returning the award acceptance form for our offer of a Commonwealth Split-site Scholarship under the CSC development theme of **Science and Technology for Development**, tenable at the University of Nottingham.Your scholarship will be funded by the Commonwealth Scholarship Commission, with the University of Nottingham making a contribution to your tuition fee. As you have now met all conditions expressed in our Notification of Award, I am able to confirm the Commission's offer of a scholarship.

Information to assist in applying for your Tier 4 student visa (for Chevening, Marshall and Commonwealth scholars) is available on the Commission website. You are strongly advised to read all of the information before making a visa application. Also, attached is your *'Confirmation of Financial Support'* letter. Please note that you must submit both letters with your visa application. You will be contacted with pre departure information by the British Council in your home country.

It is important that you familiarise yourself with what to expect when you arrive in the UK both through the 'pre-departure' materials provided to you and the information available from your university. The CSC cannot advise you on securing accommodation, you are encouraged to speak to your university's accommodation office if you have not already. You can also find useful advice on finding accommodation, as well as other useful information on studying in the UK on the UKCISA website here: www.ukcisa.org.uk . It is highly recommended that you take advantage of any 'meet and greet' service that your university provides. Information may have been provided with your university offer letter or you can contact your International Office through the university's website for instructions on how to travel to your institution.

Enclosed is a *cashcard* which will be loaded with your warm clothing allowance and your first quarterly stipend. Your cashcard can be used at bank machines around the UK. Instructions on how to use your card were provided with your confirmation email and you must ensure that the cashcard is activated before your arrival in the UK. We recommend that you carry some funds with you for your arrival in case of emergencies or any unforeseen problems with your cashcard. A sum of £200 should be sufficient for this purpose. If you have any enquiries regarding your cashcard please contact finance@cscuk.org.uk

You are now part of a community of over 27,500 people who have held a Commonwealth Scholarship or Fellowship since 1959. Commonwealth Scholarships support talented and motivated individuals who have the potential to make change. Your Commonwealth Scholarship is funded by the UK *Department for International Development* (DFID) with the aim of enabling you to gain the knowledge and skills required for sustainable development.

COMMONWEALTH SCHOLARSHIPS

DST/INT/DAAD/P-12/2019
Government of India
Ministry of Science and Technology
Department of Science & Technology
(International Bilateral Cooperation Division)

Technology Bhavan, New Mehraulli Road
New Delhi-110016
Date: 25/09/2019

ORDER

Subject: Implementation of Indo-German Joint project entitled "**Synthesis, Reactivity and Bonding in Metal and Metal-free Borylenes**" coordinated by **Dr. Ashwini Kumar Phukan, Professor, Department of Chemical Sciences, Tezpur University, Napam–784028, Assam, India** with the German partner **Dr. Holger Braunschweig** - regarding.

Sanction of the President is hereby accorded for incurring an expenditure not exceeding **Rs. 13,68,000/- (Rupees Thirteen Lakh Sixty Eight Thousand only)** for implementation of the Indo-German joint project entitled "**Synthesis, Reactivity and Bonding in Metal and Metal-free Borylenes**" coordinated by **Dr. Ashwini Kumar Phukan, Professor, Department of Chemical Sciences, Tezpur University, Napam–784028, Assam, India** in collaboration with **Dr. Holger Braunschweig, Professor, Institute of Inorganic Chemistry Lehrstuhl II, Am Hubland, Julius Maximilians - Universität Würzburg, 97074 Würzburg, Germany** for a total duration of 2 years from the date of issue of the sanction order. The detailed breakup of the grant for General as well as Capital Components are given below :-

General Component	:	Rs. **13,68,000/-**
Capital Component	:	Rs. Nil

2. As per the terms and conditions, agreed by both side, under the project the sending side will bear the cost related to the International air travel, medical insurance and visa charges whereas the receiving side shall bear the cost of accommodation, hospitality and local travels of the visiting scientist. The break-up of approved expenditure is as indicated below :-

Headings	Estimated expenditure(s)		
	1st year	2nd year	Total
1. Expenses for Indian scientists to visit Germany under the project International air-fare (by IATA excursion/ economy class by shortest route) including surface travel from place of work in India to place of work in Germany & back; Visa charges. Medical Insurance cover.	2 visits 24 & 24 days 424,000/-	2 visits 24 & 24 days 424,000/-	4 visits 96 days 848,000/-
2.Expenses for German scientists to visit India under the project (i)Per-diem charges @ Rs. 2,500/- per day (ii) Accommodation charges	2 visits 20&20 days 100,000/- 160,000/-	2 visits 20&20 days 100,000/- 160,000/-	4 visits 64 days 200,000/- 320,000/-
3. Total	684,000/-	684,000/-	13,68,000/-

3. Sanction of the President is hereby accorded for release of 1st instalment amounting of **Rs. 684,000/- (Rupees Six Lakh Eighty Four Thousand only)** to the grantee Institute. The amount of grant will drawn by the Drawing and Disbursing Officer, DST and will be disbursed to **Department of Chemical Sciences, Tezpur University, Assam**. The bank details for electronic transfer of funds through RTGS are given below:-

E-25929

No. DST/INT/BLR/P-15/2019
Government of India
Ministry of Science and Technology
Department of Science & Technology
(International Bilateral Cooperation Division)
****

Technology Bhawan,
New Mehraulli Road
New Delhi-110016
Date :- 18.09.2019

ORDER

Subject: Implementation of Indo-Portugal Joint project entitled: **"Green and sustainable catalysts based on zeolites for oxidative transformation reactions"** coordinated by Dr. Utpal Bora with the Bulgarian partner Dr Yuri Kalvachev regarding.

Sanction of the President is hereby accorded for incurring an expenditure not exceeding **Rs. 11,82,350/- (Rupees Eleven Lakh Eighty Two Thousands Three Hundred fifty)**for implementation of the Indo-Iran joint project entitled **"Green and sustainable catalysts based on zeolites for oxidative transformation reactions"** coordinated by Dr Utpal Bora Tezpur University, Department of Chemical Science, Napaam, Tezpur Assam, India -784028 India with the Dr. Yuri Kalvachev Professor Institute of Catalysis, Bulgarian Academy of Science, Acad Georgi Bonchev, Sofia Bulagaria a total duration of two years from the date of issue of the Sanction Order. The detailed breakup of the grant for General as well as Capital Components are given below:-

General Component : Rs. **11,82,350/-**
Capital Component : Rs.___N.A.__

S. No.	Budget Head	1st Year (Rs.)	2nd Year (Rs.)	Total (Rs.)
1.	Number of visits (02 per year) one of up to 21 days by PI/ CO-PI 2500*21 = one of up to 03 months from Bulgarian side by student/ young researcher PI/Co PI 50000/ month * 03	52,500 1,50,000	52,500 1,50,000	1,05,000 3,00,000
	Accommodation	1,00,000	1,00,000	2,00,000
2.	Air Travel* International travel by shortest route by lowest economy class (to and fro) for visit of Indian participants to Bulgaria including silver class medical insurance,	2,00,000	2,00,000	4,00,000

Advanced Materials Group
Materials Sciences and Technology Division
CSIR-North East Institute of Science and Technology, Jorhat
(Council of Scientific & Industrial Research)
उत्तर पूर्व विज्ञान तथा प्रौद्योगिकी संस्थान
(सी एस आई आर की अंगीभत इकाई)

NO. QSP/MR/18/MS//2019 November 20, 2019

With reference to the email from Dr Utpal Bora, Associate Professor, Department of Chemical Sciences, Tezpur University dated November 01, 2019 (copy enclosed), informed that currently their research group along with Prof. Ashim J Thakur, Professor and Dr Pankaj Bharali, Assistant Professor are facing problem for the analysis of the XPS characterization of the nanomaterials due to lack of expertise in this area. Now, they are willing to collaborate with undersigned for details XPS analysis of the nanocatalyst and intellectual input in the characterization for complete understanding of the chemical bonding and surface functionalization of the nanocatalyst. The work will be carried out without hampering the assigned tasks. The intellectual input of the XPS analysis of the nanocatalyst will be provided by the undersigned. The collaboration will open up a formulation of a new joint collaborative project in the area of nanomaterials synthesis and its application. The knowledge generation under this collaboration research work will be shared by both the institute.

Director may kindly allow to collaborate with Dr Utpal Bora and his team from Department of Chemistry, Tezpur University to provide the characterization of the nanoctalyst.

20/11/2019

(Manash R. Das)
Sr. Scientist, MSTD

20/11/19

Head, MSTD

Head, RPBD May be approved.

20/11/19

DIRECTOR

20/11/19

Organic & Biomolecular Chemistry

ROYAL SOCIETY OF CHEMISTRY

PAPER



Cite this: *Org. Biomol. Chem.*, 2021, **19**, 1344

Revisiting the synthesis of aryl nitriles: a pivotal role of CAN†

Rakhee Saikia,[a] Kwihwan Park,[b] Hayato Masuda,[b] Miki Itoh,[b] Tsuyoshi Yamada,[b] Hironao Sajiki,[b] Sanjeev P. Mahanta,[a] Ashim J. Thakur[a] and Utpal Bora *

Received 17th December 2020,
Accepted 6th January 2021

DOI: 10.1039/d0ob02518f

rsc.li/obc

Facilitated by the dual role of Ceric Ammonium Nitrate (CAN), herein we report a cost-effective approach for the cyanation of aryl iodides/bromides with CAN–DMF as an addition to the existing pool of combined cyanation sources. In addition to being an oxidant, CAN acts as a source of nitrogen in our protocol. The reaction is catalyzed by a readily available Cu(II) salt and the ability of CAN to generate ammonia in the reaction medium is utilized to eliminate the additional requirement of a nitrogen source, ligand, additive or toxic reagents. The mechanistic study suggests an evolution of CN^- leading to the synthesis of a variety of aryl nitriles in moderate to good yields. The proposed mechanism is supported by a series of control reactions and labeling experiments.

Introduction

The ubiquity of nitrile-containing frameworks and their growing demand in the clinic are attributed to the biocompatibility of the nitrile functionality.[1] Around 30 nitrile-containing compounds were suggested in a variety of medications in 2010 and an additional 20 were being considered as leads for clinical development.[2] Benzonitriles and heteroaromatic nitriles also form substructures of various fine chemicals and agrochemicals.[3] They are commercially utilized in high-performance rubbers, polymers, molecular electronics and in the synthesis of heterocycles and can be easily transformed into a number of synthetically important functionalities.[4] Hence, the search for sustainable and cost-effective processes of nitrile synthesis is always an important area of research. Transition metal-catalyzed synthesis of benzonitrile and its derivatives occupies a significant domain of synthetic organic chemistry. Although Mn,[5] Fe,[6] Co,[7] Ni,[8] Zn,[9] Pd,[10] Rh,[11] and Ru[12] have all been utilized to mediate the process of cyanation, a plethora of cyanation processes have been efficiently catalyzed by Cu, in its various homogeneous and heterogeneous modifications. This is on account of its economy and versatility.[13] The lethal effects caused by the stoichiometric use of metal cyanides and high temperature (approx. 150-250 °C) employed in the classical approaches of cyanation by Sandmeyer[14] and Rosenmund–von Braun[15] prompted the utilization of either indirect cyanide sources or *in situ* generated cyanide. Apart from the recently established indirect cyanation sources such as nitromethane,[16] acetone cyanohydrin,[17] DMF,[18] *t*-BuNC,[19] NaN_3,[20] dimethylmalononitrile,[21] formamide,[22] $K_4[Fe(CN)_6]$,[23] AIBN,[24] acetonitrile,[25] and ethyl(ethoxymethylene)cyanoacetate,[26] the extensive establishment of combinations of sources of carbon and nitrogen to generate aryl nitriles is gaining increasing importance. The combinations of NH_3(aq)–DMF,[27] NH_4I–DMF,[28] NH_4HCO_3–DMSO,[29] and NH_4HCO_3–DMF[30] are some standout examples. In this regard, the cyanation of readily available aryl iodides/bromides with these combined-reagents enables easy access to aryl nitriles. 4-(Trifluoromethyl)benzonitrile as a critical intermediate of the antidepressant 'Fluvoxamine' is readily synthesized from 4-chlorobenzotrifluoride on a ton-scale.[31]

Known to participate as a one-electron oxidant in a number of carbon–carbon bond formation reactions,[32] the importance of Ceric Ammonium Nitrate (CAN) is highlighted in the pioneering works by Trahanovsky[33] and Heiba–Dessau.[34] As a most extensively used cerium(IV) reagent, the wide popularity of CAN is based on its easy handling, cost-effectiveness, high reduction potential value (+1.61 V *vs.* NHE), low toxicity and high solubility in many organic solvents.[35]

In addition to its use as an oxidant, herein, we present a report of its capability to act as a nitrogen atom donor. Together with DMF as a carbon source (and solvent), the two easily available precursors modulate into an efficient cyanating agent for aryl iodides/bromides. The presented methodology highlights a dual role of CAN and requires no additives, no toxic reagents and involves a ligand-free Cu(II) catalyst (Fig. 1).

[a]*Department of Chemical Sciences, Tezpur University, Napaam, Tezpur, Assam, Pin-784028, India. E-mail: utbora@yahoo.co.in, ubora@tezu.ernet.in*
[b]*Laboratory of Organic Chemistry, Gifu Pharmaceutical University, 1-25-4 Daigaku-nishi, Gifu 501-1196, Japan*

†Electronic supplementary information (ESI) available. See DOI: 10.1039/d0ob02518f

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QUB Ph D SUBSISTANCE ALLOWANCE

CONTINUATION CERTIFICATE & CLAIM

This is to certify that I, Mr. GUNANGA JYOTI GOGOI, a student in the Department/ School of Mechanical and Aerospace Engineering at Queen's University, Belfast have been present for the work of the PhD program under MoU between Tezpur University and Queen's University, Belfast during the period from 08/04/2021 to 07/05/2021.

The claimed amount: UK £ 500 @ £500 pm.

Gunanga Jyoti Gogoi

Signature of the Student

Date: 07/05/2021

Counter signed by:

Chenbg

Signature of Supervisor at QUB

Date: 08.05.2021

Niranjan Karak

Signature of Co-supervisor at TU

Date: 12/05/2021

Fwd: Tezpur Students - Collaborative Research Agreements

From: "John Holbrey" <J.Holbrey@qub.ac.uk>
Subject: Fwd: Tezpur Students - Collaborative Research Agreements
Date: Fri, May 21, 2021 9:29 pm
To: "ruli@tezu.ernet.in" <ruli@tezu.ernet.in>

Dear Prof Borah,

I'm please to say that Sanskrita is settling in to Queen's well and is clearly very keen and capable. I'm excited to see how the project will develop. We have an administrative job that has to be completed, the attached Collaborative Research Agreement form requires a number of signatures from you and authorities at Tezpur and then if you could return it to me I will pass it on to have the corresponding signatures completed from administration at the QUB side.

Regards

John

--

Prof John Holbrey
School of Chemistry and Chemical Engineering
Queen's University Belfast

-------- Forwarded Message --------
From: Amber Glanfield
<A.Glanfield@qub.ac.uk<mailto:Amber%20Glanfield%20%3cA.Glanfield@qub.ac.uk%3e>>
To: John Holbrey
<J.Holbrey@qub.ac.uk<mailto:John%20Holbrey%20%3cJ.Holbrey@qub.ac.uk%3e>>, Haresh
Manyar <h.manyar@qub.ac.uk<mailto:Haresh%20Manyar%20%3ch.manyar@qub.ac.uk%3e>>
Subject: Tezpur Students - Collaborative Research Agreements
Date: Fri, 23 Apr 2021 08:36:09 +0000

Dear John and Haresh,

I was pleased to hear the Tezpur students have now arrived safely and are settling in at Elms Village.

The Graduate School has now sent through the Collaborative Research Agreements that must be signed by the students, yourselves and your Tezpur counterparts. So please could these forms be completed in collaboration with your student as you start to virtually meet with them over the next few weeks.

The name of the co-ordinator at Tezpur is Professor D K Bhattacharyya. The Tezpur co-supervisor will know who that is and can arrange their signature, Prof Bhattacharyya will in turn arrange for the Tezpur PVC to sign. Once you have all those parts and your own sections completed please return them to me I can finalise the remaining signatures on the QUB side.

Kind regards,

Amber

Amber Glanfield, PhD
Research Administrator
School of Chemistry and Chemical Engineering

[Logo]<https://eur02.safelinks.protection.outlook.com/?url=http%3A%2F%2Fwww.qub.ac.uk%2F%3Futm_source%3Dlogo%26utm_medium%3Demail%26utm_campaign%3DQUB%
[www.qub.ac.uk]<https://eur02.safelinks.protection.outlook.com/?url=http%3A%2F%2Fwww.qub.ac.uk%2F%3Futm_source%3DWUB%2520Text%2520Link%26utm_medium%3De

[Facebook]<https://eur02.safelinks.protection.outlook.com/?url=https%3A%2F%2Fwww.facebook.com%2FQUBelfast%2F&data=04%7C01%7CJ.Holbrey%40qub.ac.uk%7Cb09b
[Twitter]<https://eur02.safelinks.protection.outlook.com/?url=https%3A%2F%2Ftwitter.com%2FQUBelfast&data=04%7C01%7CJ.Holbrey%40qub.ac.uk%7Cb09b640ba4464
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[Youtube]<https://eur02.safelinks.protection.outlook.com/?url=https%3A%2F%2Fwww.youtube.com%2Fuser%2FCome2Queens&data=04%7C01%7CJ.Holbrey%40qub.ac.uk%7C
[Snapchat]<https://eur02.safelinks.protection.outlook.com/?url=https%3A%2F%2Fwww.snapchat.com%2Fadd%2Fqubelfast&data=04%7C01%7CJ.Holbrey%40qub.ac.uk%7Cb
[Spacer]
[Spacer]

[Address]

Attachments:

cid: B91011A007CDBF45BE5F6AC7E2DA5BD2@eurprd07.prod.outlook.com.html
Size: 23 k
Type: text/html

image001.jpg	
Size:	34 k
Type:	image/jpeg
Info:	image001.jpg

image002.jpg	
Size:	2.5 k
Type:	image/jpeg
Info:	image002.jpg

QUEEN'S UNIVERSITY BELFAST

Admissions and Access Service

Queen's University Belfast
Belfast
BT7 1NN
Northern Ireland
United Kingdom
Tel +44 (0)28 9097 3004
intl.student@qub.ac.uk
www.qub.ac.uk

ACADEMIC TECHNOLOGY APPROVAL SCHEME (ATAS)

Before you can apply for entry clearance to study in the UK, you will need to apply online for an ATAS clearance certificate. When you make your online application for your certificate you will need the following information. Please see the enclosed guidance notes for further details.

Name:	Ms Sanskrita Madhukailya		
Date of Birth:	04/01/1996		
Applicant Code:	40336351	**Application Number:**	00684027
CAH3 Code:	CAH07-02-01		
CAH3 Descriptor:	Chemistry		
School:	School of Chemistry & Chemical Engineering		
Qualification:	PhD	**Length of Programme:**	4 Years
Method of Study:	Full-time Research	**Date of Entry:**	March 2021

Proposed Area of Research*

New Ionic liquid materials recently developed at QUB have the potential to be used as draw fluids for forward osmosis water desalination using low grade energy to address the global challenge to provide clean, accessible drinking water. To further develop and address technical performance challenges in these materials, ionic liquids incoprorating substituted phenyltetrazolate anions will be synthesised and and investigated. This research program will provide training in synthesis, characterisation and uses of ionic liquids and ionic liquid catalysts as enabling technologies directed at addressing global challenges.

C McEachern

Cathy McEachern (Mrs)
Admissions Manager (International)

* Please note that the proposed area of research may change over time. Consequently what is set out above is not a contractual commitment to working within this particular area of research.

QUEEN'S UNIVERSITY BELFAST

Admissions and Access Service

Queen's University Belfast
Belfast
BT7 1NN
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United Kingdom
Tel + 44 (0)28 9097 3004
intl.student@qub.ac.uk
www.qub.ac.uk

Application No: 00684027

25 January 2021

Ms Sanskrita Madhukailya
Dhowapukhuri Gaon
Charigaon, Assam
Jorhat
India

Dear Ms Madhukailya

School of Chemistry and Chemical Engineering

Joint PhD - Queen's University Belfast and Tezpur University

Design of Supported Ionic Liquid Catalysts for the Synthesis of 5 Substituted Tetrazoles

I refer to your application for admission as a full-time research student to the above programme. The Selector has now considered your application and I am pleased to confirm that it will be possible to offer you a place commencing in March 2021 at Queen's University Belfast.

Your Principal Supervisor will be Professor John Holbrey, email j.holbrey@qub.ac.uk, School of Chemistry and Chemical Engineering, Queen's University Belfast and your Tezpur University (Co-Supervisor) will be Professor Ruli Borah, email ruli@tezu.ernet.in.

The Faculty of Engineering and Physical Sciences will email you a signed copy of the Collaborative Research Student Agreement as soon as it is available.

You are advised to read the additional notes concerning the project title of your research and your registration as an undifferentiated research student in the first instance. This information is also available at www.qub.ac.uk/new-students.

Please note that you will be required to obtain **ATAS clearance** before you can enrol and register at the University. The enclosed ATAS summary document will help you to complete your application for ATAS clearance and you should begin this process immediately. Full details about the scheme, including the online ATAS application, can be found at www.fco.gov.uk/atas.

Tuition Fees and Accommodation Costs – the combined tuition fee and accommodation cost for each semester in 2020/21 will be £10,777. Arrangements for the payment of tuition fees will be subject to the terms and conditions of the agreement between Queen's University Belfast and Tezpur University. Accommodation will be reserved for you in Willow Walk (a 15 minute walk from the University Campus). You will receive further information regarding this when you accept your offer.

Your start date will be Monday, 1 March 2021. Details of the admissions procedures, and support facilities for students together with useful contact details, are outlined in the Admissions and Access Service Guidance Notes. This information is also available at www.qub.ac.uk/new-students.

Replying to your offer of a place – It is essential that you let me know whether or not you wish to accept this offer, by replying online following the links through the postgraduate application portal at http://go.qub.ac.uk/pgapply.

In the meantime, if you need any further help or advice regarding your admission, please do not hesitate to contact the Admissions Service.

Yours sincerely

Cathy McEncheran

SERVICE AGREEMENT

This Service Agreement is made on the 23rd day of Feb, 2017

Between,

Finitiatives Learning India Private Ltd (hereinafter referred as 'FLIP'), a finance education company, having its registered office at # 1112, 10th Main, Gokulam 3rd Stage, Mysore - 570002, Karnataka, India

AND

School of Management Sciences, Tezpur University, an educational institute, having its registered office at Tezpur-784028, Assam, India herein after referred to as 'Client'.

Overview

This agreement covers the terms and conditions, under which the Client will utilise FLIP's online training solutions.
This agreement is effective for a period of 3 years, starting from Feb 2017.
The proposed launch date for the program is 14th day of March, 2017.

Product & Service Offerings

FLIP will offer the following products and services to the Client.

Product		
Course	**Practice and evaluation tests**	**Assessments**
Smart Banker Program with Finacle from Infosys – Part 1	Interim tests will be available post set of chapters throughout the program	Online Assessment- Part 1
Smart Banker Program with Finacle from Infosys – Part 2		Online Assessment- Part 2

- This course will be offered to the 3rd year students of M.com Integrated program with BFS Specialization & credit points will be provided as per the client rules & regulations.
- Access to the online program will be given for 6 months/batch.
- Course coverage of additional training is mentioned in Annexure 1.

FLIP will offer the following standard services (included in pricing), to complement its training solution, for the duration of the program.

Standard Services	
Service Description	**Frequency**
Kick off session (telephonic/web-based)	1 session before the start of the batch
Telephonic Doubt Clearing Sessions with faculty	2 sessions per batch. (Additional will be charged at INR 3500+Tax per session)
Technical support over email and phone	Unlimited, during the time training is active
Learning report to Client SPOC (Single Point of Contact)	Fortnightly, during the time training is active
Completion status to learners	Available to Candidate 24*7
Newsletter on happenings & terms in Banking & Finance	Weekly

IT/System Requirements
Network should allow FLASH content
The client system should have FLASH installed in the browser
Each screen of our chapter files is <200KB; **base level net connectivity** required
Our content is compatible with all browsers **except Internet Explorer (IE) 6**

Pricing & Payment Schedule

Pricing			
Course / Program Name	**Price Per License (Excluding Tax)**	**No. of Licenses**	**Total Price ***
Smart Banker Program with Finacle ONLY	INR 5,388*	42 (14 students per year for 3 years)	INR 2,26,300*

*Taxes extra as applicable.

1. 100% payment will be made upfront, within 30 days of invoicing.
2. For any additional license, it will be priced at INR 5,400+tax per licence.
3. In case, the number of licenses used are less than 42 in a span of 3 years, then the excess payment will be adjusted at the time of renewal of the contract in the 4th year.
4. In case the full payment is not made within 30 days of date of invoice, the access to program may be stopped and participants will not be able to access the program. The access will be resumed only when the full payment is made.
5. FLIP agrees and understand that the client will deduct the TDS if applicable.

Note: In case a student withdraws his/ her admission and his/ her program usage is above 5%, it will be considered as a used license. If the usage is <=5%, then it can be replaced with a new ID.

Certification:

This program is a customised version, and it do not contain certification. Students who are interested to write the certification exam must pay an additional fee for training and certification.

- Additional training fee- INR 3500+Tax for (35.5 hours)
- Additional certification fee- INR 1500+Tax each for Part 1 and Part 2 Certification exam.
- Course coverage for the additional content is mentioned in Annexure 2.
- If the participants opt for the certification exam, the student will be provided one month of additional access of the program to prepare for the certification exam.
- Students who are eligible, must write the certification exam within one year of the start of the batch.

Single Point of Contact (SPOC)

FLIP and the client will designate a Single Point of Contact (SPOC) who will coordinate to run the learning process. Both parties will also designate an escalation point.

Client	SPOC	Escalation Point
Name	Hridoy Saikia	Subhrangshu Sekhar Sarkar
Designation	Joint Registrar	Head, Department of Commerce
Email	hridoy@tezu.ernet.in	subh@tezu.ernet.in
Phone	9957184359	9435081446

FLIP	SPOC	Escalation Point
Name	Niveda UK	Payal Chatterjee
Designation	Education Institutions	Sr. Manager- Educational Institution
Email	niveda.uk@learnwithflip.com	payal.chatterjee@learnwithflip.com
Phone	7204741596	9008703436

Deliverables by FLIP

- FLIP will provide **online learning courseware, practice tests, evaluation tests & final assessment.**
- FLIP will provide unlimited technical and content support through email for the smooth functioning of course.
- FLIP will assign a **Dedicated Relationship Manager,** who will ensure smooth completion of the overall Learning process.
- FLIP will share the **learning report, fortnightly** with college. It will have details such as: percentage of completion for each individual, Practice test scores etc.

Deliverables by the Client

- Client will make this online program mandatory for the all relevant students.
- Client will ensure that all the students are adequately informed about the program.
- Client will share the list of candidates with name, email id, phone number to FLIP.
- Client will take necessary actions basis the report sent by FLIP.
- Client will conduct all the tests in a proctored environment.
- Client will allocate dedicated time for the students to study in the lab.

Best Practices for an effective e-learning experience

Step 1: Mail from the Dean/Director - to students stating the importance of the Program, and why they need to take it seriously.
This mail should be sent to the students at least 2 days before the launch.
Step 2: An introductory (kick-off) session - (via tele-conference) by FLIP (mandatory attendance for students), which will also be attended by college Dean/Director. The features of the program and any queries that the students might have will be addressed at this stage. The SPOC should emphasize why they have partnered with FLIP for the program.
Step 3: Mail from SPOC - Escalation from director (if necessary) for students who are behind schedule on a regular basis. FLIP will provide the list of students who are behind schedule.
Step 4: Assessment Tests – Should be conducted in college, and in a proctored environment; it is the responsibility of the college to ensure the same.
Step 5: Weightage in Assessment – Scores in the final assessment test should be given weightage in the semester evaluation, to ensure seriousness.

Course Structure

- The program is divided into 2 Parts, Consisting of 5 modules in Part 1 and 10 modules in Part 2
- At the end of each module, there is a practice test and an evaluation test.
- The participant needs to complete the chapters in each module, and then take the practice test.
- The participant must clear the evaluation test with a minimum score of 60%.
- Only after the participant has cleared the evaluation test, will the next set of chapters (module) gets unlocked.
- At the end of each part, a Final Assessment Test will be conducted.
- A student should complete at least 80% of the program to be eligible for the Final Assessment Test.

Study Plan and Monitoring Learning Progress

- A detailed study plan will be provided to the students with defined timelines to complete the program.
- The participant must adhere to the study plan to successfully complete the program.
- If for any reason, a student cannot adhere to the study plan, s/he should mail to FLIP marking the college SPOC.
- Fortnightly, a learning report would be sent to the SPOC. This learning report would include details of every participant's learning progress.
- FLIP will follow up with students and send reminders (call/e-mail/sms) on a regular basis.
- 1 Faculty doubt clearing session is mandatory per batch, where all the students must attend it from the classroom in a specific date and time, mentioned by FLIP. The session will be conducted either through webinar/Skype. Other sessions are optional to the students.
- Once escalation is done by FLIP, the college will follow up with students lagging behind, to ensure they adhere to the study plan and complete the program.

General Terms & Conditions

- The licensing fee paid by the client will enable it to offer FLIP's e-learning courses, as defined above, via a login ID for each user.
- The courses will be hosted on FLIP's learning platform (www.learnwithflip.com) only.
- Authorized Users include all the students of the client only.
- Each user ID is for one student only; one user ID license may not be shared across multiple users; such usage constitutes breach of contract.
- One user ID cannot be used to train multiple participants in a classroom or any other format.
- All licenses will have to be allocated within 6 months of the date of contract.
- The content belongs to FLIP and can neither be downloaded nor distributed. FLIP will provide PDF summaries of each chapter, which may be downloaded by the user.
- Users of training courses may create easily downloadable notes for their own use. No further documentation will be provided by FLIP.

Confidentiality & Use of Information

Each party will retain in confidence the terms of their agreement for Content Licences and Services and all other information and know-how of the other party disclosed to or acquired by such party ("Receiving Party") pursuant to, or in connection with this Agreement, which is either designated as proprietary or confidential or which, under the circumstances, ought in good faith to be treated as proprietary or confidential ("Confidential Information").

Signature

Signature

For Finitiatives Learning India Pvt. Ltd.	**For Tezpur University**
Name:	**Name:** H. Saikia
Designation:	**Designation:** Sr. Registrar (A
Date: _____	Date: 23/02/17

Joint Registrar
Tezpur University

5/8

Annexure 1: Course Coverage- Smart Banker Program- Finacle ONLY

Part I

Module 1 - Types of Accounts

Chapter - 1: Liability Products - Overview
Chapter - 2: Types of Accounts 1 (Individual Accounts)
Chapter - 3: Types of Accounts 2 (Non-Individual Accounts)
Chapter - 4: Types of Accounts 3 (Account Status)

Module 2 - Understanding CASA

Chapter - 1: Savings Account I
Chapter - 2: Savings Account II
Chapter - 3: Current Account I
Chapter - 4: Current Account II

Module 3 - Understanding FD

Chapter - 1 : Fixed Deposits I
Chapter - 2: Fixed Deposits II

Module 4 - FD Finacle Process

Finacle Process 1: FD Opening
Finacle Process 2: Printing an FD
Finacle Process 3: FD Extension
Finacle Process 4: FD Premature Closure
Finacle Process 5: Recurring Deposits
Finacle Process 6: SI Process

Module 5 - FD Simulation

Finacle Simulation 1: FD Opening
Finacle Simulation 2: Printing an FD
Finacle Simulation 3: FD Extension
Finacle Simulation 4: FD Premature Closure
Finacle Simulation 5: RD & SI

Part II

Module 6 - Cash

Chapter - 1: Cash

Module 7 - Cash Finacle Process

Process 1: Beginning of the day
Process 2: Cash Receipt
Process 3: Cash Withdrawal
Process 4: Opening of an FD by Cash

Module 8- Cash Simulation

Simulation 1: Beginning of the Day
Simulation 2: Cash Receipt

Simulation 3: Cash Withdrawal
Simulation 4: Opening of an FD by Cash

Module 9 - Cheque Clearing and Settlement

Chapter - 1: Electronic Clearing Services
Chapter - 2: NEFT and RTGS

Module 10- NEFT & RTGS - Process and Simulation

Process 1: NEFT & RTGS Payment
Simulation 1: NEFT
Simulation 2: RTGS

Module 11- Account Servicing

Account Servicing & Modification - Part I
Account Servicing & Modification - Part II

Module 12 - Account Servicing & Modification (Part I) - Processes

Process 1: Change in Address/Email/Mobile Number
Process 2: Updation of PAN Card
Process 3: Addition of Nominee
Process 4: Change of Account Status
Process 5: Change in Dispatch Mode of Account Statement

Module 13- Account Servicing & Modification (Part I) - Simulations

Simulation 1: Change in Address/Email/Mobile Number
Simulation 2: Updation of PAN Card
Simulation 3: Addition of Nominee
Simulation 4: Change of Account Status
Simulation 5: Change in Dispatch Mode of Account Statement

Module 14- Account Servicing & Modification (Part II) - Processes

Process 1: Issue of DD
Process 2: Cancelation of DD
Process 3: Issue of Cheque Book
Process 4: Stop payment for a cheque
Process 5: Issue of Account Statement

Module 15 - Account Servicing & Modification (Part II) - Simulations

Simulation 1: Issue of DD
Simulation 2: Cancelation of DD
Simulation 3: Issue of Cheque Book
Simulation 4: Stop payment for a cheque
Simulation 5: Issue of Account Statement

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Reg No : TA-P23032414

Date of Issue : 23rd March 2017

Valid till : 23rd March 2020

Authorised Signatory

www.tallyacademy.in

FORMAT FOR COMPLETION REPORT

PART - 1

1. Title of the project : **e-Varaha: Information System for Safe Pork Production in North Eastern India**

2. Implementing Organization : **Tezpur University, Assam**

3. Sanction No. and Date : **ITRA/15(189)/Ag&Food/e-Varaha/01, Dtd. October 28, 2016**

4. (a)Total Budget Outlay :

   Original - : **Rs.240.87L (Total)**
   Rs.26.10L (Tezpur University, Assam)
   Rs.15.05L (1st Installment to Tezpur University)

   Revised, if any: Nil

   (b)Duration of project : **26 Months**

   (c)Date of completion and reasons fordelay, if any : **31-12-2018**

5. Total funds spent under various approved Budgetary Heads/actual expenditure.Reasons for deviation, if any (as per enclosed Table 1) : **Rs. 14.58238 Lakh**

6. Details of equipment/assets acquired out of MIT funds with the name of equipment, sources of supply, total cost/whether Indianor imported (as per enclosed Table 2.1, 2.2 and 2.3) : **As enclosed**

7. Details of manpower associated with theproject: (as per enclosed Table 3) **As enclosed**

8. Details of year wise audited statement of accounts and utilization certificates submitted to MIT (as per G.F.R.19 & 19A) : **As enclosed**

25/03/2019

Professor
Department of Computer Science & Engg
Tezpur University

Finance Officer
Tezpur University

TABLE 3: MANPOWER ASSOCIATED WITH THE PROJECT:

S.No.	Name	Designation	Qualification	% of time devoted to this project	Salary drawn Y/N	Date of Joining	Date of Leaving	Total average emoluments (monthly)
1.	**Saurav Roy**	**JRF**	**MCA**	**100%**	**Y**	**06-04-2017**	**19-05-2017**	**27,500/-** (with DA)
2.	**Sukhen Shil**	**JRF**	**M. Tech (IT)**	**100%**	**Y**	**17-07-2017**	**31-12-2018**	**27,500/-** (with DA)
3.	**Kaushik Ray**	**JRF**	**M. Tech (IT)**	**100%**	**Y**	**03-04-2017**	**31-12-2018**	**25,000/- (up to 31st July, 2017, 27,500/-)**
4.	**Dr. Nityananda Sarma**	**Professor, PI**	**PhD (CSE)**	**30%**	**N**	**28-10-2016**	-	**Nil**
5.	**Dr. Sarat Saharia**	**Assoc. Professor, Co-PI**	**PhD (CSE)**	**25%**	**N**	**28-10-2016**	-	**Nil**

- (a) Institute Faculty and Staff
- (b) Staff recruited for the project
- (c) Students

25/03/2019
Professor
Department of Computer Science & Engg
Tezpur University

Principal Investigator,
Dr. Nityananda Sarma
Professor, Dept. of CSE
Tezpur University
Date : 25-03-2019 Place: **Tezpur**

Finance Officer
Tezpur University
Finance Officer
Tezpur University

25/3/19
Registrar
Tezpur University
Registrar
Tezpur University

CENTRE FOR DEVELOPMENT OF ADVANCED COMPUTING

APPROVAL NOTE

CORP: R&D: 2015:1017

May 8, 2015

Subject: Approval of your Project Proposal "Implementation of Distributed Honeynet System at NE to collect, distribute and analyze live cyber-attack data" submitted to North East Steering Committee

Dear Shri D.K.Jain,

This has reference to the project proposal "Implementation of Distributed Honeynet System at NE to collect, distribute and analyze live cyber-attack data" submitted to the North East Steering Committee which was reviewed during the meeting held on November 13, 2014.

With reference to the revised proposal received from you on March 27, 2015 and subsequent clarifications provided, we are happy to inform you that your project proposal as per the details given below has been approved.

Title of the Project:	Implementation of Distributed Honeynet System at N.E to collect, distribute and analyze live cyber-attack data
Chief Investigator:	Shri D.K Jain, Director, C-DAC, Mohali
Duration of the Project:	36 Months
Scheduled Start Date of the Project:	May 2015
Scheduled End Date of the Project:	April 2018
Total Project Outlay:	Rs. 181.24 Lakhs
C-DAC Contribution:	Rs. 181.24 Lakhs

Total Project Outlay:

l. No.	Head	1st Year	II nd Year	III rd Year	Total
1	Capital Equipment & Software(non-recurring)	38	0	0	38
2	Consumable stores (incl., maintenance, books etc.)+ Internet bandwidth	9	9	9	27
3	Manpower (CDAC-Mohali, Kolkata, Silchar, Tezpur University)	27.6	30.36	33.396	91.356
4	Travel (Internal) & Training	6	4.7	4.7	15.4
5	Overheads	4.515	2.409	2.5674	9.4914
	Total	85.115	46.469	49.6634	181.2474

We request your highest degree of commitment and dedication to ensure that the project meets all its deliverables well in time and make a significant contribution to the development of North East Region. The guidelines and terms & conditions for the execution of the project are enclosed in Annexure A and B respectively. Please go through the same and ensure they are followed.

We request you to send the acknowledgement of this Approval Note countersigned by the Chief Investigator & Executive Director to the Corporate Office, at the earliest.

This has approval of the Competent Authority.

With regards,

P. Sinha

Pradeep K.Sinha
Sr. Director (Corporate Strategy and R&D)

To: Chief Investigator

CC: 1. Director, C-DAC, Mohali
2. North East Steering Committee

Annexure A

Guidelines for the Execution of the Project

- With this Approval Note, we will be releasing the first installment of your Project Outlay.

- The schedule of the project and its timely and successful execution is extremely important and Chief Investigators shall adhere to the same.

- The Steering Committee will call for regular review of the project, normally every three months. You are expected to present a report on the progress of the project vis-à-vis the deliverables and proposed timeline at the time of every review, including financial status.

- Depending on the project requirements, progress of the project and utilization certificate for the released funds, further funds will be released.

- Chief Investigator shall provide a brief progress report (maximum one page) of the project to the Member Convener of Steering Committee (currently, Mr. Anant Kelkar – anantk@cdac.in) by 25th of every month.

- As you are aware the Central Government is attaching high priority to the development of North East Region, successful execution and implementation of the project will make a visible impact.

- If there are any hindrances or issues in timely and effective execution of the project, please bring them to the attention of the Steering Committee as soon as possible; this will help us to look at remedial action before it is too late.

Annexure B

Terms and Conditions for the Execution of the Project

i. The grant is for the specific project as approved by the Steering Committee and shall be subject to the following conditions:

   a. The grant amount shall be spent for the project within the specified time
   b. Any portion of the grant, which is not ultimately required for expenditure for the approved purposes, shall be duly surrendered to DeitY

ii. The centre shall maintain an audited record in the form of a register in the prescribed proforma for permanent, semi-permanent assets acquired solely or mainly out of this grant;

iii. The assets referred to in (ii) above will be property of DeitY and should not, without prior sanction of DeitY, be disposed off or encumbered or utilized for the purposes other than those for which the grant has been sanctioned;

iv. At the conclusion of the project, DeitY will be free to sell or otherwise dispose of the assets which are the property of DeitY and centre shall render to DeitY the necessary facilities for facilitating the sale of these assets;

v. The centre shall send a list of assets referred to in (ii) above to the Steering Committee at the time of seeking further installments of the grant;

vi. Should at any time centre cease to exist, such assets etc., shall revert to DeitY;

vii. The centre shall render an audited statement of accounts to Steering Committee;

viii. The utilization of grant for the intended purposes will be looked into by the Steering Committee;

ix. DeitY or its nominee/s will have the right to the books and accounts of the centre for which a reasonable prior notice would be given;

x. The centre should maintain separate audited account for the project. If it is found expedient to keep a part or whole of the grant in a bank account earning interest, the interest, thus earned should be reported to Steering Committee. The interest so earned will be treated as a credit to the centre to be adjusted towards future installments of the grant;

xi. Sale proceeds of components, prototype, pilot project etc. fabricated as a result of development of the project arising directly from funds granted shall be remitted to DeitY;

xii. For the know-how generated by the project, guidelines for TOT/know-how transfer as given in Chapter VI of the "Guidelines for Funding R&D Projects by DeitY" will be applicable;

xiii. DeitY will have the right to call for drawings, specifications and other data necessary to enable the transfer of know-how to other parties and the centre shall supply all the needed data at the request of DeitY;

xiv. Application by centre for any other financial assistance or receipt of grant/loan from any other Agency/Ministry/Department for this project should have the prior approval of DeitY;

xv. The centre is not allowed to entrust the implementation of this project for which grant is received to another institution and to divert the grant received as assistance to later institution.

xvi. The centre will examine IPR protection issues in consultation with IPR Cell of DeitY to file patents, register the copyrights, etc. before making it public by publishing in the technical journals and books, presenting finding in the conferences, etc.

Proposal Submitted To the NORTH EAST Steering Committee of C-DAC

Title of the Project: Implementation of Distributed Honeynet System at N.E to collect, distribute and analyze live cyber-attack data

Name of Chief Investigator: Sh. D.K.Jain

Centre: C-DAC Mohali

Contact Details:

- **E-mail ID:** dkjain@cdac.in
- **Office Telephone No.:** 0172-2237052-56
- **Mobile No.:** 8146059059

Revised date of submission: 27th March 2015

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SUMMARY SHEET

1. Title of Project: Implementation of Distributed Honeynet System at N.E to collect, distribute and analyze live cyber-attack data

.

2. Organisation

a) Name : Centre for Development of Advanced Computing, (CDAC), Mohali.

b) Address : A-34, Industrial Area, Phase – VIII, Mohali, Punjab – 160071

3. Chief Investigator & Other Investigators, Coordinators

Chief Investigator

a) Name : Sh. D.K.Jain

b) Designation : Director

c) Department : Centre Director

d) Address : A-34, Industrial Area, Phase – VIII, Mohali, Punjab – 160071

Co Investigator 1

a) Name : Sh, Rakesh Sehgal

b) Designation : Principal Engineer

c) Department : Cyber Security Tech. Div.

d) Address : A-34, Industrial Area, Phase – VIII, Mohali, Punjab – 160071

Co Investigator 2

a) Name : Dr. Nityananda Sarma

b) Designation : Professor

c) Department : Computer Science and Engineering

d) Address : Tezpur University.

Co Investigator 3

a) Name : Sh. Selvakumar

b) Designation : Joint Director

c) Department : Center Head

d) Address : CDAC Silchar.

4. **Nature of Project:** (Check one)

- ✓ Research, Development & Engineering (R,D & E) leading to production capability
- ☐ Application oriented Research, Design and Development (R,D&D) having production potential
- ☐ Basic R&D

5. **Objective of the Project**

Implementation of Distributed Honeynet System at N.E to collect, distribute and analyze live cyber-attack data

- Implementation and Deployment of Distributed Honeynet System for capturing network attacks in N.E.
- Collection of unknown spreading malware samples on internet and its categorization.
- Designing and Implementation of analysis techniques, such as data mining, network forensics, system forensics and impact analysis.

6. **Brief outline of the project with specific technology fall-outs**

Executive Summary

Cyber security threats pose an important challenge to both researchers as well as system administrators. The variety and the dynamic nature of the attack make it challenging to analyze the attacks. The same attack can be occurring at various locations and in various time frames.

Foreign agencies do not provide such data easily. Also the problem gets complicated because of the prevalence of non standard data format and privacy concerns which hinder the sharing of actual data.

Data sharing does not allow the researchers to observe the attack cycle completely because the data is collected at one location and shared with researchers after due data anonymity and filtering. So the researchers do not get the holistic view of the attacks. Thus it is important to have data collection system in place.

CDAC Mohali has been working in this area and has developed a Distributed Honeynet System. The Honeynet system allows to collects live data from the live environment. It overcomes the challenges of live data collection. It saves the time and efforts involved in multiple redeployments required for collecting data, on part of

the researchers thereby allowing the researchers to gainfully use their time for independent research and analysis perceived.

In the view of above, it is proposed to deploy Honeynet nodes at various North East locations and collect data for further research. The Honeynet system is client-server architecture and following would be provided:

- Honeynet node (Having capability of dynamically configuring various Honeypots as collection sensors)
- Central collection analysis server which can act as a analysis console for the researchers to enable further in-depth analysis.

The data is collected in a controlled and safe environment without violating the semantics of a live attack. The captured data is labeled and stored in a relational database for future analysis and research. The Attack data consist of malware binaries dropped on victim Honeypot and the captured network traffic. Here we would like to bring about that the analysis may have multiple perspectives to fully understand the nature of attack. The data diversity on each node is different based on its network provider and type of domain used, thus multiple diverse locations are proposed for collection of data through Honeynet nodes.

Establishment of Concept lab at Tezpur University:

A concept lab will be established which would demonstrate the capabilities of the Honeynet System in terms of capturing the live cyber attack data and its analysis. The concept lab would have all requisite hardware and software solutions developed by CDAC-Mohali which would be used for demonstration and training purposes. This would enable further penetration of technology to the user agencies and research community.

It will consist of Honeynet management and analysis console at server side and capturing sensors at client side (node side). The concept lab will also have the access to central repository at CDAC-Mohali to the access the data collected through other deployed nodes. The space and other infrastructure support for the concept lab would be provided by Tezpur University.

Establishment of Distributed Honeynet Node (at various NE locations for live cyber attack data capturing):

CDAC-Silchar would be hosting one node of Distributed Honeynet System which will

act as collection system. The data collected at this node would be accessible through a analysis web console which queries the data from the RDBMS server hosted at CDAC-Mohali. Another node is proposed at Tripura Institute of Technology, Govt. of Tripura). The contact details would be taken from CDAC Kolkata. A letter of intent is yet to be received from the institute).

Software support and data repository at CDAC-Mohali:

CDAC Mohali has already developed Distributed Honeynet System and few nodes for the collection of attack data are functional in CDAC centres and other academic institutions at various geographical locations. The existing resources at CADC Mohali would be used for software maintenance and data repository management. CDAC Mohali will give trainings and workshops for deployments and usage of technology. Post deployment S/w maintenance and other technical support functions would be done by CDAC Mohali centre. Analysis would also include the categorization of attack data.

7. Expected outcome in physical terms (as applicable)

- ✓ Distributed Honeynet System network establishment at three NE locations for Attack data capturing.
- ✓ Tool for attack data categorization
- ✓ Experiments and joint exercises
- ✓ Database of labeled malwares and network data
- ✓ Project Reports and Documents

Nature of documents for technology transfer

- ✓ System Requirements Specifications
- ✓ Design documents
- ✓ Testing reports
- ✓ Installation and user manuals

8. Agency with which link up is established/proposed or participating agencies and their role in the project (enclose supporting documents)

C-DAC Silchar

1. Penetration in NE region and liaison activities at NE Institutes wherever possible with technical support from CDAC Mohali.
2. Model of Distributed Honeynet System, (Client – Server) deployment for demonstration of technology and analysis outcome demo by CDAC Mohali, by giving infrastructure and logistics provision.
3. Providing services implementation & maintenance of Hardware and Network Infrastructure wherever further deployments are done in NE region.

Tezpur University

1. Providing Lab infrastructure for establishment of Honeynet Lab. Providing services for post implementation for maintenance of Hardware and Network Infrastructure.
2. Providing the Network ranges within their Univ. network where Distributed Honeynet System will be deployed for attack data collection
3. Liaison with local engineering institutes and Univ. and other agencies at NE and act a cyber-attack data dissemination center for PhD scholars and other interested researchers.
4. The Univ. will also employ their PhD scholars and faculty to work on different analysis techniques, such as data mining, network forensics, system forensics and impact analysis on a long term basis.
5. Concept lab is proposed to help students, faculties and other researchers who want to use the collected data.

Consent cum Support Letter from Participating Agency

तेजपुर विश्वविद्यालय
(केंद्रीय विश्वविद्यालय)
नपाम, तेजपुर - 784 028, असम, भारत

TEZPUR UNIVERSITY
(A Central University)
Napaam, Tezpur - 784 028, Assam, India

No. DoRD/Col.Project/10-13/5055 Dated: 12 November, 2014

To
Mr. D.K.Jain
Director,
C-DAC Mohali.

Dear Sir,

Subject: **Deployment and Maintenance of Distributed Honeynet System at N.E to collect distribute and analyze live cyber-attack data.**

Tezpur University is pleased to collaborate with C-DAC, Mohali for the proposed cyber security project of Deployment of Distributed Honeynet System in the N.E. networks for capturing of live cyber-attacks and their categorization in North East regional cyber space. For the pilot project to be successfully implemented, Tezpur University will extend the following support:

a) **Providing services for post implementation for maintenance of Hardware and Network Infrastructure.**
b) **Providing the Network ranges within their Univ. network where Distributed Honeynet System will be deployed for attack data collection**
c) **Liaison with local engineering institutes and Univ. and other agencies at NE and act a cyber-attack data dissemination center for PhD scholars and other interested researchers.**
d) **The Univ. will also employ their PhD scholars and faculty to work on different analysis techniques, such as data mining, network forensics, system forensics and impact analysis on a long term basis.**

Yours Sincerely,

Chandra Kr. Mahanta 12/11/2014
(C.L. Mahanta)
Dean, Research & Development

Dean
Research & Development
Tezpur University

9. Duration of Project:

Three years (36 Months)

10. Year-wise break-up of physical achievements with specific intermediate milestones (in terms of aims and objectives)

Year I

- Requirement study and analysis
- Implementation of Distributed Honeynet System at CDAC- Silchar, other NE academic locations.
- Training and workshops on Honeynet Technology Implementations to NE participating agencies.
- Implementation of concept lab at Tezpur University and its demonstrations
- Support and troubleshooting, bug fixing
- Development of test-bed for experimentations in network security for engineering students.

Year II

- Implementation of Distributed Honeynet System at NE region in coordination with Tezpur Univ, CDAC Silchar
- Experimentation and analysis on collected attack data.
- Development of tools and mechanisms for attack data categorizations.
- Machine learning based analysis of collected attack data
- Impact analysis and further conclusion
- Forensic analysis, enhancements and further inclusions

Year III

- Improvement in Data control techniques and its testing
- Integration of new Data Control mechanism with Distributed Honeynet System
- Evaluations and testing of tool and mechanism.
- Improvements based on University administrator's feedback and participating agencies feedback.

11. Likely End User(s):

i. Academic Universities and Engineering Institutions
ii. Security agencies.
iii. Researchers/academicians

12. Name of other organizations jointly participating in the project :

- Tezpur University, CSE Deptt., (Concept lab, research and analysis techniques development)
- Tripura Institute of Technology (Govt. of Tripura-linkages being established)
- CDAC- Silchar for model deployment, data collection and demonstration purposes.

13. Total Budget outlay (Expenditure in Rs. Lakhs)

l. No.	Head	1st Year	II nd Year	III rd Year	Total
1	Capital Equipment & Software(non-recurring)	38	0	0	38
2	Consumable stores (incl., maintenance, books etc.)+ Internet bandwidth	9	9	9	27
3	Manpower (CDAC-Mohali, Kolkata, Silchar, Tezpur University)	27.6	30.36	33.396	91.356
4	Travel (Internal) & Training	6	4.7	4.7	15.4
5	Overheads	4.515	2.409	2.5674	9.4914
	Total	85.115	46.469	49.6634	181.2474

Total : 181.2474

14. a) Contribution of Project Implementing/ & other Organization in Total Budget Outlay : Infrastructure support at their location

b) DIT Contribution : 181.2474

Signature of Chief Investigator	**Signature of Head of the Institution/Organization**
Designation	**Designation**
Date	**Date**

DETAILS OF THE PROPOSAL

PART 1: BACKGROUND INFORMATION

1. Title of Project : Implementation of Distributed Honeynet System at N.E to collect, distribute and analyze live cyber-attack data

2. Organisation

a) Name : CDAC-Mohali

b) Address : A-34, Industrial area, Phase-8, Mohali, Chandigarh-160071

3. Chief Investigator & Overall Coordinator

Chief Investigator

a) Name : Sh. D.K.Jain

b) Designation : Director

c) Department : Centre Director

d) Address : A-34, Industrial Area, Phase – VIII, Mohali, Punjab – 160071

Co Investigator 1

a) Name : Sh, Rakesh Sehgal

b) Designation : Principal Engineer

c) Department : Cyber Security Tech. Div.

d) Address : A-34, Industrial Area, Phase – VIII, Mohali, Punjab – 160071

Co Investigator 2

a) Name : Dr. Nityananda Sarma

b) Designation : Professor

c) Department : Computer Science and Engineering

d) Address : Tezpur University.

Co Investigator 3

a) Name : Sh. Selvakumar

b) Designation : Joint Director

c) Department : Center Head

d) Address : CDAC Silchar.

3. Competence of Investigators/Co-investigators:

Chief-Investigator – Sh. D.K. Jain

- After passing out from Thaper Engg College Patiala, Sh Jain started his carrier in design & Development of Opto Electronics equpt at Central Electronics Ltd Sahibabad. He worked on the projects like Photon Counter System, Synchronous Signal Averager, Hot Axle detection system and Scanning Electron Microscope.

- Subsequently he shifted to NTPC and worked in the area of Operation and maintenance of Wireless communication and captive satellite communication for in house communication system for NTPC. There he worked for about 12 years.

- Subsequent to that he shifted to ER & DCI which was merged with CDAC. He worked in the area of embedded system, Data Centre and data communication e-gov and training.

- In this organization he worked till 2010 as Additional Director at Noida, then Director at Hyderabad.

- Presently he is with CDAC Mohali and is working as Director.

Co-Investigator-1: Sh. Rakesh Sehgal

- B.E. (Electronics), M.Tech.(CS) with distinction from DAVV University Indore.
-
- Co-investigator has extensive experience in projects funded by DIT in the area of Honeynet technology. Successfully completed investigative, development projects on the honeynet technologies funded by Diety.

- Attended DIT sponsored seven weeks training program on Cyber Security conducted by IISc Bangalore in 2003 under a National initiative by the DIT.

- Conducted Workshops on Information Security, for army staff, Banking professionals and Govt. officers.
  .
- Given lectures as expert Faculty in National Institute of Technical Teachers Training and Research (NITTTR, Chd) on Network Perimeter security and wireless security. Guest Lecturer for Information and Network Security at the Mahatma Gandhi State Institute of Public Administration, Chd.

- Conducted workshop for Integrated Defense Services staff –DAIRA, Defense CERTs and CERT-in officials on “Honeynets- Attack data capturing & analysis”.

- Participated in round table discussions on “Information Assurance in Cyber Battlespace” at HQ Army Training command, Shimla.

- Invited Speaker at Information Warfare Seminar for defense officers at HQ Central Command, Lucknow.

Publications:

1. Data Diversity of a Distributed Honeynet based Malware Collection System, Saurabh Chamotra, Dr. Raj Kamal, Rakesh Kumar Sehgal, J.S.Bhatia, (ETNCC2011)

2. Honeynet Based Botnet Detection Using Command Signatures, J.S. Bhatia, R.K. Sehgal, and Sanjeev Kumar, WiMoA 2011/ICCSEA 2011, CCIS 154, pp. 69–78, 2011. © Springer-Verlag Berlin Heidelberg 2011

3. Botnet Command Detection using Virtual Honeynet, J.S.Bhatia #1, R.K.Sehgal*2, Sanjeev Kumar#3, International Journal of Network Security & Its Applications (IJNSA), Vol.3, No.5, Sep 2011, DOI : 10.5121/ijnsa.2011.3514

4. Distributed Honeynet System using Gen III Virtual Honeynet, Sanjeev Kumar, Paramdeep Singh, Rakesh Sehgal, and J. S. Bhatia, 978-1-4577-0174-0/11/$26.00 ©2011 IEEE

5. An Integrated Framework for Malware Collection and Analysis for Botnet Tracking , Rakesh Kumar Sehgal, D. S. Bhilare , Saurabh Chamotra , Special Issue of International Journal of Computer Applications (0975 – 8887) on Communication Security, No.10. Mar.2012, www.ijcaonline.org

6. Distributed Honeynet System using Gen III Virtual Honeynet, Sanjeev Kumar, Paramdeep Singh, Rakesh Sehgal, and J. S. Bhatia, International Journal of Computer Theory and Engineering, Vol. 4, No. 4, August 2012

7. Honeysand: An Open Source Tools Based Sandbox Environment for Bot Analysis and Botnet Tracking, Saurabh Chamotra, Rakesh Kumar Sehgal, Dr. Raj Kamal , Special Issue of International Journal of Computer Applications (0975 – 8887) on Communication , Security, No.7. Mar.2012, www.ijcaonline.org

8. Hybrid Honeypot Framework for Malware Collection and Analysis , Sanjeev Kumar , Rakesh Sehgal, J.S.Bhatia, 978-1-4673-2605-6/12/ © 2012 IEEE, (ICIIS), 2012 IIT Chennai

9. Malicious data classification using structural information and behavioral specifications in executables , Kumar, S. ; Rama Krishna, C. ; Aggarwal, N. ; Sehgal, R., Chamotra, S.Recent Advances in Engineering and Computational Sciences (RAECS), 2014 , IEEE Explore

10. UAC: A Lightweight and Scalable Approach to Detect Malicious Web Pages. Harneet Kaur, Sanjay Madan, Rakesh Kumar Sehgal, Advances in Intelligent Systems and Computing, (CSOC 2014), Springer Volume 285, 2014, DOI 10.1007/978-3-319-06740-7_21

Co-investigator- 2: Prof. N. Sarma (Tezpur University)

1. Name and Designation: Dr. Nityananda Sarma, Professor in Computer Science & Engineering

2. Date of Birth: 28-05-1968

3. Name and Address of the Institution: Tezpur University, Napaam,
Tezpur – 784028, Assam
Phone: 03712-275356(O)
+919435084063 (M)
Fax: 03712-267005
Email: nitya@tezu.ernet.in

4. Department: Computer Science & Engineering

5. Field of Specialization: Mobile & Wireless Networks, Network Security, Optical Networks,

6. Academic Qualifications: BE (CSE) from Dibrugarh University, M.Tech (CSE) from IIT Kharagpur,

PhD (CSE) from IIT Guwahati

7. R&D and Teaching Experience: 22+ Yrs

8. Industrial Experience, if any: NIL

9. Awards / Prizes / Certificates etc. won by the Investigator/Co-Investigator:

1. IETE KS-Krishnan Memorial Award for the year 2009.
2. Elected as Fellow of IETE
3. Merit scholarship in BE level from Govt. of Assam.
4. Silver Jubilee award for being all branch topper (from 2nd year to 3rd year) in B.E. level
5. Best paper award in ADCOM-2007
6. Best paper award in ICIT 2014

10. Publications:

a) Journal/Conference/Workshop:

i) N Sarma and D K Bhattacharyya, A Secure Payment Protocol, in the Proc. of International Conference RTPS-2003, January, 2003, Guwahati, India
ii) D K Bhattacharyya and N Sarma, WISE: Web-based Interactive System of Education, in the Proc. of National Seminar on Information Technology – 24th April, 2003, Itanagar, India
iii) I Hussain and N Sarma, A Fault Tolerant Task Scheduling for Real Time Multiprocessor systems, in the Proc. of International Conference on Information Technology, pp.308-313, 23-25 December, 2003, Bhuboneswar, India
iv) N Sarma and B Bora, Provisioning QoS in Mobile Ad Hoc Networks, in the Proc. of National Workshop on Trends and Issues in Wireless Network and Mobile Computing (WNMC'04), pp.45-57, August 26-28, 2004, Tezpur, India
v) N Sarma and S K Singh, A Mutual Exclusion Algorithm for Mobile Ad Hoc Networks, in the Proc. of National Workshop on Trends and Issues in Wireless Network and Mobile Computing (WNMC'04), pp.17-24, August 26-28, 2004, Tezpur, India

vi) N Sarma and S Nandi, Enhanced QoS Support in Mobile Ad Hoc Networks, in the proc of Int'l Conference on Telecommunications and Networks (TeNe'05), Bridgeport University, USA, pp. 267-273, 10-20 Dec, 2005

vii) S S Satapathy and N Sarma, A Power Efficient Multi-zone data gathering protocol for WSNs, in the proc. of National Workshop on Trends in Advanced Computing 2006 (NWTAC'06), pp.23-32, 23-24 January, 2006.

viii) R Tripathy and N Sarma, Route Recovery Mechanism for QoS routing in MANETs, in the proc. of National Workshop on Trends in Advanced Computing 2006 (NWTAC'06), pp.39-45, 23-24 January, 2006.

ix) P S Reddy, N Sarma and R. K. Das ,Fault Tolerant distributed ME Algorithms: A Survey, in the proc. of National Workshop on Trends in Advanced Computing 2006 (NWTAC'06), pp.92-99, 23-24 January, 2006.

x) N Dutta and N Sarma, Providing QoS for Real-Time Multimedia Applications by Bandwidth Adjustment, in the proc. of National Workshop on Trends in Advanced Computing 2006 (NWTAC'06), pp.113-119, 23-24 January, 2006.

xi) N Sarma and S Nandi, QoS Support in Mobile Ad Hoc Networks, Third IEEE/IFIP International Conference on Wireless and Optical Communications Networks (WOCN 2006), Bangalore, India, 11-13 April, 2006.

xii) S S Satapathy and N Sarma, TREEPSI: TRee based Energy Efficient Protocol for Sensor Information, Third IEEE/IFIP International Conference on Wireless and Optical Communications Networks (WOCN 2006), Bangalore, India, 11-13 April, 2006.

xiii) P. S Reddy, N Sarma, R K Das, Enhancing Fault-Tolerance in a Distributed Mutual Exclusion Algorithm, ICIT, pp. 56-59, 9th International Conference on Information Technology (ICIT'06), Dec, 2006.

xiv) R. Tripathy and N Sarma, Efficient Route Recovery in QoS-based AODV Routing Protocol, International Conference on Intelligent Systems & Networks (IISN-2007), pp. 140-145, Feb 23-25, 2007, Haryana, India.

xv) M. Jims and N. Sarma, QoS-Aware MAC Protocols in MANETs, National Conference on Trends in Advanced Computing (NCTAC-2007), pp. 8-13, March 22-23, 2007, Tezpur, India.

xvi) M. P. Baruah and N. Sarma, Quality of Service Routing in MANETs, National Conference on Trends in Advanced Computing (NCTAC-2007), pp. 35-42, March 22-23, 2007, Tezpur, India.

xvii) D. Acharjee and N. Sarma, An Alternate Failure Tolerant Model for Multipath Routing of Sensor Networks, National Conference on Trends in Advanced Computing (NCTAC-2007), pp. 236-238, March 22-23, 2007, Tezpur, India.

xviii) M Jims, N Sarma, and S Nandi, "Priority Based Fairness Provisioning QoS-Aware MAC Protocol", The 15th International Conference on Advanced Computing and Communications, December 2007 (IEEE Press).

xix) N Sarma, and S Nandi, "Route Stability based QoS Routing (RSQR) in MANETs", The 10th International Symposium on Wireless Personal Multimedia Communications, December 2007.

xx) D Acharjee and N Sarma, Slope Based Shortest Path Routing For Wireless Sensor Network, The 15th International Conference on Advanced Computing and Communications, December 2007

xxi) N Sarma, S Nandi and R Tripathi, "Enhancing Route Recovery for QAODV Routing in Mobile Ad Hoc Networks," The 9th International Symposium on Parallel Architectures, Algorithms, and Networks (I-SPAN 2008), pp. 39-44, 2008.

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b) Edited Volumes

i). Network Security: Issues, Challenges and Techniques co-edited with D K Bhattacharyya and published by Narosa Publishing House Pvt Ltd., June, 2010; ISBN No 978-81-8487-088-6.

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11. R&D Projects: 3 completed, 1 Ongoing.

Co-investigator-3: Sh. Selvakumar

Academic Qualification:

- Has a basic degree in Agriculture with post-graduation in Genetics and a doctorate in Microiology. He also has a PGDST of erstwhile NCST.

Work Experience:

- Dr Selvakumar has been working in C-DAC (erstwhile NCST) since 1998. During his tenure in C-DAC he was:

- He was a key member in the Development Gateway Foundation funded project, "Bridging the Digital Divide using ICT". Involved in the project from project proposal preparatioin, implementation and roll out. He was the team lead for 2 initiatives under the project, viz., Vyapar – a web based system for information exchange between buyers and sellers, and E-forms - an easy-to-use powerful tool for creating HTML forms on the fly and for data collection. Currently Vyapar is hosted in the InDG portal and e-Forms (renamed as Fulcrum) is being used in the J&K project.

- He was in charge of Project Management in the DeitY funded project, "Delivery of e-Governance services through State Portal, e-Forms and SSDG in Jammu & Kashmir", in which the state portal of J&K and the initial e-Governance services were developed and launched.

- He has been the Center Head of the newly started C-DAC center in the North East located in Silchar since February, 2014.

4. Other Commitments of the Chief Investigator and Co-Investigators:

25% of time of the Chief-Investigator and 25% time of Co-Investigators would be devoted for this project, 100% time of project staff would be devoted to the project.

5. Infrastructure and other facilities available at the institute for undertaking this project :

a) List of major equipment along with model numbers, specifications etc.

CDAC has state-of-the-art infrastructure complimented with good computing facilities consisting of servers and workstations distributed in its campus. Cyber Security Technology Lab, which is established at C-DAC, Mohali will be used for carrying out the data collection, analysis and also for testing the security solution.

Equipment Type/ Qty. Number	Specification
Mid Range Server -2 Nos	HP Server Proliant ML 350 G4p Xeon 3 GHz Processor,1GB RAM,2 x36 GB HDD,DVD
Mid Range Server -1 Nos	IBM(236 xSeries) Xeon 3.6 GHz Dual Processor,8GB RAM,2 x146 GB HDD,DVD
Mid Range Server -4 Nos	HP Server Proliant ML 350 Xeon 2.33 GHz Processor,4GB RAM,3 x72 GB HDD,DVD
Mid Range Server -4 Nos	HP Server Proliant ML 150 G6 Xeon 2.0 GHz Processor,8GB RAM, 2 x146 GB HDD,DVD
High End Storage, Server -1 Nos	HP Server DL 580G5 Xeon 2.13 GHz Processor,16GB RAM, 2 x300 GB HDD, HP Storage P4300, 8 x 300 GB HDD(2.4 TB)
Desktop Computer -12 Nos.	IBM 8143-MQH (Pentium 4)
UPS	5 KVA -1
Firewall	Sonic wall NSA 3500
Mid Range Router	2600 Series
Managed Switch -1	2950 Series
Leased Line Internet	4MBPS
Visual Display	Sony 119 cms/ 47 inches

a) Existing manpower and other personnel with names available for the project on full-time basis.

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6. Expensive Equipment /facilities available elsewhere which could be made use of for the project :

a) Distributed Honeynet System- Central Server Repository for continuous attack data collection and analysis at CDAC Mohali to be used.

7. Details of collaborating agencies necessary (As this would vary from project to project, details may be given as appropriate) :

i. Tezpur University, CSE Deptt.

ii. Tripura Institute of Technology (Govt.of Tripura)

8. Additional information, if any.

PART II: TECHNICAL INFORMATION

1. Aim and Scope of the project (in terms of specific physical achievement)

The aim of the project is deployment of earlier developed system in North-East geographic networks to collect, distribute and analyze live cyber attacks. This would give situation awareness for the network where it would be deployed as well as attack data for further analysis and research. Currently the collected data would be attack records in the form of network traffic and malware samples by initiating the networks, network services, operating systems or applications. This can also be enhanced by adding certain more features for attack data collection like memory state monitoring.

- ✓ Implementation of Distributed Honeynet System for capturing network attacks in N.E.
- ✓ Collection of unknown spreading malware samples on internet and its categorization.
- ✓ Designing and Implementation of analysis techniques, such as data mining, network forensics, system forensics and impact analysis on a long term basis.

2. Detailed description of the Project:

Technology Fallouts:

Background:

A Honeynet is a highly controlled network that interacts with attackers in cyber space to gather the attack data, to collect intelligence on attack techniques and behaviors of the black hat community. When we look into other network security devices like Firewalls, IDS etc. they are usually based on some predefined signatures to collect the attacks and there has been large amount of work done in the field of network intrusion detection over the years. With the consideration of unknown attack detection, intrusion detection is becoming not useful to detect these kinds of attacks spreading in the network and intrusion detection based attack detection is becoming a very challenging process. Honeynets are enabling us in providing the deep understanding of attacks patterns which are bypassed by the network intrusion detection system. Therefore the Honeynets are becoming very useful to collect the unknown attacks.

Further Distributed Honeynet System which include a combination of various low and high interaction honeypots play an important role in cyber-space threat discovery and measurement; we have developed and deployed Distributed Honeynet by integrating low-interaction and high-interaction honeypot technologies, and use it for daily measurement of the specific threats on the Cyber Space. The system has a capability to collect malwares propagating in the internet. Further we can deploy this system at various heterogeneous locations to collect the latest trends of the malwares and further by doing analysis of collected malwares we can generate the data set for the validations and generation of the signature for Intrusion Detection System (IDS). The repository of the data set collected is maintained at central site which has also the context i.e environment details on which the data is captured for the analysis of the malwares.

In the earlier projects, CDAC Mohali team had developed a Distributed Honeynet System (DHS) with Dynamically Configurable Honeypot framework (DCH) as nodes which are deployed and functional at few geographical locations which include the mixture of low and high interaction Honeypots. The developed DHS system is capable of collecting autonomously spreading malware which propagates by exploiting the vulnerabilities in the operating system and network services. The Distributed Honeynet System has been deployed on the public IP address ranges of Enterprise Networks, ISPs as well as organizational public IPs and captures the attacks that are performed on the deployed IP ranges.

In the subsequent project the team had extended the scope of malware collection by development of Active Honeypot system for collection of malware that propagate by drive-by-download technique. The Active Honeypot system has been integrated with the Distributed Honeynet System.

The Honeynet can play a vital role in the detecting malware and botnet attacks on end user systems by deploying Honeypots in networks to collect attack payload and malware samples for further analysis and devising appropriate security solutions.

The main purpose of honeypots is to collect the attack data in the form of malicious network traffic and malwares. Honeypots are used to collect malwares and attack related

information such as source of infections and attackers, but there are several **limitations** of the current honeypot based detections.

1. The honeypot does not categorize the various attacks without sufficient in depth analysis. For example, there is no difference between a drive-by-download attack, P2P botnet attack, Centralized botnet attack, Worm infection and any other known forms of attacks, all of which share malware as the focal point of attacks.

2. The honeypot takes way too much time to figure out the kind of attack. This involves analysis at various levels including memory level, file level, network level and any other system related analysis. All of which do not specifically give an idea as to the nature of the attack.

Currently the categorization of attack classes is dependent upon the open source tool Snort IDS, and malware labeling is being done by Virus Total.

Our Proposal:

As part of the implementation of Distributed Honeynet System, the distributed honeynet node (which contain the different types of honeypots such as low interaction, high interaction and web application honeypots) for NE networks would be undertaken in which the honeynet nodes (with DCH framework) would be deployed at various heterogeneous networks of North-East. Further the attack data collected by deploying the distributed honeynet sensors would be analyzed. It is proposed to apply the machine learning algorithm based attack categorization to categorize the attack data.

In the proposed project, the main agenda is deployment of earlier developed system in North-East geographic networks to collect, distribute and analyze live cyber attacks. This would give situation awareness for the network where it would be deployed as well as attack data for further analysis and research. Currently the collected data would be attack records in the form of network traffic and malware samples by initiating the networks, network services, operating systems or applications. This can also be enhanced by adding certain more features for attack data collection like memory state monitoring.

The project envisages the implementation of Distributed Honeynet System in NE region and categorization of collected attack data. Further there is a requirement to categorize the honeypots traffic for fast attack categorization.

Proposed Approach: Key ideas:

Categorization: We propose a application protocol level flow mining approach to solve these problems. Each attack can be characterized by certain traffic features extracted from known samples.

Semantic Analysis: We propose a feature to attack semantic mapping by mapping protocol interactions to the statistical features. This will enable us to see the nature of the attack and the patterns they exhibit. To do this: we will extract the protocol request/response messages to the statistical features. For instance: how many messages need to be exchanged to download a binary successfully? Or how many packets need to be sent to do a neighborhood mapping?

Technical Approach:

A. Attack Categorization:

We will create a classifier for each possible attack using multiple traffic features. During run-time, for each traffic interaction, these features will be extracted during the interaction phase and classified appropriately. Further, for malwares that evade Honeypot by detecting them, the honeypot will be modified to send out packets containing dummy payload, which have similar entropy as a real payload and of same size. This will ensure that the appropriate protocol interaction is completed by the attacking node.

B. Semantic Analysis

We will identify messages by using the TCP/HTTP or any high level protocol request response patterns. Within a given request/response cycle we will examine the features extracted and isolate those features which show significant change. These features will then be correlated with the specific messages that contributed towards these features. Since this correlation cannot be hidden, even if the payload is encrypted, the mapping will

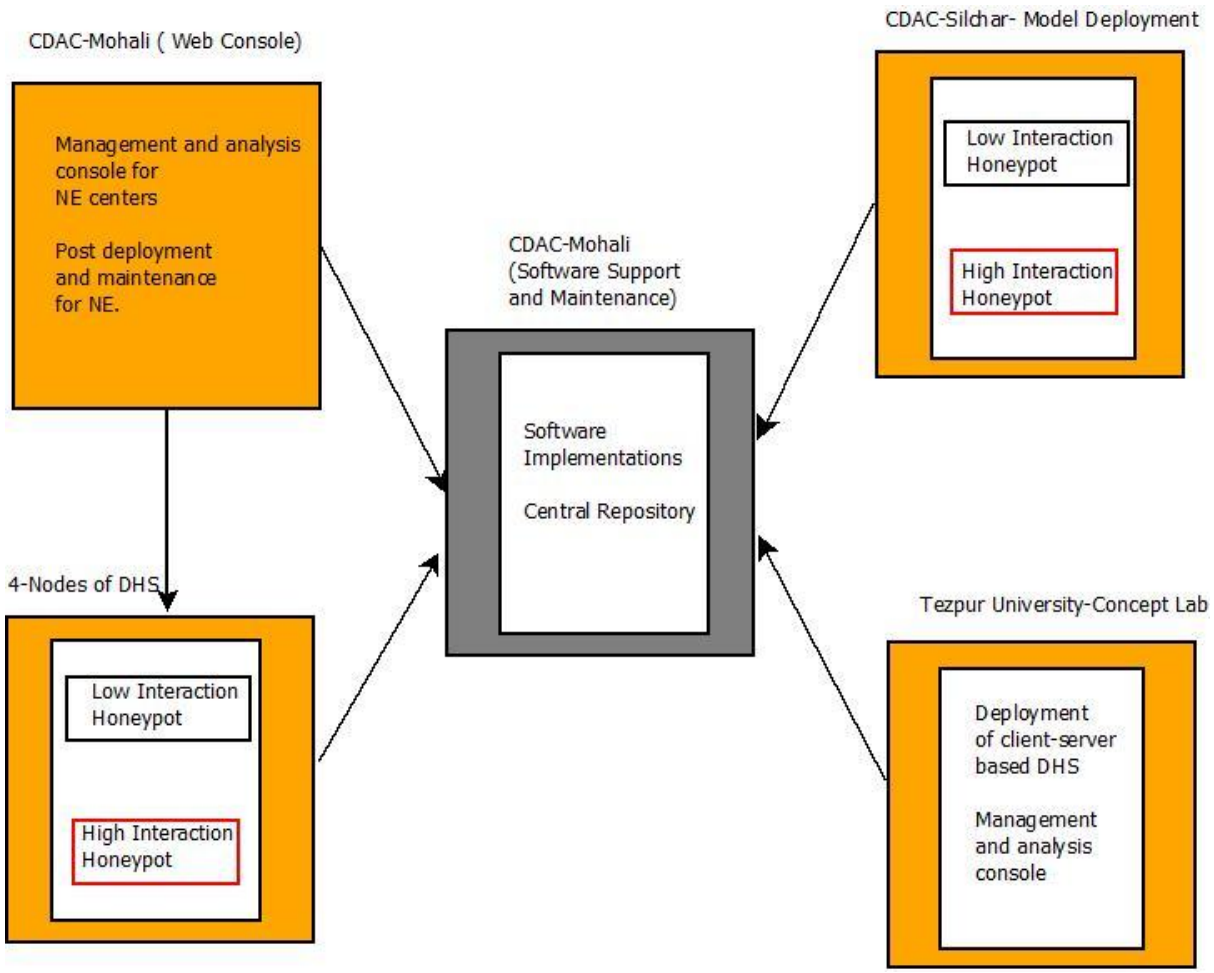
enable us to explain how the features are helpful in identifying and isolating the specific sub-attack from within the entire cycle. This will allow us to identify only very specific features that really contribute towards exact isolation of the sub-attacks and thereby, the entire attack. This approach allows us to "explain" the contribution of the features to sub-attacks and the contribution of sub-attacks to the main attack. If a similar attack is launched, then we can look at the number of sub-attacks in common and identify the nature of the damage caused by the malware. This "explainable" features is missing so far from existing data analytic frameworks in particular.

Advantages:

1. This approach does not rely on signatures but only on traffic features. No need to examine changes to the file system or any other related information. This would be fast and accurate.

2. The explainable nature allows analysts to know the exact nature of the damage and initiate remedial measures. The explanation will also establish that an attack actually happened as will be statistically proven through the presence of several sub-attacks.

Proposed Implementation Plan:



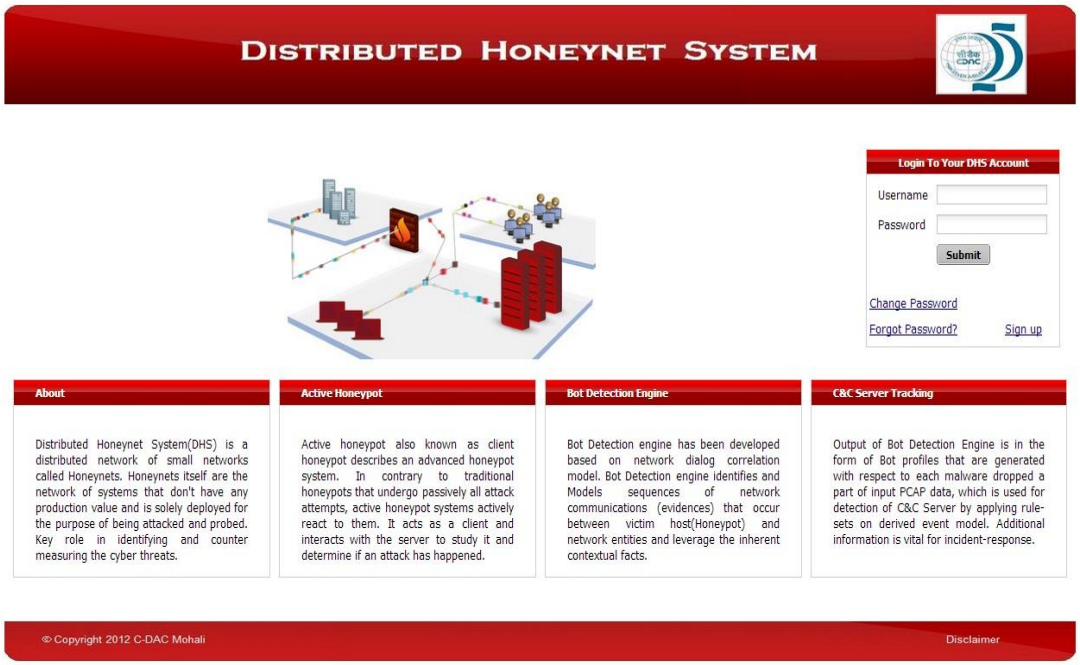
The source of data feed for our network traffic analysis module is network dumps collected through various deployed distributed Honeypot system. For collecting this input data we propose an infrastructure of Distributed Honeynet System which comprises to two main parts; Distributed Honeynet Sensors (Remote Nodes) and Analysis Console at center. Distributed Honeynet framework allows simultaneous deployments of virtual machines with Honeypot sensors which can be a mix of low and high interaction honeypots. Analysis module will comprise of a suitable machine learning based classification system.

The user of the node of Distributed Honeynet System will get the login credential to the collected data and its statistics. This information will be helpful for the network and system security administrators as well as the researchers for applying any analysis on the collected

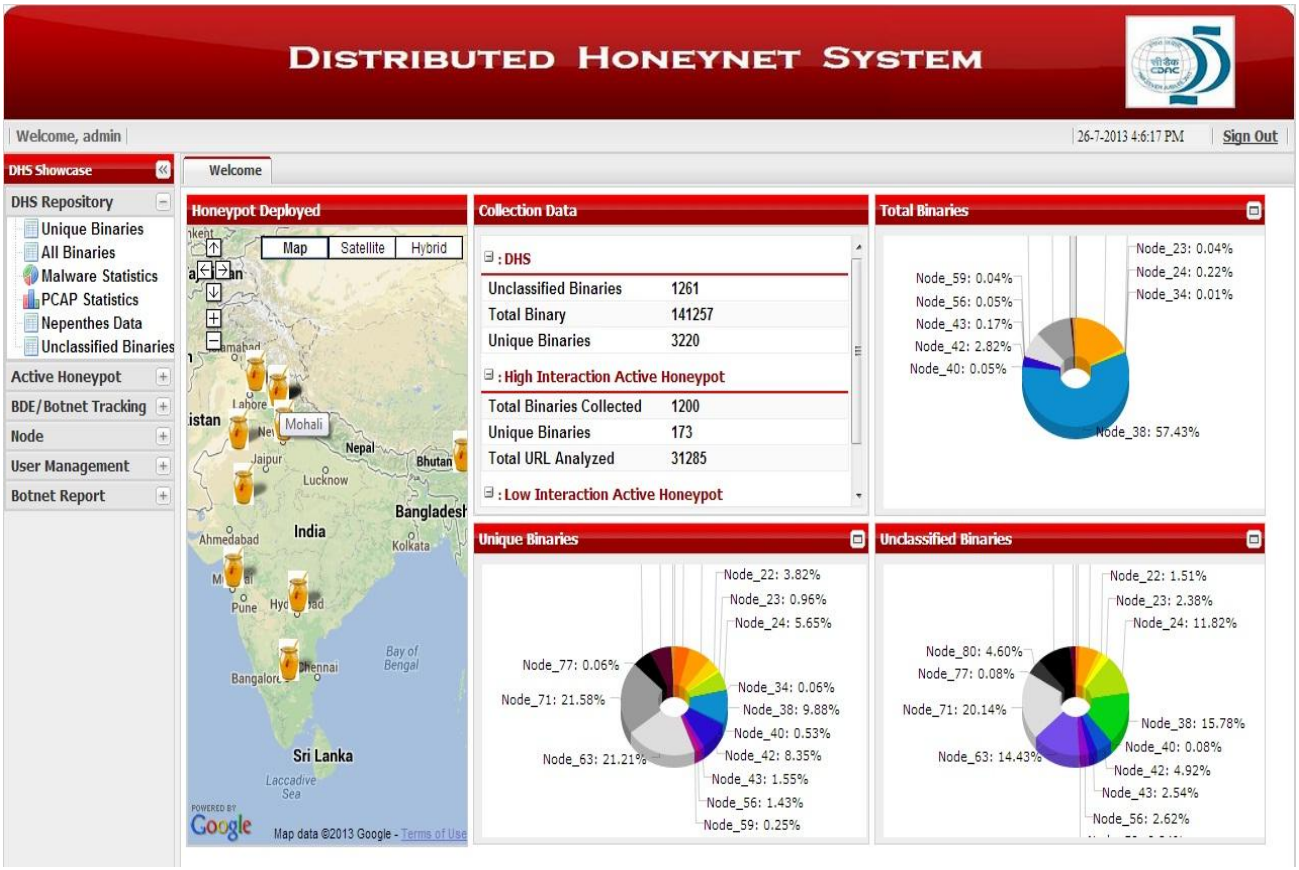
attack data. With such sort of information, the users can be categorized into following ways:

- System/network administrators
- Researchers which will use the data for research purposes
- Security agencies and defense

The node user will get the login credentials to see the collected attack data on the web interface. Following snapshot depict the screenshot of the Distributed Honeynet System.



Further, the user's can see the collected statistics through GUI of Distributed Honeynet System which will give the complete collected statistics and analysis results of the collected attack data on the distributed honeynet node deployed. Following screenshot depict the dashboard of the developed GUI which display the set of total collected binaries, unique binaries among the collected binaries as well as unclassified binaries which are not classified by any of the popular anti-virus scanner. As shown in the left pane, the DHS repository displays the unique binaries, all binaries, Malware statistics, PCAP statistics, and unclassified binaries.



a. USER AGENCY Responsibilities

USER AGENCY – **Tezpur University, Tripura Institute of Technology**
Responsibilities would be to enable the project equipment deployment at the user-agency location by providing required institute infrastructure for deployment (building space, electricity and water). After the deployment of the DHS, the agency would undertake the training of the deployed project staff for its usage and enable continuous collection of attack data for investigation. They will participate in data analysis and use the attack collection facility for further research.

b. **C-DAC Responsibilities (list for each centre separately, in case of multi centre projects)**

CDAC Mohali:

- Requirement study
- Implementation and Deployment of the technology at Tezpur University, CDAC Silchar, Tripura Institute of Technology
- Conduct of training workshops and handholding
- Software maintenance and Support

- Inclusion of innovative categorization of Network Traffic mechanism
- Providing services implementation & maintenance of Hardware and Network Infrastructure wherever further deployments are done in NE region

CDAC Silchar:

- Provision of lab infrastructure to establish the model deployment at CDAC Silchar.
- Liaison with engineering institutes and Univ. and other agencies at NE

3. **Need, forecast and urgency for the technology proposed to be developed/deployed with significance to North East Region:**

This would provide leadership in anti-threat intelligence, research and innovation, enable to conducts incident response during malware-related incidents. This would foster the growth of forensics, AV technology and product development in the region.

4. **Specific manner in which know-how generated here is envisaged to be translated into production.**

The competence developed would lead to emergence of service sector in the region.

Prototype test bed for data capturing & collection can be used as a kit for production.

5. **a) Name of production agencies willing to productionise / use and market surveys if any made by them regarding demand for the product :**

Non commercial product

b) Alternative production/user agencies. :

6. **Details of work already done by present investigators / R&D team in this or other areas:**

Details in annexure A

Project 1:

i) **Project Title :** Attack Methodology analysis, Network attack modeling and survivability simulation

ii) Funding Agency (or Internal funding) : MCIT

iii) Brief earlier Project Summary

The project envisaged R&D work in the area of attack methodologies, their analysis leading to representing Intruder methodology, developing analysis tool kits. In this

project, Honeynet Gen III was designed and deployed for information gathering concerning the tactics, techniques and procedures of hacker community. It is an effective research tool to achieve Research in areas of information Assurance and Intrusion Detection. Network attack scenario generation software was developed and modeling for attack plan recognition/detection researched.

iv) Technical Status vis-a-vis objectives : Annexure A

v) Financial Status (Total Project outlay, expenditure to date): 125.3 Lacs

vi) Duration and year of initiation : : Three Yrs, Feb. 2005

vii) Date of completion : : 28 Feb 2008

Project 2:

i) **Project Title :** Development of Distributed Honeynet System

ii) Funding Agency (or Internal funding) : MCIT

iii) Brief earlier Project Summary

The Distributed Honeynet system helps to detect large-scale coordinated attacks. In this project, a Distributed Honeynet system with following capabilities was developed.

- Collection of latest malware / bot samples.
- Detection of botnets.
- Finding and Confirming the Latest attack trends

iv) Technical Status vis-a-vis objectives : Annexure A

v) Financial Status (Total Project outlay, expenditure to date): 145 Lacs

vi) Duration and year of initiation : 2 years, Oct. 2008

vii) Date of completion: 31 Jan 2011

Project 3:

i) **Project Title:** Development of Active Honeypot for Distributed Honeynet System

ii) Funding Agency (or internal funding): MCIT

iii) Brief earlier Project Summary

a. The proposed Active Honeypot will actively browse the URL using either emulated browser or real browser and then capture the response of the web site. The ultimate goal of the active Honeypots is to detect suspicious DOM elements including iframes,

malicious and obfuscated Javascripts malicious contents, malware downloaded as part of drive by download malware distribution site, dynamic domain redirection chain, and Malicious HTTP commands. Researches of client side Honeypot mostly aim at finding malicious websites.

iv) Technical Status vis-a-vis objectives : (Annexure A)

v) Financial Status (Total Project outlay, expenditure to date): 203.5 Lacs

vi) Duration and year of initiation : 2 years

vii) Date of completion: 31st July, 2013

7. Summary of similar work being done elsewhere in the country

None, to our knowledge

8. a) Specific problems, hold-ups and difficulties foreseen in the implementation of the project :

None

b) If the answer is not Nil to 8(a), how does Chief Investigator propose to overcome them?

--

9. Detailed PERT/BAR Chart:

Please Refer Annexure A.

10. Details of possible alternative arrangements if the Chief Investigator leaves institution or is unable for any other reason to continue on this project:

Co-investigator shall take over.

11. List the personnel already working in the organization who would be transferred to work full time on this project.

---

12. Name of experts whom the Chief Investigator would invite to join the project team as full time/part time member.

NIL.

PART III – FINANCIAL DETAILS

Table – 1: Yearly break-up (Expenditure in Rs. Lakhs)

l. No.	Head	1st Year	II nd Year	III rd Year	Total
1	Capital Equipment & Software(non-recurring)	38	0	0	38
2	Consumable stores (incl., maintenance, books etc.)+ Internet bandwidth	9	9	9	27
3	Manpower (CDAC-Mohali, Kolkata, Silchar, Tezpur University)	27.6	30.36	33.396	91.356
4	Travel (Internal) & Training	6	4.7	4.7	15.4
5	Overheads	4.515	2.409	2.5674	9.4914
	Total	85.115	46.469	49.6634	181.2474

Individual breakup for each centre

CDAC Mohali

Breakup

Sr.No.	Head	1st Year Budget	2nd year budget	3rd year budget	Total
1	Capital(Hardware) and Software	2	0	0	2
2	Manpower	6	6.6	7.26	19.86
3	Travel and Training	2.5	1.2	1.2	4.9
4	Consumables(including Network connections)	2	2	2	6
5	Over Head	0	0	0	0
6	Total	12.5	9.8	10.46	32.76

Manpower Breakup CDAC Mohali

Sl. No.	Designation	Nos.	Monthly Salary (INR in Lakhs)	Commitment	Cost	1st year	2nd year	3rd year
1	Project Engineer	1	0.50 ~~0.80~~	100%	**0.50**	**6.0**	**6.6**	**7.26**
Total (INR in Lakh)					**19.86**			

Capital/Travel & Training/Consumable Justifications

Sr. No	Item Descriptions(including test equipment, components, material etc.	Purpose	Total
1	Mid Range Server- 1	Honeynet Configuration and Implementation	2.00L
2	Travel & Training	Trainings and workshops	4.90L
3	Internet Bandwidth	Internet Connection	6.00L (upto 4Mbps) – for 3 years

Details Expenditure for NE Nodes deployments :

1. Tripura Institute of Technology (Govt. of Tripura),

Sr.No.	Head	1st Year Budget	2nd year budget	3rd year budget	Total
1	Capital(Hardware) and Software	4	0	0	4
2	Manpower	6	6.6	7.26	19.86
3	Travel and Training	1	1	1	3
4	Consumables(including Network connections)	1.5	1.5	1.5	4.5
5	Over Head	0	0	0	0
6	Total	12.5	9.1	9.76	31.36

Sl. No.	Designation	Nos.	Monthly Salary (INR in Lakhs)	Commitment	Cost	1st year	2nd year	3rd year
1	Project Engineer	1	0.50	100%	**0.50**	**6.0**	**6.6**	**7.26**
Total (INR in Lakh)					**19.86**			

CDAC Silchar

Breakup

Sr.No.	Head	1st Year Budget	2nd year budget	3rd year budget	Total
1	Capital(Hardware) and Software	17	0	0	17
2	Manpower	6	6.6	7.26	19.86
3	Travel and Training	1	1	1	3
4	Consumables(including Network connections)	1.5	1.5	1.5	4.5
5	Over Head	0	0	0	0
6	Total	25.5	9.1	9.76	44.36

Manpower Breakup CDAC Silchar

Sl. No.	Designation	Nos.	Monthly Salary (INR in Lakhs)	Commitment	Cost	1st year	2nd year	3rd year
1	Project Engineer	1	0.50	100%	0.30	6.00L	6.60L	7.26L
Total (INR in Lakh)					19.86L			

Sr. No	Item Descriptions(including test equipment, components, material etc.	Purpose	Total
1	Mid Range Server- 2	Distributed Honeynet Node Client and Server	4.0L
2	GFI Sandbox	Analysis	13.0L
2	Travel & Training	Trainings and workshops	3.00L
3	Internet Bandwidth	For analysis and experiments	4.50L (for 3 years)

Tezpur University

Breakup

Sr.No.	Head	1st Year Budget	2nd year budget	3rd year budget	Total
1	Capital(Hardware) and Software	15	0	0	15
2	Manpower	9.6	10.56	11.616	31.776
3	Travel and Training	1.5	1.5	1.5	4.5
4	Consumables(including Network connections)	4	4	4	12
5	Over Head	4.515	2.409	2.5674	9.4914
6	Total	34.615	18.469	19.6834	72.7674

Manpower Breakup Tezpur University

Sl. No.	Designation	Nos.	Monthly Salary (INR in Lakhs)	Commitment	Cost	1st year	2nd year	3rd year
1	Project Engineer-I	1	0.50	100%	0.50	6 L	6.6L	10.164L
2	Project Assistant	1	0.30	100%	0.30	3.6L	3.96L	4.356L
Total (INR in Lakh)					**31.78**			

Sr. No	Item Descriptions(including test equipment, components, material etc.	Purpose	Total
1	High Range Server- 3 (Client Node, Server and Database server)	Distributed Honeynet Client and Server	10.00L
3	Analysis Softwares and license	Malware and Network analysis	5.00L
3	Travel & Training	Trainings and workshops	3.60L
4	Internet Bandwidth	For analysis and experiments	4.00L per year

Annexure A

Details of Completed Projects

Project title: To investigate network attack methodologies using Honeynet and to simulate/model network attacks

In earlier project, C-DAC Mohali team had studied the requirements of a Honeynet and various Honeynet architectures (Gen I, Gen II and Gen III) available in the literature and had implemented GEN II & Gen III Honeynet architecture. Further enhancements to Gen III architecture implementation including integration of Host IDS and Application IDS were made to increase attack detection domain. Suitable luring tactics were identified and appropriate vulnerabilities were incorporated in the implementation to lure the attackers.

The team had developed and implemented mechanisms and schemes for attack analysis, attack methodology and attack modeling. An attack tree is constructed by considering all possible actions required for a given goal using attack methodology graph. The attack modeling using attack tree is useful for high level attack analysis and their in-depth understanding. It also serves as an input for early warning system.

To establish the latest trend analysis of Honeynet data and provide a framework for Threat Model Generation, an Online Honeynet Statistical data generator has been developed that provides: IP based distribution , Service wise (port based) distribution, alert based distribution, country based distribution and time wise distribution of Attacks, Alerts and scans.

The team has been interacting with CERT-in on continuous basis for validation of data collected. The enhanced Gen III Honeynet architecture implementation merges the output format of various tools operating at network and system level into a common and correlated format by "HFLOWD" Perl script for data fusion. The standard Gen III Honeynet Architecture has a single detection sensor i.e. Snort (NIDS) but the experimental results revealed the retrieval of attack evidences from system traces and application logs. Hence, OSSEC (HIDS) and Mod-security (AIDS) had been integrated in stealth mode into Gen III Honeynet Architecture through network correlation (based on identifiable network parameters) and “HFLOWD” data fusion enhancements. The final

output can be visualized on our improved Walleye Analysis Console as a separate area and in correlation with established network sessions.

Current Implementation GEN III Architecture including enhancements is given below):

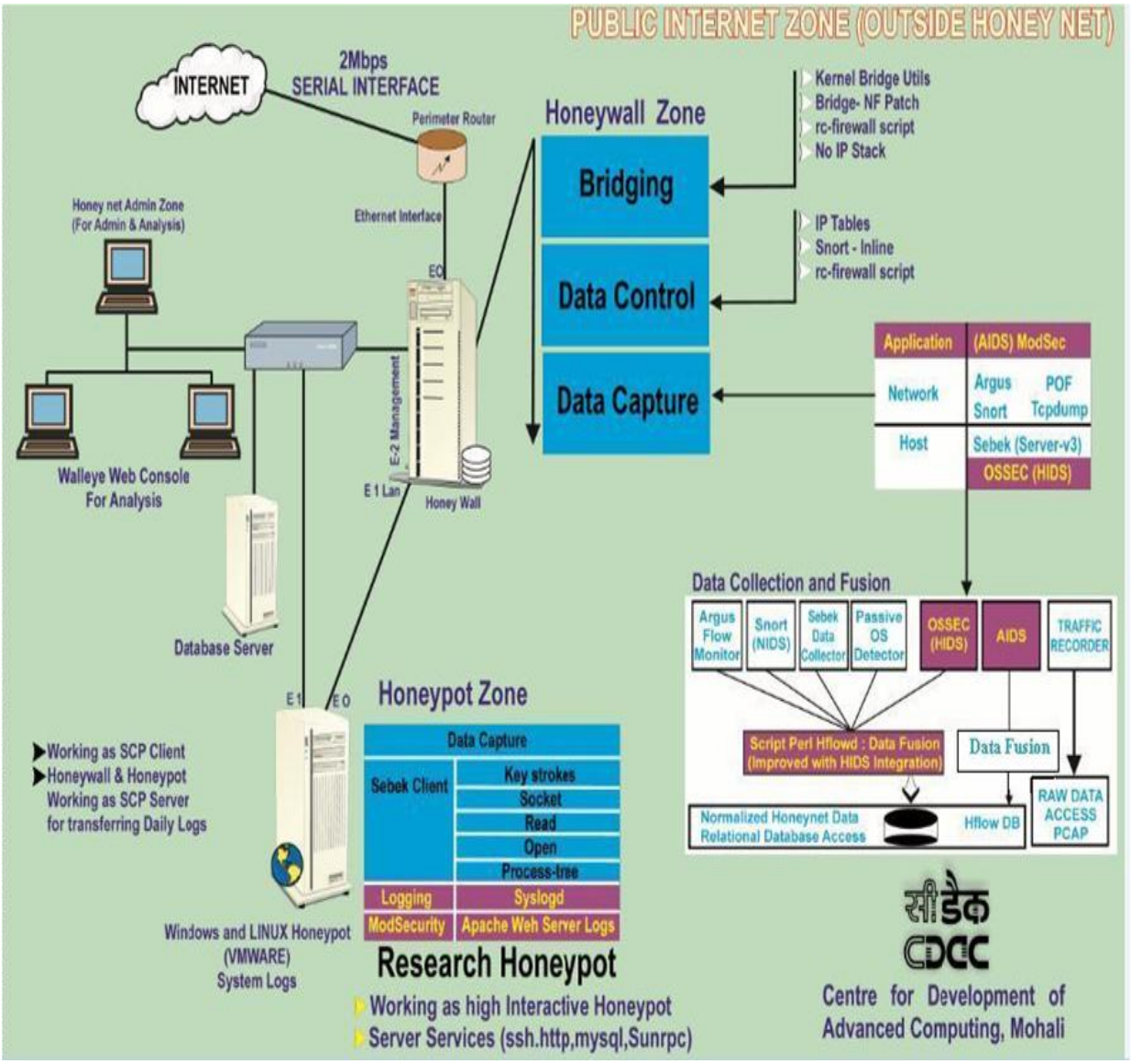


Figure: Enhanced GEN III Honeynet Architecture

The following were the benefits established from the enhanced Gen- III Honeynet Architecture:

- Enhanced attack classification depicting degree of access attempt into system and attack severity.

- Enhanced attack detection supplemented at host layer (detection of SSH attacks, login access attempts, brute force attacks, buffer overflow attacks, etc) and application layer (SQL injection, XSS attacks, etc).
- Considerable percentage reduction of unknown attack domain as the known network, host and application attacks have been separated from Honeynet data with HIDS and AIDS integration.
- Improved attack analysis as the HIDS and AIDS alerts can be graphically visualized in a separate area in correlation with network sessions.

Project Title: Development of Distributed Honeynet System, with dynamically configurable Honeypots as distributed nodes.

The system has the capability for

- Collection of malware and BOT samples
- Detection of Botnets
- Finding and confirming the latest attack trends

Targets Achieved:

- Dynamically configurable Honeypot framework has been developed
- Three tier architecture based Distributed Honeynet system has been developed
    - Tier 1: Data capturing sensors
    - Tier 2: Data collection
    - Tier 3: Data storage and data analysis
- A full working prototype of Distributed Honeynet System is functional with dynamically configurable Honey net nodes deployed at eight geographically distributed locations lying in different IP domains.
- Bot Detector Engine V1.0 developed
- Botnet Tracking Engine has been developed.
    - A web interface also has been developed in the project for
        - accessing the functionality of distributed Honeynet system
        - sharing the data and analysis results with participating institutes
        - monitoring the health of the remote nodes

- Collected malware samples along with analysis results were being shared with CERT-in on regular basis. The malware samples collected are labeled and archived in the malware database along with the related information. Some of the Major Incidents captured and shared with CERT-in

Following figure gives an abstract picture of the three tier architecture of Distributed Honeynet System

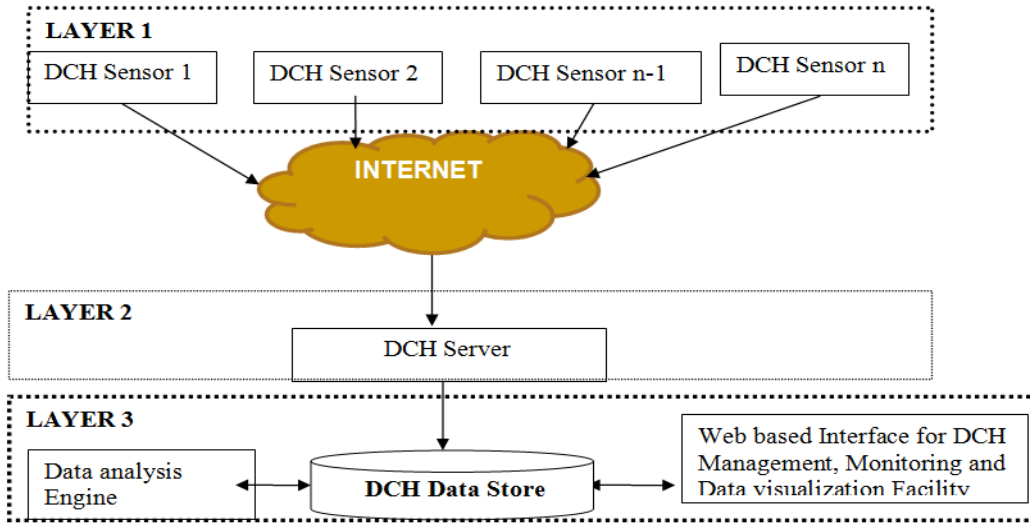


Figure: Distributed Honeynet System

Layer1: incorporates Dynamically Configurable Honeypot framework as the collection sensor. Dynamically configurable Honeypot framework provides a collection of high interaction and low interaction honeypots in a virtualized network environment as capturing sensors. Following is the details of these capturing sensors and the data that is captured at every Distributed Honeynet remote node.

Captured Data from DHS:

- From low interaction Honeypot (Nepenthes)
  - binary files/EXE files
- From High interaction Honeypots
  - Exe files (exe files created/dropped)
  - Log files that has been modified

- Log files that has been deleted
- PCAP files (network packets)
- Network flow data

The Dynamically Configurable Honeypot Framework (DCH) is a client server based model where client nodes are deployed at layer 1 of Distributed Honeynet system and server node is placed at layer 2.

The DCH client is designed as a single threaded daemon. This single threaded design avoids any kind of deadlocks and hence the need of applying mutexing which reduces the code complexity. Further two programmable interrupt based timers are used which initiates the process of data collection and loading the fresh image of the Honeypot in the Honeynet. This node performs following functions:

- Client registration.
- Configuration /reconfiguration of Honeynet system.
- Data collection

Further this DCH client node is dynamically configurable in terms of:

1) Configuration(operating system) of Honeypots
2) Contents of Honeypots
3) IP addresses

Being dynamic in terms of above mentioned parameters means that the Honeynet will change these parameters without going off line or rebooting. This feature reduces the down time of the collection framework and hence minimize the data loss during the reconfiguring the honeypots.

The dynamism of the remote node further depends upon two factors.

1. The availability of resources at remote node.
2. The capability of the Operating system image present at remote node.

The availability of resources refers to the availability of resources such as IP ranges, registered domain names at remote node. Whereas the image capabilities refer to the Operating system Image features. In virtualization every running operating system is treated as a file which is referred as Image of that operating system. This image is the complete operating system in itself in the form of a file in VDI format.

The configurability of remote node depends upon the factors such as availability of the

suitable image file with required software's installed. Every node incorporates a database of such Image files. These image files in the database are crafted in to a special hierarchal format devised by us to express the capability of the image in terms of operating system configurability and software packages available in that image. Every time a configuration is sent by the central node administrator from remote server (layer 2) the client configuration manager (at layer 1) parses the configuration and checks the availability of required resources and images. If the image is available it creates the clone of that image which is a copy of the image file and configures it and starts it.

To extend this dynamism to cover more Operating system with diverse software packages and vulnerability .There is a provision to remotely upload the Image files in to the remote nodes image database. There is also a provision to locally add the new image file in the standard format in to the nodes image database.

Further for the purpose of data capturing and collections at remote DCH client nodes techniques like Cross-time-diff and active file system monitoring are used. In cross-time-diff technique the running image of the operating system is compared with the original image that is saved in the image database and the difference in terms of new files added, edited, deleted is extracted. This extracted data if further filtered, compressed and sent to the DCH central server deployed at layer 2. To optimize the bandwidth utilization during the data transfer the collected data at node is compressed before sending it to central server. For this a lossless data compression technique which is a variant DEFLATE data compression algorithm is used.

The compressed data is sent using SSL/TLS channel to the DHS server this channel is further strengthened by the use of Digital certificates. This ensures data integrity and confidentiality.

Layer 2 incorporates Distributed Honeynet Server which performs following activities for remote client nodes:

1. Registration of the new nodes.
2. Providing the configuration data to the remote nodes.
3. Processes data sent by remote nodes by fusing the data with the configuration information of Honey pots and converting it in to a relational data base format.

At server end the binary sample data sent by the remote nodes is uncompressed, parsed

and fused with Meta data and is converted in to the Relational data base format and stored in MYSQL data base. All these mechanism are developed using GNU tool chain.

Layer 3 consist the Database which acts as a data source for analysis engine and DCH web interface. The data base is designed to minimize the data redundancy and optimize the data retrieval. The database has the provision to maintain the information related to the honeypots configuration with time stamp during which the configuration was deployed. This information helps the analyst to segregate the malware sample based upon the configuration from which the malwares were collected.

Further this database stores the MD5 string of the binary sample with respect to the time of its collection. The binary file is only saved in data base if it founds to be unique. Hence it reduces the data redundancy as repeatedly coming binary samples are stored once but the date and time of their repeated collection is saved with their MD5 string. This reduces the redundant data without any information loss.

At layer 3 there a web based interface for Distributed Honeynet system management, data visualization and dynamically configuring the remote Honeynet nodes has been developed. This web interface provides following features:

1. Data and analysis results display
2. Monitoring the health of the remote nodes
3. Sharing data with the participating institutes
4. Configuring and reconfiguring the Honeynet nodes

Bot Detection Engine:

Bot detection engine developed by us identifies those binary samples which uses the IRC (Internet relay Chat) commands for communication with the BOT C&C server. As IRC (Internet relay Chat) channels are greatly used by the Botnets for the communication with the bot systems hence any malware sample which uses the IRC commands such as USER, NICK, and PASSWORD etc. are assumed to be a bot sample.

The binary classifier developed by us uses the Native API call sequence mining technique for the segregation of the malware samples that uses the IRC protocol from the rest of the malware samples collected by the Distributed Honeynet system.

Following steps are performed during bot detection process:

- The bot detection engine fetches the malware samples from the malware repository.
- The binary samples are then executed in the sandbox environment and Native API traces are collected.
- These API traces are given as an input to the bot detection engine
- The bot samples identified by the bot detection engine are further submitted to the botnet tracking engine.

Botnet tracking Engine:

The Botnet tracking engine detects the Botnet C&C server by clustering the network traffic of the bot samples. The clustering is performed based upon the features such as:

1. IRC, HTTP commands (USER, NICK, and PASSWORD etc)
2. Domain names resolved
3. IP addresses

We have developed a system to detect the Botnet C & C server based on Botnet commands signatures matching in payload of packet captured. This is useful for IRC and HTTP based Botnet C & C server detection.

Project Title:

Development of an Active Honeypot for Distributed Honeynet System (DHS)

Scope

a) To design and develop a web based low interaction Honeypot (**Active Honeypot**) for the collection of malware samples which uses "drive-by-download" technique as an attack vector to propagate.
b) Incorporation of Active Honeypot system as a collection sensor in the dynamically configurable Honeynet framework and its deployment in distributed Honeynet system.
c) Development of a Bot detection engine to detect the HTTP bots.
d) Development of botnet tracking engine to detect live C & C server

Target Achieved:

1. Development of Active Honeypot System (AHP):

- **Drive-by-Download Attack Scenario**

In drive-by-download attack, a user visits a web page (or redirects to another URL) that hosts a web exploitation kit. Following the client's request for HTTP page, the web server implements some server side logic that assesses user details such as user location, browser's version, browser type, etc. Based upon these details, the server sends the exploit code to the user as part of HTTP response. This exploit code attacks the client, and if successful, executes a downloader component without the user's consent or notice. The downloader, in turn, makes a follow-on request to download and execute a piece of malware from the URL specified by the attacker. Once the malware is downloaded and executed, the attacker has complete control of the client machine and can steal sensitive information.

- **Deriving Problem Statement from Attack Scenario**

Being a type of client-side attack, detection of Drive-By-Download attacks needs to be addressed at client-side. This requires URL execution in browser environment to detect the browser exploitation to further affect system changes (malware drop). The team presented the translation of attack scenario to problem statement. The problem statement can be stated as the development of Active Honeypot for (a) Overcoming the challenge of multiple browser-OS combinations to detect actual system exploit (b) Capturing static and dynamic webpage contents (c) Inspection of dynamic Javascript behavior to detect mal code and/or redirections (d) Actual exploitation of system to detect and capture system changes (specifically malware dropped) (e) Determination of malware distribution site and the chain of dynamic URL redirections.

- **Solutions Developed to address Problem Statement**

To address the above mentioned challenges (problem statement), a hybrid prototype Active Honeypot System has been developed that includes two components **(1) Low Interaction Active Honeypot** to address problems (a), (b), and (c). It is a browser independent solution using emulated browser and Javascript engine that facilitates the execution of URL and Javascripts in safe emulated environment (protected from self-exploitation) without the need to configure browser-specific environment. Use of emulation enables the capturing of static and dynamically generated web contents including mal iframes and mal links. Use of Javascript engine enables the inspection of dynamic javascript behavior thus defeating the mechanisms of obfuscation and other

code-hiding techniques used by attackers. (**2) High Interaction Active Honeypot** to address problems (d) and (e) stated above. High Interaction Active Honeypot is a browser-specific solution since it performs URL execution using a real browser and a real system to identify actual exploitation. It is based on a Client-Server Analysis Framework to identify persistent state changes (particularly file system) on client machine during active URL visit. It utilizes the mechanism of HTTP Session reconstruction to re-establish the complete HTTP Session from PCAP data (captured during run time execution of URL on client machine) to detect the chain of dynamic auto-redirections and malware distribution site.

Project Efforts:

- **Active Honeypot Modules:**

Active Honeypot has four modules

a. Seeder to seed input URLs to Active Honeypot (Currently manual seeder)
b. Crawler for active URL hunt
c. Low Interaction Active Honeypot (based on emulated browser)
d. High Interaction Active Honeypot (based on real browser)

- **Design Details**

(a) Low Interaction Active Honeypot

- Basic Functionality is to analyze and classify URLs using emulated browser (and SpiderMonkey Javascript Engine).
- Analysis domain includes following parameters:
  - ***URL Hunt*:** Crawling to extract web links from seed URLs (Incorporated in Desktop Application) with user-driven specification of crawling depth.
  - ***Inspection of Structural Aspects*:** Monitoring elements of webpage using DOM Parsing to declare potentially suspicious iFrames & Malicious Links (statically embedded and dynamically generated) based on their structural and syntactical traits:
  - ***Javascript Analysis*** to detect
    - ***Obfuscated Scripts*** based on n-grams, entropy, entropy density and Longest String Length (Machine-Learned Model Comparison)

- ***Scripts that exhibit mal behavior*** (JS runtime Behavior Extraction) based on frequency and sequence mining (Machine-Learning based models) of function calls and the logical categories of function calls.
- Signature based detection Scanning incorporating self-crafted signatures, iscanner signatures, signatures from Honeysift (Open source-Low Interaction Honeyclient).
- Exploit declaration added for Java Scripts based on AV signature scanning.

- ***Extraction of Redirection domains*** are provided as additional output parameters (extracted from DNS queries) which gives the visibility of auto redirections.
- ***Cross-validation of results*** through simultaneous result display of Google-Safebrowsing for every analyzed URL on same output window.
- HTML DOM graph display in PNG format for visualizing DOM structure of the webpage.

(b) High Interaction Active Honeypot

- The basic functionality is to identify malicious servers by interacting with potentially malicious servers using a real browser on a dedicated virtual machine and observing the system for unauthorized state changes. If such state (File system, registry, process) changes are detected, the interacted server is classified as malicious. Following are the main features:
  - ***Browser-based solution*** for detection of malicious URLs. It is based on a Client-Server Analysis Framework to identify persistent state changes during active URL visit.
  - ***Controller Module*** that provides seed URL to the client virtual machine for execution, logs state changes on client machine, and processes analysis logs received from client, and restarts the analysis process with fresh clean virtual machine.
  - ***State-Change Monitor*** Module that actively monitors state changes of client machine during URL execution in real browser. The state change recorder maintains process white-listing to identify dynamically generated suspicious activities (excluding while-listed processes) on

client machine. This also enables to capture malware dropped during URL visit.

- ***Network Monitor*** to inspect HTTP communications between client machine and Web server based on Snort and Emerging threat rules.
- ***Auto-Redirection tracking*** that actively tracks and logs dynamic domain redirections during URL execution.
- ***HTTP Session reconstruction*** to re-establish the complete HTTP Session from collected PCAP data (during run time analysis of URL) to detect the malware distribution site.
- ***Malware Classification module*** that classifies the dropped binary using virus-total labeling.

- High interaction Active Honeypot has been benchmarked and evaluated with respect to GFI Sandbox.

- **Integration of Active Honeypot with Distributed Honeynet System**

Low and High Interaction Active Honeypot have been integrated with DHS Web Interface. Now the user has the flexibility to feed URL to Low or High or both Active Honeypot and view the results on DHS Web interface.

Development of Bot Detection Engine/Botnet Tracking

Main Functionality and Features

The main approach for bot detection is to identify and model sequences of network communications (evidences) that occur between victims host (Honeypot) and network entities and leverage the inherent contextual facts.

Input Dataset: PCAP Dump (Packet Data Collected on DHS)

Output: Bot Infections and associated following results

(a) Events Generated and the actual communications logged in raw format

(b) C&C IP, Attacking IP, and Egg Download IP

(c) IP Profiling to determine ISP, Country, and Address

(d) Event sequence Graph

Achievements of Bot Detection Engine:

- HTTP Bot detection and Botnet (C&C) tracking module has been developed (Details Annexure IV) and integrated with DHS Web Interface.
- Unit testing of all the sub modules, integration testing and smoke testing of the complete DHS with HTTP bot detection engine and botnet tracking engine has been done.
- Performance evaluation of the Bot detection engine has been done in terms of latency involved in processing of data.
- Comparison of the results of bot detection engine with the similar open source tools (Botflex, Bothunter, Rishi, Botnet systems) has been done.
- Case Studies for the following specific bot infection instances has been done
    - IRC Bot with multiple Malware samples
    - SDbot
    - Zeusbot or Zbot (HTTP BOT sample)
    - Encrypted Bot samples

Annexure B: Gantt chart

Sl No	Activity	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Q12
1	Requirement study and analysis												
2	Implementation of Distributed Honeynet System at CDAC- Silchar												
3	Training and workshops on Honeynet Technology Implementations to NE participating agencies.												
4	Implementation of concept lab at Tezpur University												
5	Support and trouble-shooting, bug fixing												
6	Development of test-bed for experimentation in network security for engineering students.												
7	Implementation of Distributed Honeynet System at NE region in coordination with CDAC-Kolkata												
8	Experimentation and analysis on collected attack data.												
9	Development of tools and mechanisms for attack data												

	categorizations.												
10	Machine learning based analysis of collected attack data												
11	Impact analysis and further conclusion												
12	Forensic analysis, enhancements and further inclusions												
13	Improvement in Data control techniques and its testing												
14	Integration of new Data Control mechanism with Distributed Honeynet System												
15	Evaluations and testing of tool and mechanism.												
16	Improvements based on University administrator's feedback and participations												

BIOTECH CONSORTIUM INDIA LIMITED

BCIL

5th Floor, Anuvrat Bhawan, 210, Deen Dayal Upadhyaya Marg, New Delhi-110002
CIN: U73100DL1990PLC041486 **Tel** : 2321 9064 - 67 **Fax** : 011- 2321 9063
Email : info bcil@nic in **Website** : http //www bcil nic in

BCIL/NER-BPMC/2016/ October 21, 2016

The Registrar
Tezpur University
Naapam, Tezpur
Dist -Sonitpur-784028
Assam

Dear Sir/Madam,

Sub.: Release of grant to the implementing agency for the project entitled "Integrating Genome scale metabolic analysis of model plant pathogen *Ralstonia solanacearum* with RNAseq and fluxomics" under DBT's Twinning Program for the NE

As you may be kindly aware, the Department of Biotechnology (DBT), Government of India, has setup a North Eastern Region-Biotechnology Programme Management Cell (NER-BPMC) for promotion of Biotechnology in the North Eastern Region of India through Biotech Consortium India Limited (BCIL), a company promoted by the Department of Biotechnology.

We are happy to inform you that the above proposal submitted by your organization has been sanctioned by DBT. Please find enclosed herewith a copy of the Sanction Order No. BT/PR16361/NER/95/192/2015 dated October 18, 2016 of the DBT along with terms and conditions of the grant, format of UC/SE, format of asset acquired, format of memorandum of agreement, etc.

Our Accounts Division has transferred an amount of Rs. 13.79 lakhs as first year release towards implementation of the above project at your organization as per the following bank details:

Account Holder's Name: Registrar, Tezpur University
Bank and branch: State Bank of India, Tezpur Branch
Account No. : 30448821505
IFSC code: SBIN0000195

We request you to please send the signed Memorandum of Agreement (MOA) which has to be executed on Rs. 100/- non-judicial stamp paper as per the format enclosed. You are also requested to send a signed copy of the terms and conditions of the grant. These signed documents may please be sent to **Dr. T. Madhan Mohan, Adviser, Department of Biotechnology (DBT), Block No. 2, 7th Floor, CGO Complex, Lodi Road, New Delhi-110003** within a month's time from the date of release of funds to your organization, latest by November 21, 2016.

Kindly acknowledge the receipt.

Thanking you,

Yours faithfully

(Vinod Kumar)
Manager

CC for kind information to:

1) The Vice-Chancellor, Tezpur University, Napaam- 784028, Tezpur, Assam.
2) Dr. Siddhartha Satapathy, Associate Professor, Department of Computer Science & Engineering, Tezpur University, Napaam- 784028, Tezpur, Assam
3) Dr. Suvendra Ray, Associate Professor, Department of Molecular Biology and, Biotechnology, Tezpur University, Napaam- 784028, Assam

Scanned by TapScanner

165-166

Administrative
(Application No: AGRI/2015/33)

File No. BT/PR16361/NER/95/192/2015
GOVERNMENT OF INDIA
MINISTRY OF SCIENCE & TECHNOLOGY
DEPARTMENT OF BIOTECHNOLOGY
(NER-BPMC)
****

Block-2, 7th Floor,
CGO Complex, Lodhi Road
New Delhi-110003
Dated: 18 / 10 / 2016

ORDER

Sanction of the President is hereby accorded under Rule 18 of the Delegation of Financial Powers Rules, 1978 for the implementation of the project under **'DBT's Twinning programme for the NE'** titled **"Integrating Genome scale metabolic analysis of model plant pathogen *Ralstonia solanacearum* with RNAseq and fluxomics"** by Dr. Siddhartha Satapathy, Tezpur University, Tezpur and Dr. Shyam Masakapalli, Indian Institute of Technology, Mandi, Himachal Pradesh at a total cost of **₹ 57.40 lakhs (Rupees Fifty Seven lakhs and Forty thousand only)** for a period of three years on the terms and conditions detailed as under:

2.0 The Project:

2.1 Project Title: Integrating Genome scale metabolic analysis of model plant pathogen *Ralstonia solanacearum* with RNAseq and fluxomics

2.2 Project Investigators

2.2.1 Principal Investigator: (Parent Institute)	**Dr. Siddhartha Satapathy** Associate Professor Department of Computer Science & Engineering Tezpur University, Napaam- 784028, Tezpur, Assam
2.2.2 Co-PI Name: (Parent Institute)	**Dr. Suvendra Ray** Associate Professor Department of Molecular Biology and Biotechnology Tezpur University, Napaam- 784028, Assam
2.2.3 Principal Investigator: (Collaboration Institute)	**Dr. Shyam Masakapalli** Assistant Professor School of Basic Sciences Indian Institute of Technology, Academic Block, Near Vallabh College Campus, Mandi- 175001, Himachal Pradesh.

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OFFICE OF THE DEAN R&D
TEZPUR UNIVERSITY
DATE OF RECEIPT 17/3/2020
RECEIPT NO 4322
SIGNATURE

Office of the Registrar
Receipt No 8515
Date 16/03/20
Received by D.O

ICMR
INDIAN COUNCIL OF MEDICAL RESEARCH

भारतीय आयुर्विज्ञान अनुसंधान परिषद्
स्वास्थ्य अनुसंधान विभाग, स्वास्थ्य और परिवार कल्याण मंत्रालय, भारत सरकार
Indian Council of Medical Research
Department of Health Research, Ministry of Health and Family Welfare, Government of India

Dean, R&D / for info
PO.

File No: 5/7/1679/NER/Ad-hoc/2019-RBMCH

Date: 06.03.2020

The Registrar,
Tezpur University,
Napaam, Tezpur,
Assam - 784028

Subject: Sanction of budget allotment for the NER Research proposal entitled "Randiomics with Machine Learning Methods towards Prediction of Gall Bladder Cancer"

Sir,

The Director General of the ICMR sanctions the above mentioned research scheme initially for the period of one year initially from 31.03.2020 subject to extension up to the total duration specified in para 4 below:-

1. The Director General of the ICMR also sanctions the budget allotment of Rs. 12,67,448/- as detailed in the attached statement for the period from 31.03.2020 to 30.03.2021. The grant-in-aid will be given subject to the following conditions.
2. The payment of the grant will be made in lump-sum to the Head of the Institute. The first installment of the grant will be paid generally as soon as report regarding appointment of the staff is received by the council. The Staff appointed on the project should be paid as indicated in the budget statement.
3. The staff on the project will be recruited as per the rules and procedure of the hot institute and second part of the undertaking be obtained from the employees of the project. The staff grant will not be released unless the required undertaking [part-II] from Head of the Institute is received in this Office.
4. The demand for payment of the subsequent installment of the grant should be placed with the Council in the prescribed Performa. The approved duration of the scheme is TWO YEAR. The annual extension will be given after review of the work done on the scheme during the previous year.
5. Five copies of the annual progress report should be submitted to the ICMR every year after completion of ten months of the project giving complete actual details of the research work done. Failure to submit the report in time may lead to termination of project.
6. Subject to the condition that the grant will be utilize after following the previsions laid down in the GFTs-2017& Ta Rules please keep the fund in the separate saving Bank Account opened for ICMR funded Research projects so that interest earned thereon is credited into the accounts.

The receipt of this letter may please be acknowledge.

Yours faithfully

(Harjeet Kaur Bajaj)
Admn. Officer
For Director General

Coppy together with a copy of the budget statement forward to information to:-

1. Dr. Rosy Sarmah, Assistant Professor, Deptt. Of CSE, Tezpur University, Napaam, Tezpur, Assam - 784028
2. Accounts. V for information.
3. IRIS Cell No. 2019-7821
4. Budget Section
5. Sh. B.S. Yadav, Sr. To, ICMR, New Delhi.

HD
Mar 1.
17/3/20

File No: 5/7/1679/NER/Ad-hoc/2019-RBMCH
RFC NO. RBMCH/NER/8/2019-20 Dt- 26/02/2020
Date of Start: 31.03.2020
Duration: 2 Year

Subject:- Research Project entitled "Randiomics with Machine Learning Methods towards Prediction of Gall Bladder Cancer"

1st Year Budget
(31.03.2020 to 30.03.2021)

Sl.NO.	BUDGET HEAD	1st YEAR
A.	**Salary**	
(1)	Junior Research Fellow – 31000/- p.m + HRA 8% = 2480 (31000+2480) 33480/-p.m.	4,01,760/-
(2)	Project Technician – 1 @ 16000/- p.m 10% annual increment for 2nd Year (16000 + 710) =16710/- p.m	1,92,000
	Sub Total	**5,93,760/-**
B.	**Recurring (Contingency)**	
(i)	Stationery	10,000/-
(ii)	Publication charges /seminars /workshops	20,000/-
(iii)	Consumables	25,000/-
(iv)	Risk management & Miscellaneous	25,000/-
	Sub Total - B	**80,000/-**
C	**Non – Recurring**	
(i)	High end workstation (computer with accessories) = (Quantity-2)- (Rs. 2,50,000 x 2)	5,00,000/-
(ii)	Network Printer (1)	20,000/-
	Sub Total - D	**5,20,000/-**
D.	Travel	40,000/-
E.	Overhead charges 5% (A+B)	33,688/-
	Grand Total (A+B+C+D+E)	**12,67,448/-**

Total Budget allotment of Rs. 12,67,448/- (Rupees Twelve Lakhs Sixty Seven Thousand Four Hundred Forty Eight Only)

Administrative Officer
For Director General

भारतीय आयुर्विज्ञान अनुसंधान परिषद
स्वास्थ्य अनुसंधान विभाग, स्वास्थ्य और परिवार
कल्याण मंत्रालय, भारत सरकार

Indian Council of Medical Research
Department of Health Research, Ministry of Health
and Family Welfare, Government of India

File No: 5/7/1679/NER/Ad-hoc/2019-RBMCH

Date: 06.03.2020

Subject:- Payment for the project entitled "Randiomics with Machine Learning Methods towards Prediction of Gall Bladder Cancer"

MEMORANDUM

The Director General of the ICMR sanctions the payment of Rs. 12,67,448/- (Rupees Twelve Lakhs Sixty Seven Thousand Four Hundred Forty Eight Only) as the 1st & final installment of the 1st year grant for the period from 31.03.2020 to 30.03.2021 for incurring expenditure in connection with the above mentioned project.

The amount of Rs. 12,67,448/- may be debited to the provision made of Rs. 12,67,448/- on the above mentioned for the year 2019-2020.

A formal bill for Rs. 12,67,448/- is sent herewith for payment released by NEFT/RTGs in favour of the Registrar, Tezpur University, Tezpur, Assam, with issued the concurrence of the Finance Divn., Vide RFC No. RBMCH/NER/8/2019-20 Dated - 26/02/2020

(Harjeet Kaur Bajaj)
Admn, Officer
For Director General

Accounts V. ICMR
Copy to:

1. The Registrar, Tezpur University, Tezpur, Assam. The grant has been sanctioned on the conditions laid down in our letter referred to above.
2. Dr. Rosy Sarmah, Assistant Professor, Deptt. Of CSE, Tezpur University, Napaam, Tezpur, Assam- 784028.
3. IRIS Cell No. 2019-7821

Admn. Officer
For Director General

प्रगत संगणन विकास केंद्र
CENTRE FOR DEVELOPMENT OF ADVANCED COMPUTING

संचार एवं सूचना प्रौद्योगिकी मंत्रालय की वैज्ञानिक संस्था, भारत सरकार
A Scientific Society of the Ministry of Communications and Information Technology, Government of India

सी डैक
CDAC

पुणे विश्वविद्यालय परिसर,
गणेशखिंड, पुणे - 411 007, भारत
Pune University Campus,
Ganeshkhind, Pune 411 007, India.
फ़ोन / Tel: +91-20- 2570 4100,
फैक्स / Fax : +91-20 -2569 4004
www.cdac.in

CORP: R&D: 2015

May 12, 2015

Prof. Mihir K.Chaudhury
Vice Chancellor
Tezpur University
Tezpur – 784028
Assam, India

Subject: Approval of the Project Proposal 'Establishment of State-of-the-art High Performance Computing Infrastructure for Scientific Computing in North-East at Tezpur University, Napaam, Sonitpur, Assam'

Reference: Your letter No/TU/VC/I(1)/2015 dated 14.1. 2015

Dear Prof. Chaudhury,

We are pleased to inform you that the project proposal titled 'Establishment of State-of-the-art High Performance Computing Infrastructure for Scientific Computing in North-East at Tezpur University, Napaam, Sonitpur, Assam' submitted to the C-DAC's North East Steering Committee has been approved (vide Approval Note Ref. No. CORP: R&D: 2015:1020 dated May 8, 2015).

As mentioned in the project proposal, following are the Chief Investigator and Co-Investigators:

1. Shri Ashish Ranjan, C-DAC, Pune – Chief Investigator
2. Shri Deepu C V, C-DAC, Pune – Co-Investigator
3. Prof. Dhruba K. Bhattacharyya, Tezpur University, Assam – Co-Investigator

This is for your kind information.

We are thankful to you for extending your full support to this important national initiative and shall look forward to fruitful association with your University.

P. K. Sinha
Sr. Director (Corporate Strategy and R&D)

Encl: Approval Note dated May 8, 2015

Cc: 1. Prof. Dhruba K. Bhattacharyya, Dean, Tezpur University, Assam
2. Dr. Hemant Darbari, Executive Director, C-DAC, Pune

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CDAC

CENTRE FOR DEVELOPMENT OF ADVANCED COMPUTING

APPROVAL NOTE

CORP: R&D: 2015:1020

May 8, 2015

Subject: Approval of your Project Proposal 'Establishment of State-of-the-art High Performance Computing Infrastructure for Scientific Computing in North-East at Tezpur University, Napaam, Sonitpur, Assam' submitted to North East Steering Committee

Dear Shri Ashish Ranjan,

This has reference to the project proposal 'Establishment of State-of-the-art High Performance Computing Infrastructure for Scientific Computing in North-East at Tezpur University, Napaam, Sonitpur, Assam' submitted to the North East Steering Committee which was reviewed during the meeting held on November 13, 2014.

With reference to the revised proposal received from you on January 19, 2015 and subsequent clarifications provided, we are happy to inform you that your project proposal as per the details given below has been approved.

Title of the Project:	Establishment of State-of-the-art High Performance Computing Infrastructure for Scientific Computing in North-East at Tezpur University, Napaam, Sonitpur, Assam
Chief Investigator:	Shri Ashish Ranjan, C-DAC, Pune
Duration of the Project:	24 Months
Scheduled Start Date of the Project:	May 2015
Scheduled End Date of the Project:	April 2017
Total Project Outlay:	Rs. 122.50 Lakhs

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CDAC

Total Budget Outlay (Rs. in Lakhs):

Particulars	Year 1	Year 2	Total
Capital Equipment & Software (non-recurring)	90.00	0.00	90.00
Consumable stores (incl., maintenance, books etc.)	2.00	1.00	3.00
Manpower	9.76	10.74	20.50
Travel (Internal)	3.00	2.00	5.00
Training & Workshop	2.00	2.00	4.00
Total	**106.76**	**15.74**	**122.50**

We request your highest degree of commitment and dedication to ensure that the project meets all its deliverables well in time and make a significant contribution to the development of North East Region. The guidelines and terms & conditions for the execution of the project are enclosed in Annexure A and B respectively. Please go through the same and ensure they are followed.

We request you to send the acknowledgement of this Approval Note countersigned by the Chief Investigator and the Centre Head to the Corporate Office, at the earliest.

This has approval of the Competent Authority.

With regards,

P. Sinha

Pradeep K.Sinha
Sr. Director (Corporate Strategy and R&D)

To: Chief Investigator

CC: 1. Executive Director, C-DAC, Pune
2. North East Steering Committee

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CDAC

Annexure A

Guidelines for the Execution of the Project

- With this Approval Note, we will be releasing the first installment of your Project Outlay.
- The schedule of the project and its timely and successful execution is extremely important and Chief Investigators shall adhere to the same.
- The Steering Committee will call for regular review of the project, normally every three months. You are expected to present a report on the progress of the project vis-à-vis the deliverables and proposed timeline at the time of every review, including financial status.
- Depending on the project requirements, progress of the project and utilization certificate for the released funds, further funds will be released.
- Chief Investigator shall provide a brief progress report (maximum one page) of the project to the Member Convener of Steering Committee (currently, Mr. Anant Kelkar – anantk@cdac.in) by 25th of every month.
- As you are aware the Central Government is attaching high priority to the development of North East Region, successful execution and implementation of the project will make a visible impact.
- If there are any hindrances or issues in timely and effective execution of the project, please bring them to the attention of the Steering Committee as soon as possible; this will help us to look at remedial action before it is too late.

Media Lab Asia
innovating for Digital Inclusion

(A Section 25 Company of Department of Electronics and Information Technology, Ministry of Communications and Information Technology, Govt. of India)
CIN : U72900MH2001NPL133410
ISO 9001 - 2008 Certified Organisation

Corporate & Registered Office :
Samruddhi Venture Park
No.2, 4th Floor
Central M.I.D.C. Road
Andheri (East), Mumbai - 400093
Tel.: +91(22) 28312931 / 28327505
Fax: +91(22) 28379158

Information Technology Research Academy (ITRA)

Administrative Approval

Ref. No.: ITRA/15(189)/Ag&Food/e-Varaha/01 **Date: October 28, 2016**

Subject: Administrative Approval for execution of the project "e-Varaha: Information System for Safe Pork Production in North Eastern India."

Implementing Agency's:

(I) Indian Institute of Technology (IIT) Guwahati, Guwahati – Lead Institution- IT
- **(i) Tezpur University (TUA), Assam-PIN-IT;**
- **(ii) Manipur University (MUM), Imphal, Manipur- PIN-IT;**

(II) ICAR-National Research Centre on Pig (NRCP), Guwahati – Lead Institution-AF;
- **(i) ICAR-Research Complex for North Eastern Hilly Region (ICAR-RCNEHR-M), Barapani, Meghalaya- PIN-AF;**
- **(ii) College of Veterinary Sciences & Animal Husbandry, Central Agricultural University (CVS), Aizawl, Mizoram-PIN-AF;**
- **(iii) Veterinary College, Karnataka Veterinary, Animal and Fisheries Sciences University (VC-KVAFSU), Karnataka– PIN-AF;**

Dear Sir,

I am directed to convey the Administrative Approval of the Competent Authority for the implementation of the above named project by the team of 7 institutions with Indian Institute of Technology (IIT), Guwahati as (Lead Institution-IT) and ICAR- National Research Centre on Pig (ICAR-NRCP), Guwahati (Lead Institution-AF). The total estimated cost of the project is Rs. 240.87 Lakhs (Rupees Two Crore Forty Lakhs and Eighty Seven Thousand Only) as Grant from ITRA, Media Lab Asia. **The duration of the project is 24 months.**

2. ITRA, Media Lab Asia shall provide funds for undertaking the Project through a grant of Rs. 240.87 Lakhs (Rupees Two Crore Forty Lakhs and Eighty Seven Thousand Only) for the implementation of the above project. The details of the project are given in the Annexure-I enclosed. Also enclosed is Annexure-II containing the **terms & conditions** for the implementation of the said project.

3. You are instructed to send the confirmation of acceptance of **terms & conditions** attached with the Administrative approval in respect of the project. The release of funds shall be subject to receiving the duly signed copy of the terms & conditions by Media Lab Asia.

Media Lab Asia — Ministry of Communications & Information Technology — Govt. of India — Dept. of Electronics & Information Technology

George Arakal
Director (Admin. & Finance)

media Lab ASIA
innovating for Digital Inclusion

ISO-9001-2008 CERTIFIED

To:

1. Prof. S.R.M. Prasanna, Dean, Research & Development, Indian Institute of Technology Guwahati, Guwahati - 781039.
2. Prof. Mihir Kanti Chaudhuri, Vice-Chancellor, Tezpur University, Assam - 784028.
3. Prof N Lokendra Singh, Registrar, Manipur University, Canchipur, Imphal - 795003.
4. Dr. Dilip Kumar Sarma, Director, ICAR-National Research Center on Pig, Guwahati - 781131.
5. Dr. S.V. Ngachan, Director, ICAR-Research Complex for NEH Region, Umroi Road, Umiam Meghalaya - 793103.
6. Dr. D. Kathiresan, Dean, College of Veterinary Sciences & Animal Husbandry, Central Agricultural University, Selesih, Aizawl, Mizrom - 796014.
7. Dr. Chandrapal Singh, Registrar, Veterinary College, Karnataka Veterinary, Animal and Fisheries Sciences University (VC-KVAFSU), Nandinagar, Bidar-585 401

cc:

1. Dr. Sukumar Nandi, Professor, Department of Computer Science & Engineering, Indian Institute of Technology, Guwahati - 781039.
2. Dr. Nityananda Sarma, Professor, Department of Computer Science & Engineering, Tezpur University, Assam - 784028.
3. Dr. Khumukcham Robindro Singh, Assistant Professor, Department of Computer Science, Manipur University, Manipur - 795003.
4. Dr. Girish Patil S., Senior Scientist (Livestock Products Technology), ICAR-National Research Center on Pig, Indian Council of Agricultural Research, Guwahati - 781131.
5. Dr. Arnab Sen, Principal Scientist & Head, Division of Animal Health, ICAR-Research Complex for NEH Region, Barapani, Meghalaya - 793103.
6. Dr. Tapan Kumar Dutta, Associate Professor, Department of Veterinary Microbiology, College of Veterinary Sciences & Animal Husbandry, Central Agricultural University, Selesih, Aizawl, Mizrom - 796014
7. Dr, Nagappa Karabasanavara, Assistant Professor, Department of Veterinary Public Health and Epidemiology, Veterinary College, Karnataka Veterinary, Animal and Fisheries Sciences University (VC-KVAFSU), Shivamogga, Karnataka - 577204.
8. Dr. R. V. Prasad, Dean, Veterinary College, Karnataka Veterinary, Animal and Fisheries Sciences University (VC-KVAFSU), Shivamogga, Karnataka - 577204.
9. Dr. M.R. Anand, MD & CEO, Media Lab Asia - New Delhi
10. Prof. Narendra Ahuja, Director, ITRA - New Delhi
11. Mr. K.P. Sivadas, Manager Accounts, Media Lab Asia - Mumbai
12. Project File



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Annexure-I to the Administrative Approval No.: ITRA/15(189)/Ag&Food/e-Varaha/01 **Date : 28.10.2016**

1.	Name of the Project	:	**e-Varaha: Information system for Safe Pork Production in North Eastern India**
2.	Objective	:	i) Study of the routes of pig- to- pork in the north eastern region ii) Development of IT based systems for examining health status of slaughter purpose pigs and post mortem examination of carcasses iii) Development of tools for identification and traceability of pork iv) Development of piggery information management system for traceability and pork safety
3.	Area of Coverage	:	IT based Transformations in Indian Agriculture and Food *(ITRA-Ag&Food)*
4.	Names of the Implementing Agencies and Status	:	(i) Indian Institute of Technology (IIT), Guwahati- Lead Institution- IT (ii) Tezpur University (TUA), Assam-PIN-IT; (iii) Manipur University (MUM), Imphal, Manipur-PIN-IT; (iv) ICAR-National Research Centre on Pig (NRCP), Guwahati – Lead Institution-AF; (v) ICAR-Research Complex for North Eastern Hilly Region (ICAR-RCNEHR-M), Barapani, Meghalaya- PIN-AF; (vi) College of Veterinary Sciences & Animal Husbandry, Central Agricultural University (CVS), Aizawl, Mizoram-PIN-AF; and (vii) Veterinary College, Karnataka Veterinary, Animal and Fisheries Sciences University (VC-KVAFSU), Karnataka – PIN-AF
5.	Locations	:	(i) IIT Guwahati: Guwahati, Assam (ii) TUA: Tezpur, Assam (iii) MUM: Imphal, Manipur (iv) ICAR-NRCP: Guwahati, Assam (v) ICAR-RCNEHR-M: Barapani, Meghalaya (vi) CVS: Aizawl, Mizoram (vii) VC-KVAFSU - Karnataka
6.	Principal Investigators	:	(i) Dr. Sukumar Nandi, Professor, Dept. of Computer Science & Engineering, IIT Guwahati (ii) Dr. Nityananda Sarma, Professor, Dept. of Computer Science & Engineering, TUA (iii) Dr. Khumukcham Robindro Singh, Assistant Professor, Dept. of Computer Science, MUM (iv) Dr. Girish Patil, S., Senior Scientist (Livestock Products Technology), ICAR-NRCP (v) Dr. Arnab Sen, Principal Scientist & Head, Division of Animal Health, ICAR-RCNEHR-M (vi) Dr. Tapan Kumar Dutta, Associate Professor, Dept. of Veterinary Microbiology, CVS (vii) Dr. Nagappa, Karabasanavar, Assistant Professor, Dept. of Veterinary Public Health and Epidemiology, VC-KVAFSU - Karnataka
7.	Total project duration	:	24 Months
	i) Expected date of commencement	:	October 28, 2016
	ii) Expected date of completion	:	October 27, 2018

MEDIA LAB ASIA - IIT

8. Total project outlay :

a. Year wise and item-wise break up of expenditure (Rs. In Lakhs)

Budget Head	1st Year	2nd Year	Total
1. Capital Equipment and Works	57.50	1.00	58.50
2. Consumable stores	13.33	12.01	25.34
3. Manpower*	53.90	51.75	105.65
4. Training & Development	0.00	1.50	1.50
5. Workshop / Conference / Seminar	0.00	5.00	5.00
6. Technology and Entrepreneurship Development	0.00	0.00	0.00
7. Travel	12.00	16.00	28.00
8. Miscellaneous/Contingencies	0.47	0.47	0.94
9. Overheads	7.97	7.97	15.94
Total	**145.17**	**95.70**	**240.87**

b. Institution wise Budget outlay – Summary (Rs. in Lakhs)

Institute Name	1st year	2nd year	Total
IIT Guwahati	40.50	27.00	67.50
TUA, Assam	15.05	11.05	26.10
MUM, Imphal	14.75	10.50	25.25
ICAR-NRCP, Guwahati	48.67	26.87	75.54
ICAR-RCNEHR-M, Meghalaya	7.76	6.76	14.52
CVS, Aizawl	8.15	6.98	15.13
VC-KVAFSU, Karnataka	10.29	6.54	16.83
Total	**145.17**	**95.70**	**240.87**

c. Institute wise Budget (Rs. in Lakhs):

c.1 **IIT Guwahati: LIN-IT** (Rs. in Lakhs)

Budget Head		1st year	2nd year	Total
Capital Equipment and works		14.00	1.00	15.00
Consumable Stores	LFC	3.00	3.00	6.00
	S&C	0.50	0.50	1.00
Manpower	RP	15.50	13.00	28.50
	OPSL	0.00	0.00	0.00
	PIP	1.50	1.00	2.50
Training & Development	RMC	0.00	0.00	0.00
Workshop/Conference /Seminar	SSD	0.00	1.00	1.00
	OA	0.00	0.00	0.00
	SPPS	0.00	1.00	1.00
Technology and Entrepreneurship Development	TED	0.00	0.00	0.00
Travel	Domestic	2.00	2.00	4.00
	International	1.50	2.00	3.50
Miscellaneous & Contingency	Misc	0.20	0.20	0.40
Overhead	Overhead	2.30	2.30	4.60
GRAND TOTAL		**40.50**	**27.00**	**67.50**

MEDIA LAB ASIA - INDIA

c.2 **TUA, Assam: PIN-IT** (Rs. in Lakhs)

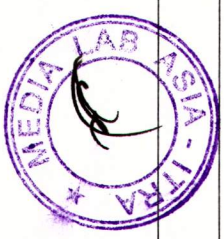
Budget Head		1st year	2nd year	Total
Capital Equipment and works		5.50	0.00	5.50
Consumable Stores	LFC	0.25	0.50	0.75
	S&C	0.25	0.50	0.75
Manpower	RP	6.60	6.60	13.20
	OPSL	0.00	0.00	0.00
	PIP	0.00	0.00	0.00
Training & Development	RMC	0.00	0.00	0.00
Workshop/Conference /Seminar	SSD	0.00	0.50	0.50
	OA	0.00	0.00	0.00
	SPPS	0.00	0.50	0.50
Technology and Entrepreneurship Development	TED	0.00	0.00	0.00
Travel	Domestic	1.50	1.50	3.00
	International	0.00	0.00	0.00
Miscellaneous & Contingency	Misc	0.00	0.00	0.00
Overhead	Overhead	0.95	0.95	1.90
GRAND TOTAL		**15.05**	**11.05**	**26.10**

c.3 **MUM, Imphal: PIN-IT** (Rs. in Lakhs)

Budget Head		1st year	2nd year	Total
Capital Equipment and works		5.50	0.00	5.50
Consumable Stores	LFC	0.50	0.25	0.75
	S&C	0.25	0.25	0.50
Manpower	RP	6.60	6.60	13.20
	OPSL	0.00	0.00	0.00
	PIP	0.00	0.00	0.00
Training & Development	RMC	0.00	0.00	0.00
Workshop/Conference /Seminar	SSD	0.00	0.50	0.50
	OA	0.00	0.00	0.00
	SPPS	0.00	0.50	0.50
Technology and Entrepreneurship Development	TED	0.00	0.00	0.00
Travel	Domestic	1.00	1.50	2.50
	International	0.00	0.00	0.00
Miscellaneous & Contingency	Misc	0.00	0.00	0.00
Overhead	Overhead	0.90	0.90	1.80
GRAND TOTAL		**14.75**	**10.50**	**25.25**

c.4 **ICAR-NRCP, Guwahati: LIN-AF** (Rs. in Lakhs)

Budget Head		1st year	2nd year	Total
Capital Equipment and works		27.00	0.00	27.00
Consumable Stores	LFC	1.50	1.25	2.75
	S&C	0.15	0.25	0.40
Manpower	RP	7.20	8.06	15.26
	OPSL	3.96	3.96	7.92
	PIP	3.84	3.83	7.67
Training & Development	RMC	0.00	0.50	0.50
Workshop/Conference /Seminar	SSD	0.00	1.00	1.00
	OA	0.00	0.00	0.00
	SPPS	0.00	0.00	0.00



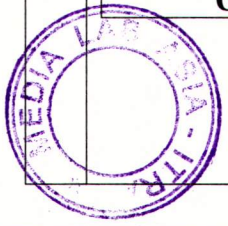
Technology and Entrepreneurship Development	TED	0.00	0.00	0.00
Travel	Domestic	2.50	2.50	5.00
	International	0.00	3.00	3.00
Miscellaneous & Contingency	Misc	0.20	0.20	0.40
Overhead	Overhead	2.32	2.32	4.64
GRAND TOTAL		**48.67**	**26.87**	**75.54**

c.5. **ICAR-RCNEHR-M, Meghalaya: PIN-AF** (Rs. in Lakhs)

Budget Head		**1st year**	**2nd year**	**Total**
Capital Equipment and works		1.00	0.00	1.00
Consumable Stores	LFC	1.50	1.00	2.50
	S&C	0.00	0.00	0.00
Manpower	RP	3.60	3.60	7.20
	OPSL	0.00	0.00	0.00
	PIP	0.00	0.00	0.00
Training & Development	RMC	0.00	0.50	0.50
Workshop/Conference /Seminar	SSD	0.00	0.00	0.00
	OA	0.00	0.00	0.00
	SPPS	0.00	0.00	0.00
Technology and Entrepreneurship Development	TED	0.00	0.00	0.00
Travel	Domestic	1.00	1.00	2.00
	International	0.00	0.00	0.00
Miscellaneous & Contingency	Misc	0.04	0.04	0.08
Overhead	Overhead	0.62	0.62	1.24
GRAND TOTAL		**7.76**	**6.76**	**14.52**

c.6. **CVS, Aizawl: PIN-AF** (Rs. in Lakhs)

Budget Head		**1st year**	**2nd year**	**Total**
Capital Equipment and works		1.00	0.00	1.00
Consumable Stores	LFC	4.33	3.66	7.99
	S&C	0.10	0.10	0.20
Manpower	RP	0.00	0.00	0.00
	OPSL	0.60	0.60	1.20
	PIP	0.00	0.00	0.00
Training & Development	RMC	0.00	0.50	0.50
Workshop/Conference /Seminar	SSD	0.00	0.00	0.00
	OA	0.00	0.00	0.00
	SPPS	0.00	0.00	0.00
Technology and Entrepreneurship Development	TED	0.00	0.00	0.00
Travel	Domestic	1.50	1.50	3.00
	International	0.00	0.00	0.00
Miscellaneous & Contingency	Misc	0.03	0.03	0.06
Overhead	Overhead	0.59	0.59	1.18
GRAND TOTAL		**8.15**	**6.98**	**15.13**



c.7. **VC-KVAFSU, Karnataka: PIN-AF** (Rs. in Lakhs)

Budget Head		1st year	2nd year	Total
Capital Equipment and works		3.50	0.00	3.50
Consumable Stores	LFC	0.85	0.65	1.50
	S&C	0.15	0.10	0.25
Manpower	RP	3.60	3.60	7.20
	OPSL	0.90	0.90	1.80
	PIP	0.00	0.00	0.00
Training & Development	RMC	0.00	0.00	0.00
Workshop/Conference /Seminar	SSD	0.00	0.00	0.00
	OA	0.00	0.00	0.00
	SPPS	0.00	0.00	0.00
Technology and Entrepreneurship Development	TED	0.00	0.00	0.00
Travel	Domestic	1.00	1.00	2.00
	International	0.00	0.00	0.00
Miscellaneous & Contingency	Misc	0.00	0.00	0.00
Overhead	Overhead	0.29	0.29	0.58
GRAND TOTAL		**10.29**	**6.54**	**16.83**

9)	Mode and extent of funding		Amount (Rupees in Lakhs)
i)	Financial support from ITRA-MLAsia	:	
	(a) Grant		240.87
	(b) Loan		NIL
ii)	Internal generation	:	NIL
iii)	Other funding Agencies	:	NIL
	(a) Internal		
	(b) External		

10) Stages for release of Grants

Release No.	Stages of implementation	Documentation to be submitted by implementation Agency	Amount to be released (Rupees in Lakhs)
1	First Installment	• Acceptance of the terms and conditions governing Grants and surety bond, as applicable.	145.17
2	Second Installment on recommendation of the Project Review and Steering Group (PRSG)	• Technical progress report • Statement of Account • Utilization Certificate for the previous Grants • Request letter for next release	95.70
	Total		240.87

MEDIA LAB ASIA - ITRA

11) Notes:

General to all teams:

(i) The budget mentioned for the Year-2 in this Administrative Approval Order is only indicative. The actual level of funding shall be decided based on the 1st Year performance and achievement of the assignments by the institutions.

(ii)

i. The revised budget includes provision for international travel of ITRA Researchers (faculty & students).

ii. The earmarked funds meant for the team are being released to Lead Institution. The proposed travel abroad using ITRA funds should be routed through the PI of the Lead Institution and the laid down procedures on the subject shall be followed.

iii. Any international travel, out of the funds earmarked by ITRA, for the project, would require explicit written approval of ITRA, on case to case basis, before undertaking any such travel.

iv. The international travel shall be as per the relevant rules of the grantee institution once the approval is granted by ITRA.

(iii) The international travel of Advisor / Mentors from abroad is being separately managed by ITRA as per GOI norms.

(iv) The Project Review and Steering Group, may from time to time, recommend changes to enhance the performance of the project. These may include changing the scope of work, adding new institutions, etc., and these shall be carried out by the teams with the approval of ITRA.

(v) It is assumed that the project work, through the entire project period, will be carried out by the Project Investigators listed in the project documents with their current institutional affiliations. Changes will generally be not entertained

(vi) Fellowship amounts for PhD students shall be in line with orders issued to this effect by Media Lab Asia – ITRA.

12) Legend:

LFC	Lab & Field consumables
S&C	Stationary & Communication
RP	Research Personnel
OPSL	Other Project Staff/Labour
PIP	Periodically Involved Personnel(Mentors & Experts)
RMC	Research Motivated Carricula
SSD	Societal Sensitivity Development
OA	Outreach Activites
SPPS	Special Programmes to attract PhD Students from UG and PG Students
TED	Technology and Entrepreneurship Development
Dom.	Domestic Travel
Intl.	International Travel
Misc	Miscellaneous & Contingency

Media Lab Asia
Govt. of India
Ministry of Communications & Information Technology
Dept. of Electronics & Information technology

George Arakal
Director (Admin. & Finance)

Report on

Global Lecture Series 2020-Under Faculty Exchange Programme as the part of MoU between Tezpur University and MSU, Malaysia

Organized by

Department of Education
School of Humanities and Social Sciences Tezpur University
Date: 9th to 11th November, 2020

A REPORT

A Global Lecture Series was organized in virtual mode by the Department of Education, Tezpur University during 9th to 11th November. It was the part of prescribed and mutually agreed activities under the MoU between Department of Education, Tezpur University (http://www.tezu.ernet.in/dedu/) and School of Education, Management and Science University, Malaysia (https://msu.edu.my/school-of-education-and-social-sciences).

The Programme was organized in fully virtual/Online mode by using the Google Meet Platform due to the world wide pandemic situation of Covid-19 in which International movement from one country to another is strictly controlled and permissible only under the prescribed guidelines of World Health Organization for the sack of health reasons.

In spite of such challenging and tough circumstances academic activities are going on throughout the globe. The Department of Education is also on the pattern of 'the show must go on' did not leave any stone unturned in fulfilling its responsibilities as a continuous chain of academic activities for the cause of education and development.

First Day: 9th November, 2020

The Google Meet link was open for the students and faculty members at 1:30 P.M. IST. By 1:40 as much as 109 students joined the Meet and the Programme started with the Welcome Address of Prof. Nil Ratan Roy, Head Department of Education, Tezpur University. On the other part Dr. Shabina Rahman, Head, School of Education, MSU, Malaysia joined and exchanged the talk. Prof. Roy in his welcome address appreciated the humbleness and extended cooperation of the authorities of both the Universities to allow for such international collaboration for the mutual exchange of ideas, values and knowledge. He also shared his efforts and thanks for the cooperation of Dr. Shabina Rahman as Head and Incharge of International Collaboration, in actualization of this academic dream. Prof. Roy explained the idea and objectives of this Memorandum of Understanding between the two renowned Universities and also put the plans for future activities. He thanked to all the faculty members and students for the active cooperation and particularly Prof. V.K. Jain, Hon'ble Vice Chancellor, Tezpur University for his kind cooperation.

After the welcome note of the head of the Department, Dr. M. S .Siddiqui introduced the first speaker Dr. Shabina Rahman. She delivered her lecture on the topic: Turning on the 'turned-out child'-Motivation in Teaching & Learning.' Major points of her Lecture are summarized as below-

1. To Children there are three major sources of Motivation-Parents, Peers and Teachers.
2. Three environments are also important to influence the motivation on the part of child-Home environment, Influence of School and The teacher himself/herself.
3. Motivational principles get their fuel from the Theories like Maslow's Hierarchy of Needs, contemporary theory of Eccles, Wigfield and Tonk.
4. In different countries the Motivation is also influenced by the cultural and social factors, financial situations and local traditions.
5. Dealings with Education in different nations also influence the motivation among children at schools or home.
6. She especially emphasized the social and family situations in different societies that influence Motivation with an example that 'Divorce rate is very high in Malaysia' she linked its direct effect on the motivation of children in the family who face the consequences of such situations.
7. She elaborated various skills for teachers to boost up the morale of students and suggested various motivational tricks and strategies.
8. She especially gives emphasis on teaching with honesty, dedication and full involvement. It should come from heart.
9. The role of expectations of success is also important in motivation.
10. The core point of her lecture was- Teaching comes from within-from heart.

The students and faculty members interacted and put their queries and questions at the end of the Lecture. The guest speakers addressed them all enthusiastically and satisfactorily.
The Session ended with the vote of thanks by Moderator at 3:30 P.M. IST.

Second Day: 10th November, 2020

The students joined at Meet link by 1:40 P.M. IST. Dr. Shabina Rahman also joined to introduce her faculty member to the audience and listeners. Dr. M. S. Siddiqui formally introduced the guest speaker Dr. Nirmala Devi Rajoo, Senior Lecturer at the School of Education MSU. She delivered her lecture on the topic: Education Assessment-Addressing Skills for the 21st Century'

The Lecture was highly appreciated by the listeners throughout. She elaborated various skills and practices required in 21st Century Classrooms and as part of teaching and learning. She elaborated three important skills for 21st century learners- Thinking Skills, Action Skills and Living Skills. Each of the skills has group of certain other sub-skills that need to be developed. She also explained various techniques of assessing and evaluating the three skills in order to know the development of the learner. The core point of her lecture was- Students communicate and collaborate every day.

The students put their questions at the end of the lecture. The Moderator put the questions one by one for the response of the guest speaker. She addressed them all in well manner to the satisfaction of the questioner.

The session ended at 3:30 PM IST with the vote of thanks to the audience and speaker by the Moderator. Dr. Shabina Rahman at the end also thanks and showed her humble gratitude for successful organization of this Session.

Third Day: 11th November, 2020

The students joined the Meet by 1:30 PM. Dr. Shabina Rahman also joined and introduced her faculty member who was the guest speaker for the day. Dr. M. S. Siddiqui, as moderator, formally introduced the speaker, Dr. Frankie Anak Subon, who teaches Research Methods, sociology of Education and English Language at MSU. He delivered his lecture on the topic: Working with others in an Educational Setting."

His lecture pointed out a numbers of techniques and Skills for the successful collaboration and human relations with others in relation to mutual exchange of ideas, values and knowledge with others at work place in an Educational Organization. He also laid emphasis on the development of various skills like communication and interaction for good human relations as well as virtues like ethical and moral uplift, honesty, humbleness etc. The core point of his lecture says- 'Success required working with people more than equipment, processes and products.'

The lecture was warmly welcome and students as well faculty members interacted with the guest speaker during the question-answer Session.

The three day Global Lecture Series has been moderated by Dr. M.S. Siddiqui, Associate Professor, Department of Education and finally extended vote of thanks to the heads of the Department of both the universities, students and faculty members for the making the event a great success by their active participation and interaction. The Session ended at 3:30 PM IST.

Recorded Lecture Series videos can be accessed from below links:

Day1:https://drive.google.com/file/d/1oQah5KQ0hnsGWL6pbJ1MGpuo_WwKirGN/view?usp=sharing

Day2:https://drive.google.com/file/d/1somENoKo9Tl0eKNdp8d8u9P4VZm6zcfa/view?usp=sharing

Day3:https://drive.google.com/file/d/1YjCgPYtnKlyWqrPPCOYLhHdAalnSLWxV/view?usp=sharing

Sd/-
(Prof. Nil Ratan Roy)
Head, Department of Education
Tezpur University

From: "Zeonics Systech" <zeonicssys@gmail.com>
Subject: Re: Proposal for Industry-Academia Collaboration
Date: Wed, February 26, 2020 10:02 am
To: "Dr M R Mahboob" <rahat@tezu.ernet.in>
Cc: manab@tezu.ernet.in,tufacee@tezu.ernet.in

Dear Dr. Mahboob,
WE are open to this & you can send your students maximum 4 Nos. at a time.

Thanking you,
Warm regards

Dr. Z. H. Sholapurwala
Managing Director

Zeonics Systech Defence & Aerospace Engineers Pvt. Ltd.
"ADAR" #3, 10th Main, 7th Cross, Maruthinagar,
New Thippasandra P.O., Bangalore - 560075.
Ph: +91 080 25241900, 25241901, 25240523
+91 080-50092601, 080-50092602, 080-50092603
Email: zeonicssys@gmail.com ; contact@zeonicssystech.com
Web Sites : www.systechcapacitors.com ; www.zeonicssystech.com
Our Google page: https://g.co/kgs/eDeTs7
Our Youtube page: https://m.youtube.com/channel/UCCNmmGnbPpWe-blXo1alumQ

On Tue, Feb 25, 2020 at 4:00 PM Dr M R Mahboob <rahat@tezu.ernet.in> wrote:

> Dear Sir,
>
> As per our telephonic conversation, i am hereby attaching a formal
> proposal for the collaboration.
>
> We look forward to a successful collaboration.
> Anticipating your kind consideration and reply.
>
> Note: A hard copy of the proposal is attached herewith
>
> regards
>
> Dr. M. R. Mahboob
> Assistant Professor,
> Electrical Engineering Department,
> School of Engineering,
> Tezpur University,
> Sonitpur, Assam-784028,
> India
> Phone No. 03712-275295
>
>
> * * * D I S C L A I M E R * * *
> This e-mail may contain privileged information and is intended solely for
> the individual named. If you are not the named addressee you should not
> disseminate, distribute or copy this e-mail. Please notify the sender
> immediately by e-mail if you have received this e-mail in error and destroy
> it from your system. Though considerable effort has been made to deliver

> error free e-mail messages but it can not be guaranteed to be secure or
> error-free as information could be intercepted, corrupted, lost, destroyed,
> delayed, or may contain viruses. The recipient must verify the integrity of
> this e-mail message.

Attachments:

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Size:	4.8 k
Type:	text/html

TEZPUR UNIVERSITY

(A Central University)

Visitor's Best University Award, 2016 and NIRF India Ranking 2016: No.05

DEPARTMENT OF ELECTRICAL ENGINEERING
Napaam, Tezpur, Assam - 784 028

Dr. M. Bhuyan
Professor and Head

Phone: (03712)267008(5252)
E-mail: hod_ee@tezu.ernet.in

To
Managing Director
Zeonics Systech Defence & Aerospace Engineers Pvt. Ltd.
"ADAR" #3, 10th Main, 7th Cross, Maruthinagar,
New Thippasandra P.O.,
Bangalore - 560075.

Dated: 25/02/2020

Subject: Proposal for Industry-Academia Collaboration

Respected Sir,

In light of your visit to Tezpur University and subsequent corresponding discussion with Dr. M. R. Mahboob, the Department of Electrical Engineering, Tezpur University would like to request your organization to be a partner with the EE dept., for training our B.TECH students in problems pertaining to Industry which are current and emerging. The department discussed this possible venture and it was decided that due to the academic engagement of the students, we propose the following schedule for our students to undergo training at your organization:

6^{th} semester students: at summer vacation, 2020: 45 days
4^{th} semester students: at summer vacation, 2021: 45 days
6^{th} semester students: at summer vacation, 2021: 45 days

We look forward to a successful collaboration. Anticipating your kind consideration.

With Regards,

(Prof. M. Bhuyan)
Head
Deptt. Of Electrical Engineering,
Tezpur University, Tezpur, Assam

भारतीय उष्णदेशीय मौसम विज्ञान संस्थान
(पृथ्वी विज्ञान मंत्रालय, भारत सरकार का एक स्वायत्त संस्थान)
डॉ. होमी भाभा मार्ग पाषाण, पुणे- ४११ ००८
IITM
INDIAN INSTITUTE OF TROPICAL METEOROLOGY
(An Autonomous Institute of the Ministry of Earth Sciences, Govt. of India)
Dr. Homi Bhabha Road, Pashan, Pune - 411 008. India
ESSO

प्रो. रवि एस. नन्जुनडैया
निदेशक
Prof. Ravi S. Nanjundiah
Director

150 YEARS OF CELEBRATING THE MAHATMA

Ref: IITM/Metflux/NG/2019-20

Date: 16 September 2019

The Registrar
Tezpur University
Tezpur-784082, Assam

Sub: Sanction of project on "Soil-Plant-Atmosphere Study in Relation to Net CO_2 Flux from Terrestrial Ecosystem of Assam"

Sir,

The Advisory Committee for the MetFlux India Project made an assessment of the project in a meeting held at the Mausam Bhavan, IMD, New Delhi on 9th July 2019. The recommendations are given below.

The data presented by the PI showed relatively high values of positive NEE during the night time, which may have been triggered by high soil respiration. Hence the PI should arrange to install a Soil CO_2 sensor to quantify the heterotrophic respiration. An EC sensor within the canopy should be used to characterise the under story carbon fluxes. Other bio-physical parameters such as, plant biomass, leaf are index, fresh litter fall etc. should also be measured on a regular basis. The Committee advised to carry out rigorous estimation of the NEE and its portioning into the GPP and ecosystem respiration. The relation between NEE and various biophysical parameters should be examined. Since a large area of the forest turns to wetland on a seasonal basis, the Methane fluxes may also be measured. The Committee, in principle, recommends allocating funds to meet the operational costs for the next three years, viz, 2019-2020, 2020-2021, and 2021-2022.

Based on the recommendation of the Committee, the Director, IITM, Pune hereby approves the continuation of the above mentioned project for a period of three years (2019-2022). An amount of Rs. 17,75,400/- is being sanctioned for the current financial year (Apr 2019 to Mar 2020).

....2/-

दूरभाष / Telephone: (020) 25865143 / 25904200
ई-मेल / e-mail : ravisn@tropmet.res.in
फैक्स / Fax : (020) 25865142
वेब / Web : www.tropmet.res.in

The year wise break up details of funds under the above mentioned project is furnished below.

S/N	Description	1st Year	2nd Year	3rd Year
01	Manpower; Project Assistant – 01 (salary may be revised as per the MoES notification)	2,64,000	2,64,000	2,64,000
02	Contingency/consumable including a portable cabin to house the sensors, battery etc.	8,00,000	4,00,000	4,00,000
03	Travel	2,00,000	2,00,000	2,00,000
04	Other Cost (Battery replacement/ Additional panels for new systems/ Fencing maintenance/Repainting of tower)	3,50,000	2,50,000	2,50,000
05	Sub-Total	16,14,000	11,14,000	11,14,000
06	Overhead (10 % of (01+02+03+04)	1,61,400	1,11,400	1,11,400
09	TOTAL (05+06)	17,75,400	12,25,400	12,25,400

The basic terms and conditions are appended in the annexure.

Kindly acknowledge the receipt of this letter.

Sincerely yours,

K. S. Nanjundiah

(Prof. Ravi Nanjundiah)

Annexure:
Terms and Conditions:

1. Duration of the project will be until 31 Mar 2022. Further extension of the project is based on the satisfactory progress and at the discretion of the Director, IITM.

2. Salary/Wages to be decided as per the MoEs Rules.

3. Allocation of budget is tentative and subject to change after reviewing progress, both physical and financial as recommended by the Advisory Committee.

4. A list of equipment procured and other assets created for this project shall be provided to IITM. All equipment/assets procured shall be maintained in the stock register of IITM.

5. All the instruments provided by IITM and the data generated will always remain the property of IITM and will be returnable after the expiry of the agreement.

6. While implementing the program all relevant procedures will be followed and IITM shall be apprised of the progress of the project from time to time.

7. Utilization certificate and statement of expenditure for the expenditure incurred on the project shall be submitted at the end of each financial year and also at the end of the project in the prescribed Pro forma.

8. The detailed project report has to be submitted in triplicate within one month after completion of the project in addition to monthly progress report.

9. The PI/Co-PI should be responsible in getting the approval from the concerned forest department to carry out the observational work at the current tower site until the tenure of the project. A copy of the approval letter should be submitted to the Director, IITM.

Program Implementation:
The project under consideration will be carried out by Dr. (Mrs.) Nirmali Gogoi and her team involved in the eco-system flux study under the Metflux India Project. IITM will provide full monetary support while the Tezpur University will be responsible in arranging administrative and logistic requirements. Progress will be monitored by a team constituted by the Director, IITM.



Dr. S. Chakraborty
Dy. Project Director, Metflux Project

CC:
1. Dr. (Mrs.) nirmali Gogoi
Department of Environmental Science
Tezpur University, Tezpur-784082
2. Account Section, IITM, Pune
3. Office Copy

Admin

404

App. No.: BMB/2017/87

No. BT/PR25738/NER/95/1329/2017
GOVERNMENT OF INDIA
MINISTRY OF SCIENCE & TECHNOLOGY
DEPARTMENT OF BIOTECHNOLOGY
NER-BPMC

Block 2, 6-8th Floors
CGO Complex, Lodhi Road,
New Delhi- 110 003
Dated: 24/12/2018

ORDER

Sanction of the President is hereby accorded, under Rule 18 of the Delegation of Financial Powers Rules ,1978 , for the implementation of the project entitled: "**Earthworm Gut Microbes Mediated Carcinogenic Polycyclic Aromatic Hydrocarbons (CPAHs) Remediation in Contaminated Soil**" for a period of 3 Year 0 Month at a total cost of Rs. **5639200** (Rupees Fifty Six Lakhs Thirty Nine Thousand Two Hundred Only) on the terms and conditions detailed here under:-

2 The Project :

2.1 Title : **"Earthworm Gut Microbes Mediated Carcinogenic Polycyclic Aromatic Hydrocarbons (CPAHs) Remediation in Contaminated Soil"**

2.2 Details of the Investigatiors:

Project Cordinator

Prof. Raza Rafiqul Hoque
Professor
Environmental Science
Tezpur University
Department of Environmental Science
Tezpur University, Tezpur, Assam, 784028

Principal Investigators:

Vandana

Generated through eProMIS

Prof. Raza Rafiqul Hoque
Professor
Environmental Science
Tezpur University
Department of Environmental Science
Tezpur University, Tezpur,Assam, 784028

Dr. Srinivasan Balachandran
Assistant Professor
Environmental Studies
Visva Bharati University
Dr. Srinivasan Balachandran,
Department of Environmental Studies,
Siksha Bhavana
Visva Bharati
Santiniketan-731235
West Bengal, ,West Bengal, 731235

Prof. Raza Rafiqul Hoque
Professor
Environmental Science
Tezpur University
Department of Environmental Science
Tezpur University, Tezpur,Assam, 784028

CO-PI:

Dr. Nayanmoni Gogoi
Assistant Professor
Environmental Science
Tezpur University
Department of Environmental Science,
Tezpur University, Napaam,
Tezpur – 784028, Assam, Tezpur - 784028, Assam

Prof. Narayan Chandra Mandal
Professor
Botany
Visva Bharati University
Department of Botany,
Siksha Bhavana, Visva-Bharati, Santiniketan
Dist.:Birbhum West Bengal, Santiniketan - 731235, West Bengal

Vaidehi

2.3 Objectives:

Overall Objectives:

1. To evaluate concentration and variation of PAHs in selected contaminated soils in different seasons.
2. To identify the possible sources and health risk (carcinogenic) of PAHs in selected soil samples.
3. To isolate, characterize and identify carcinogenic PAHs degrading earthworm gut microbial diversity.
4. To evaluate the CPAHs degrading enzymatic capability of selected isolates.
5. To optimize the physical and nutritional conditions for utmost CPAHs degrading enzyme production by the gut microbes.
6. To assess selected CPAHs contaminated soil remediation with earthworm as well as its gut bacteria in in vitro condition.
7. To quantify the PAHs accumulation in the tested agronomic plant systems [leaf, stem, seed and root] before and after microbial remediation process at the selected field.

Institute wise Objectives:

Tezpur University

1. To evaluate concentration and variation of PAHs in selected contaminated soils in different seasons.
2. To identify the possible sources and health risk (carcinogenic) of PAHs in selected soil samples.
3. To isolate, characterize and identify carcinogenic PAHs degrading earthworm gut microbial diversity.
4. To evaluate CPAHs degrading enzymatic capability of selected isolates and carcinogenicity assay of the enzymatic products.
5. To optimize the physical and nutritional conditions for utmost CPAHs degrading enzyme production by the gut microbes.
6. To assess selected CPAHs contaminated soil remediation with earthworm as well as its gut bacteria in in vitro condition.
7. To quantify the PAHs accumulation in the tested agronomic plant systems [leaf, stem, seed and root] before and after microbial remediation process in pot culture.

Visva Bharati University

1. To evaluate concentration and variation of PAHs in selected contaminated soils in different seasons.
2. To isolate, characterize and identify carcinogenic PAHs degrading earthworm gut microbial diversity.
3. To evaluate CPAHs degrading enzymatic capability of selected isolates and carcinogenicity assay of the enzymatic products.
4. To optimize the physical and nutritional conditions for utmost CPAHs degrading enzyme production by the gut microbes.
5. To assess selected CPAHs contaminated soil remediation with earthworm as well as its gut bacteria in in vitro condition.
6. To quantify the PAHs accumulation in the tested agronomic plant systems [leaf, stem, seed and root] before and after microbial remediation process in pot culture.

2.4 Time Schedule:

The duration of the project is 3 Year 0 Month from the date of this sanction order.

2.5 Project Cost:

The total cost of the project is Rs. **5639200**/-(Rupees Fifty Six Lakhs Thirty Nine Thousand Two Hundred Only) as per details given below :

Institute	Year I	Year II	Year III	Total Cost(Rs.)
1. Tezpur University	1135000	1105000	1144600	**3384600**
2. Visva Bharati University	805000	705000	744600	**2254600**
Total (Rs.)	**1940000**	**1810000**	**1889200**	**5639200**

Institute wise details are:

Budget Head	Year I	Year II	Year III	Total(Rs.)
1. Tezpur University				
Manpower	330000.00	330000.00	369600.00	**1029600.00**
Contingency	50000.00	50000.00	50000.00	**150000.00**
Travel	50000.00	50000.00	50000.00	**150000.00**
Consumables	575000.00	575000.00	575000.00	**1725000.00**
Overhead	130000.00	100000.00	100000.00	**330000.00**
Total (Rs.)	**1135000.00**	**1105000.00**	**1144600.00**	**3384600.00**
2. Visva Bharati University				
Manpower	330000.00	330000.00	369600.00	**1029600.00**
Contingency	20000.00	20000.00	20000.00	**60000.00**
Overhead	30000.00	30000.00	30000.00	**90000.00**
Travel	25000.00	25000.00	25000.00	**75000.00**
Consumables	400000.00	300000.00	300000.00	**1000000.00**
Total (Rs.)	**805000.00**	**705000.00**	**744600.00**	**2254600.00**

2.6 Equipment:

The details of the equipment sanctioned for the implementation of the project at **Annexure-I**

2.7 Manpower:

The details of the manpower sanctioned for the implementation of the project at **Annexure-II**

Vaidehi

3. **Head of Account:**

The **Recurring** expenditure involved is debitable to:

Demand No. 85	**Department of Biotechnology**
3425	Other Scientific Research 2018-2019
3425.60	Others (Sub Major Head)
3425.60.200	Assistance to other Scientific Bodies (Minor Head)
3425.60.200.29	Biotechnology Research and Development
3425.60.200.29.17	Assistance for Research and Development
3425.60.200.29.17.31	Grants-in-Aid General

4. Terms & Conditions:

Additional Terms and Conditions, specific for Twinning R and D program for NER
a. Both NER and Rest of India (RoI), Institutions scientists should work together for the objectives stated in the sanction of the project and any deviation from this would attract closure of the project at any point of time.

b. In the project review meetings, both the PIs from NER and RoI Institutions should participate and make presentation.

c. The outcomes of the project such as research papers, patents, copy rights etc. should be made jointly.

d. The NER Scientists are to be trained at the collaborating institute appropriately to empower the NER Scientists.

e. The project personal such as Research Associate, JRF or SRF, Research Assistant are also to be trained at least once in the collaborating national institute.

f. The collaborating institute scientist should visit NER institutions more frequently to guide NER scientists in design and conduct of experiments.

4.1 The other terms and conditions governing this sanction are attached at Annexure- III.

4.2 A Memorandum of Agreement (MoA) will be signed between the Department of Biotechnology and the grantee institution on Non-Judicial stamp paper Rs. 100/- in the enclosed format and the second release/installment will be made only after signing of MoA by the grantee institutions and its acceptance by DBT. In case of NGO or Private Institution, MOA signed is mandatory first release. A format of the MoA is enclosed in Annexure-IV

4.3 The Institute/Agency will keep the whole of the grant in a Bank Account earning interest, and the interest so earned should be reported to DBT in the Utilisation Certificate and Statement of Expenditure. The Interest so earned will be treated as created to the institute/Agency and shall be adjusted towards further installment of the grant and or at the time of Final Settlement of Accounts.

5. No International Travel will be undertaken from the sanctioned project grant unless specified otherwise.

6. The Registrar, Tezpur University, Tezpur, Assam and The Registrar, Visva Bharati University, Birbhum, West Bengal would be responsible for submission of Statements of Expenditure (SoE), utilization certificates (UC), Assets Certificates, Manpower staffing & expenditure details in prescribed DBT formats to DBT in respect of grants released in this project from time to time.

7. PI's of DBT sponsored projects can consider appointment of JRF from Category-II merit list of DBT-BET exam so that candidates can be paid fellowships at par with NET/GATE/BET qualified candidates as per DST OM No. A.SR/S9/Z-09/2012 dated on 21 Oct 2014. However, there is no compulsion on PI's to select candidates for JRF in their projects from Category-II of DBT-BET.

8. As per Rule 236 (1) of GFR 2017, the accounts of all Grantee Institutions or Organisations shall be open to inspection by the sanctioning authority and audit, both by the Comptroller and Auditor General of India under the provision of CAG(DPC) Act 1971 and internal audit by the Principal Accounts Office of the Ministry or Department, whenever the Institution or Organisation is called upon to do so.

9. If the Research Project involves biological resource, the obligations under the Biological Diversity Act 2002 as applicable shall be complied with by the Project Investigator, the details of such obligations can be accessed at www.nbaindia.org

10. This issues under the power delegated to this Department and with the concurrence of IFD vide their SAN No.**102/IFD/SAN/2573/2018-2019** dated **September, 24 2018.**

11. This sanction order has been noted at serial no. 404 in the Register of Grants.

Vaishali Panjabi

(Dr. Vaishali Panjabi)
Scientist 'E'

To,
The Pay & Accounts Officer,
Department of Biotechnology,
New Delhi – 110 003.

Copy to:

1 The Principal Director of Audit (Scientific Departments), DACR Building, New Delhi- 110 002.
2 Prof. Raza Rafiqul Hoque(Project Co-ordinator), Department of Environmental Science Tezpur University, Tezpur, Assam, Pin- 784028
3 The Registrar, Tezpur University, Tezpur University, Napaam, Tezpur, Sonitpur, Tezpur - 784028, Assam
4 The Registrar, Visva Bharati University, Santiniketan, Birbhum, Birbhum - 731235, West Bengal
5 Dr. Nayanmoni Gogoi, Assistant Professor, Environmental Science, Tezpur University, Department of Environmental Science, Tezpur University, Napaam, Tezpur – 784028, Assam, Tezpur - 784028, Assam
6 Dr. Srinivasan Balachandran, Assistant Professor, Environmental Studies, Visva Bharati University, Department of Environmental Studies, Siksha Bhavana, Visva Bharati Santiniketan-731235, West Bengal
7 Prof. Narayan Chandra Mandal, Professor, Botany, Visva Bharati University, Department

of Botany, Siksha Bhavana, Visva-Bharati, Santiniketan, Dist.:Birbhum West Bengal, Santiniketan - 731235, West Bengal

8 Prof. Raza Rafiqul Hoque, Professor, Environmental Science, Tezpur University, Department of Environmental Science, Tezpur University, Tezpur - 784028, Assam

9 Cash Section, DBT (2 copies).

10 Sanction Folder.

11 File Copy.

Vaishali Panjabi

(Dr. Vaishali Panjabi)
Scientist 'E'

Annexure -II

Details of the manpower sanctioned for the implemention of the project titled **"Earthworm Gut Microbes Mediated Carcinogenic Polycyclic Aromatic Hydrocarbons (CPAHs) Remediation in Contaminated Soil":**

Head	No. of Position	Year I	Year II	Year III	Total (Rs.)
1. Tezpur University					
Junior Research Fellow JRF	1	330000.00	330000.00		**660000.00**
Senior Research Fellow SRF	1			369600.00	**369600.00**
Total(Rs.)		**330000.00**	**330000.00**	**369600.00**	**1029600.00**
2. Visva Bharati University					
Junior Research Fellow Will be working at Visva-Bharati for 10 months and two months at Tezpur University	1	330000.00	330000.00		**660000.00**
Senior Research Fellow Will be working at Visva-Bharati for 10 months and two months at Tezpur University	1			369600.00	**369600.00**
Total(Rs.)		**330000.00**	**330000.00**	**369600.00**	**1029600.00**

Emoluments detail of research personal(s) mentioned in table(s) of Annexure-II shall be applicable only if candidate(s) met educational qualification and eligibility criteria as per DST OM No.SR/S9/Z-09/2012 dated 21.10.2014.

(Dr. Vaishali Panjabi)
Scientist 'E'

EARTH SYSTEM SCIENCE ORGANIZATION (ESSO)
INDIAN INSTITUTE OF TROPICAL METEOROLOGY (IITM), PUNE
(An Autonomous Institute of the Ministry of Earth Sciences (MoES), Govt. of India)

Dr. Homi Bhabha Road, Pashan
Pune – 411008, Maharashtra, India

Ref. No. IITM/MAPAN/12

Date: August' 30, 2013

To,
The Registrar,
Tezpur University,
Tezpur, Assam, India.

Subject: Project entitled 'Modelling Atmospheric Pollution and Networking (MAPAN)' – Tezpur (MAPAN – 12)

PI: Dr. Raza R. Hoque, Tezpur University, Tezpur – 784 028.

Approval of the Director, IITM of the above project at a cost of Rs. 20, 32, 000/- (Twenty Lac thirty two thousand only) is conveyed to you for implementation in a period of 4 years as per the break up given below.

Sanction of the Director IITM is also hereby conveyed to release an amount of Rs. 4, 84,000/- (Rupees Four Lac eighty four thousand only) towards the recurring grant in aid for the first year.

Sr. No.	Items	Amount (For first year)	Amount (For 2nd year)	Amount (For 3rd year)	Amount (For 4th year)	Total Amount in Rs.
1.	**Emoluments (Salaries) for Manpower**					
	2 Research Fellows (@16000 p.m. for first 2 years and @18,000 p.m. for next 2 year)	3,84,000	3,84,000	4,32,000	4,32,000	16,32,000
2.	Travel and Contingency	1,00,000	1,00,000	1,00,000	1,00,000	4,00,000
	Total	4,84,000	4,84,000	5,32,000	5,32,000	20,32,000

The Grant will be regulated in accordance with the provisions of IITM and Ministry of Earth Sciences and purely based on the MOU signed between IITM and your Institution. The Grants in aid are also subject to particular conditions mentioned below:

1. The amount towards the head "Research Fellows" shall be utilised towards the payment of stipend of the 2 JRF's employed under the MAPAN Program as per the T&C stated below. This will be a consolidated amount and no other allowance will be applicable.

2. The costs towards the travel and contingency shall be utilised mainly for travel by PI's and the JRF's only (a) to attend MAPAN related meetings held at any of the MAPAN centre in India and (b) research related training and discussion to IITM, Pune and (c) for inviting experts for training and discussion from nodal group at IITM.
3. The JRF's are required to register for Ph.D along with a co-guide from IITM as per MoU and work towards the completion of the same within the stipulated time period. The project under consideration will be carried out by all MAPAN partners in collaboration with the Nodal officer of the project 'Metropolitan Air Quality and Weather services for cities, sports and tourism" of the Ministry of Earth Sciences and SAFAR program at IITM.
4. That the grantee shall present yearly Progress Reports on the Project to IITM. The yearly progress of the project will be monitored by the MAPAN project monitoring committee constituted by Director, IITM during the MAPAN workshop or in a special meeting. The project will be mainly accessed on the progress of specific thematic problem assigned to a particular MAPAN station.
5. The yearly grant will be sanctioned for next 4 years each, after submitting the utilization certificate (UC) and statement of expenditure (SE) in the prescribed format duly certified by the competent authorities, which should be send immediately after the end of each financial year.
6. The MAPAN Centre will spend the Grants in aid exclusively in pursuance of the objectives envisaged and as per the terms and conditions and agreement (MOU) signed by the MAPAN institutions with IITM, Pune and within the prescribed period only.
7. That the IITM, Pune Administration would have no liability whatsoever for the absorption of staff after the completion of the project.
8. The accounts of the MAPAN centre at grantee institution shall be audited by the Accounts officer or by any person authorized by him on his behalf in accordance with the provisions and shall be open to inspection by the sanctioning authority whenever it is called upon to do so.
9. The expenditure involved will be met from within the sanctioned budget grant and to be debited from the IITM budget head under the scheme 'Metropolitan Air Quality and Weather services for cities, sports and tourism" of the Ministry of Earth Sciences.
10. The grantee shall not commercially exploit the results of the project in any manner.
11. The administration reserves the right to terminate the funds at any stage, if it is satisfied that the funds have not been or are not being properly utilized for the purpose for which these were sanctioned or that the progress of the work is not satisfactory. On cancellation, the amount of expenditure already incurred, along with the unspent portion of the funds shall be remitted by the grantee within one month to the notice from IITM.
12. If the Principal Investigator to whom a grant for a project has been sanctioned wishes to leave, the Institution where the project is based, the institute / Investigator will inform the same to the IITM and, in consultation with the IITM, the institute shall evolve with IITM, steps to ensure successful completion of the Project before relieving the Principal Investigator.
13. That on the completion of the project any unspent / unutilized balance out of the funds shall be refunded by the grantee to the IITM Administration forthwith.
14. The Director, IITM hereby authorise to prepare & submit bill for the sanctioned amount to Accounts officer (Joint Manager) IITM, Pune who will direct the State Bank of India, PUNE MAIN Branch, Pune to make the payment preferably electronically i.e. through RTGS to the grantee, Whose bank details should be provided in the following format:

Name of the Beneficiary	Name of the Bank where A/C of the Org. is held.	Address/Code of the Bank Branch	A/c No.	MICR Code and the IFSC Code

15. Any of the terms and conditions set herein may be added / deleted / altered / relaxed at the discretion of the Director IITM, Pune Administration.

Kindly acknowledge the receipt of the letter.

Thanking you,

Yours Sincerely,

10.9.13

(S. M. Hendre)
Joint Manager
For Director, IITM

Copy forwarded for information and necessary action to:

1. Registrar,
   Tezpur University,
   Tezpur – 784 028,
   Assam,India
2. Dr. Raza R. Hoque - PI
   Tezpur University,
   Tezpur – 784 028,
   Assam,India
3. Program Director (MAPAN)
   Dr. Gufran Beig, Program Director MAPAN, IITM, Pune
4. Joint Manager - Accounts (IITM, Pune)

10.9.13

(S.M.Hendre)
Joint Manager,
For Director, IITM

Admin

No. BT/PR41637/NER/95/1753/2021

GOVERNMENT OF INDIA
MINISTRY OF SCIENCE & TECHNOLOGY
DEPARTMENT OF BIOTECHNOLOGY

Block 2, 6-8th Floors
CGO Complex, Lodhi Road,
New Delhi- 110 003
Dated:31/03/2021

ORDER

Sanction of the President is hereby accorded, under Rule 18 of the Delegation of Financial Powers Rules ,1978 , for the implementation of the project entitled: "**Exploiting chemical ecology for IPM: Deciphering the phyto-semiochemicals involved in Insect-Plant interactions of major crop pests of North East Region-India**" for a period of 3 Year 0 Month at a total cost of Rs. **47579782** (Rupees Four Crores Seventy Five Lakhs Seventy Nine Thousand Seven Hundred and Eighty Two Only) on the terms and conditions detailed here under:-

2 The Project :

2.1 Title : **"Exploiting chemical ecology for IPM: Deciphering the phyto-semiochemicals involved in Insect-Plant interactions of major crop pests of North East Region-India"**

2.2 Details of the Investigators:

Project Coordinator

Dr. KAMALA JAYANTHI P D
PI
CROP PROTECTION, IIHR, Bangalore
Indian Institute of Horticultural Research
Department of Entomology and Nematology, Indian Institute of Horticultural Research, Hesseraghatta Lake PO, Bangalore 560089, Bangalore, Karnataka, 560089

Principal Investigators:

Vaishali

डॉ. वैशाली पंजाबी / Dr. VAISHALI PANJABI
वैज्ञानिक 'ई' / Scientist 'E'
बायोटेक्नोलॉजी विभाग / Deptt. of Biotechnology
विज्ञान और प्रोद्यो. मंत्रालय / M/o Science & Tech.
भारत सरकार, नई दिल्ली / Govt. of India, N. Delhi

Dr. KAMALA JAYANTHI P D
PI
CROP PROTECTION, IIHR, Bangalore
Indian Institute of Horticultural Research
Dr P D Kamala Jayanthi, National Fellow, Department of
Entomology and Nematology, Indian Institute of Horticultural

Dr. Badal Bhattacharyya
Principal Scientist
Entomology
Assam Agricultural University
Principal Scientist, All India Network Project on Soil
Arthropod Pests, Assam Agricultural University, Jorhat,

Dr. D Srinivasa Reddy
Director
Natural Product Chemistry Division
CSIR Indian Institute of Integrative Medicine canal road Jammu
,Jammu And Kashmir (UT), 180001

Dr. Sikha Deka
Junior Entomologist
Department of Entomology
Citrus Research Station, Assam Agricultural University Gelapukhuri Road,
District Tinsukia,, Tinsukia - 786125,

Dr. Suvendra Kumar Ray
Professor
Department of Molecular Biology and Biotechnology
Tezpur University
Tezpur,Assam, 784028

CO-PI:

Dr. Robin Chandra Boro
Assistant Professor
Department of Agricultural Biotechnology
Assam Agricultural University
Assam

Dr. Inee Gogoi
Assistant Professor
Department of Agricultural Entomology
Assam Agricultural University
Assam

Dr. Kritideepan Sarmah
Junior Scientist
Biochemistry and Agricultural Chemistry
Assam Agricultural University
785013, Assam

Dr. Santa Kalita
Assistant Professor
Environmental Science
Tezpur University
Assam

Dr. Raaj Kumar Kakoti
Principal Scientist
Plant Pathology

डॉ. वैशाली पंजाबी / Dr. VAISHALI PANJABI
वैज्ञानिक 'ई' / Scientist 'E'
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Dr. Debaraj Mukherjee
Principal Scientist
Natural Product Chemistry Division
CSIR - Indian Institute of Integrative Medicine,Canal Road, Jammu – 180001

Dr. Sumit G. Gandhi
Principal Scientist
CSIR Indian Institute of Integrative Medicine canal road Jammu
CSIR - Indian Institute of Integrative Medicine,Canal Road, Jammu – 180001, Jammu

Vaishali.

डॉ. वैशाली पंजाबी / Dr. VAISHALI PANJABI
वैज्ञानिक 'ई' / Scientist 'E'
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विज्ञान और प्रोद्यो. मंत्रालय / M/o Science & Tech.
भारत सरकार, नई दिल्ली / Govt. of India, N. Delhi

ANNEXURE-II: Core Group (RESPECT)		
Name	**Email**	**Organization**
Advisor		
Dr. Arbinda Mitra	Scientific secretary, O/o PSA to GoI	
Prof. T.G. Sitharam	Director, IIT Guwahati	
Shri P.P. Shrivastav	Advisor, NDMA	
Member from IIT Guwahati		
Prof. A.K.Sarma	aks@iitg.ac.in	Department of Civil Engineering, IIT Guwahati
Prof. Rajib Bhattacharjya	rkbc@iitg.ac.in	Department of Civil Engineering, IIT Guwahati
Prof. Chandan Mahanta	chandan@iitg.ac.in	Department of Civil Engineering, IIT Guwahati
Prof. Subhasish Dutta	subashisa@iitg.ac.in	Department of Civil Engineering, IIT Guwahati
Dr. Sreeja P.	sreeja@iitg.ac.in	Department of Civil Engineering, IIT Guwahati
Prof. Anamika Barua	abarua@iitg.ac.in	Department of HSS, IIT Guwahati
Members from other institutes/ organizations		
Prof. Rajendra Singh	rsingh@agfe.iitkgp.ac.in	Brahmaputra Chair Prof. for Water Resources, IIT Kharagpur
Dr. S.K. Jain	Skj.nihr@gov.in	Director, NIH
Dr. Manabendra Saharia	msaharia@iitd.ac.in	Dept. of Civil Engineering, IIT Delhi
Prof. Shikhar Kumar Sarma	sks@gauhati.ac.in	Director, Centre for Brahmaputra Studies, GU
Dr. Bipul Talukdar	bipulaec@gmail.com	Dept. of Civil Engineering, Assam Engineering College
Smti. Mandira Buragohain	mandiraburagohaingogoi@gmail.com	SDMA, Assam
Prof. S.P. Biswas	spbsdu@dibru.ac.in	Professor, Dept. of Life Sciences, Dibrugarh University
Swapan Mehra	swapan@ioraecological.com	CEO, IORA Ecological Solutions
Dr. Diganta Barman	diganta_isro1@yahoo.co.in	Scientist/ Engineer 'SF', NESAC
Mr. Pulendra Dutta	pulendutta@gmail.com	Dept. of Civil Engineering, Jorhat Engineering College

	pulendutta@iitg.ac.in	
Dr. Hriday Mani Kalita	hriday@nitm.ac.in	Dept. of Civil Engineering, NIT-Meghalaya
Dr. Jumrik Taipodia	jukrik@nitap.ac.in	Dept. of Civil Engineering, NIT-AP
Dr. Nayanmoni Gogoi	nayanmoni@tezu.ernet.in	Dept. of Environment Sciences, Tezpur University
Prof. Apurba Kr. Das	aburba@tezu.ernet.in	Dept. of Environment Sciences, Tezpur University
Partha Jyoti Das	parthajdas@gmail.com	Aaranyak
Dr. Arup Kumar Misra	astec@rediffmail.com	Director, ASTEC
Shri Gopal Chetri	gopal.chetri@gov.in	Adviser (Irrigation Flood Control & Water Management), NEC
Dr. Uzzal M. Hazarika	uzzal_hazarika@yahoo.com	WRE, NERIWALM
Prof. Tomo Riba	tomo.riba@rgu.ac.in	Dept. of Geography, Rajib Gandhi University, AP
Shri Ranjit Deka	rdeka77@gmail.com	Director, NEHARI, Brahmaputra Board
Dr. Arnab Bandyopadhyay	arnabbandyo@yahoo.co.in arnabbandyo@gmail.com arnabbandyo@nerist.ac.in	Dept. of Agricultural Engineering, NERIST
S.K. Rajan	tcodte@nic.in	Director, TCD, CWC
Dr. Juran Ali Ahmed	juranahmed@gmail.com	Asst. Exe. Engineer, Hydraulic Research, W.R. Division, Assam
Er. H. Duhkima	hduhkima@gmail.com	SE, PHED, Mizoram
Er. Likar Angu	angulikar767@gmail.com cewrditanagar@gmail.com	Chief Engineer (Western Zone), WRD, AP
Er. Mongsi Mog		EE, PWD (WRE), Tripura
Ms. Dipima Sarma	dipimasarma13@gmail.com	Research Assoc III, Climate Change Cell, Dept. of Forest, Assam
Er. D B Syngkon	dbsyngkon@gmail.com	SE, WRD, Meghalaya
Jiten Malakar	jitenbmalakar2014@gmail.com	Addl. Director, Soil Conservation Dept., Assam
Ms. Marbakor Mary Lynrah	lynrahmarbakor@gmail.com	Project Scientist, Meghalaya Climate Change Centre

No. BT/PR40047/NER/95/1662/2020

GOVERNMENT OF INDIA
MINISTRY OF SCIENCE & TECHNOLOGY
DEPARTMENT OF BIOTECHNOLOGY

Block 2, 6-8th Floors
CGO Complex, Lodhi Road,
New Delhi- 110 003
Dated: / /

ORDER

Sanction of the President is hereby accorded, under Rule 18 of the Delegation of Financial Powers Rules ,1978 , for the implementation of the project entitled: "**Harnessing Endophytes and Arbuscular Mycorrhizal Fungi from Citrus Microbiome for Plant and Soil Health Management in North East India**" for a period of 3 Year 0 Month at a total cost of Rs. **11335464** (Rupees One Crores Thirteen Lakhs Thirty Five Thousand Four Hundred and Sixty Four Only) on the terms and conditions detailed here under:-

2 The Project :

2.1 Title : "**Harnessing Endophytes and Arbuscular Mycorrhizal Fungi from Citrus Microbiome for Plant and Soil Health Management in North East India**"

2.2 Details of the Investigatiors:

Project Cordinator

Dr. Anjuma Gayan
Assistant Professor
Soil Science
Asaam Agricultural University
College of Sericulture,Soil Science department,
Assam Agricultural University,Jorhat-13, Jorhat, Assam, 785013

Principal Investigators:

Dr. Anjuma Gayan
Assistant Professor
Soil Science
Asaam Agricultural University
College of Sericulture,Soil Science department,

Dr. Arun Kumar Rai
Assistant Professor
Botany
Sikkim University
Department of Botany, Sikkim University (Central University),

Dr. Kiran Sunar
Assistant Professor
Botany
Balurghat Mahila Mahavidyalaya
Balurghat Mahila Mahavidyalaya (Govt. aided college), Balurghat,

Dr. Nirmali Gogoi
Assistant Professor
Environmental Science
Tezpur University
Department of Environmental Science Tezpur University

CO-PI:

Dr. Dhruba Jyoti Nath
Principal Scientist
Soil Science
Assam Agricultural University
Department of Soil Science
Assam Agricultural University
Jorhat, Jorhat - 785013, Assam

Mr. Himadri Shekhar Datta
Assistant Professor
Horticulture
Asaam Agricultural University
College of Horticulture
Assam Agricultural University
Jorhat, Jorhat - 785013, Assam

2.3 Objectives:

Overall Objectives:

1. Isolation, characterization and identification of endophytes (bacterial and fungal), Arbuscular Mycorrhizal Fungi (AMF) and plant growth promoting rhizobacteria (PGPR) from Citrus microbiome of NE region of India including Sikkim.
2. Elucidation of the mechanisms involved in growth promotion and disease suppression by endophytes, AM fungi and PGPR.
3. Development of a cost-effective microbial consortia of selected potential endophytes AM fungi and PGPR.
4. Validation of the potential microbial consortia in plant and soil health management of commercial citrus varieties.

Institute wise Objectives:

Asaam Agricultural University

1. Isolation, characterization and identification of endophytes (bacterial and fungal), Arbuscular Mycorrhizal Fungi (AMF) and plant growth promoting rhizobacteria (PGPR) from Citrus limon and Citrus reticulata of NE region of India
2. Elucidation of the mechanisms involved in growth promotion by endophytes (bacteria and fungi), AM fungi and PGPR
3. Development of a cost-effective microbial consortium of selected endophytes (bacterial and fungal), AM fungi and PGPR
4. Validation of the potential microbial consortia in plant and soil health management of Citrus limon and Citrus reticulata

Balurghat Mahila Mahavidyalaya

1. Isolation, characterization and identification of endophytes (bacterial and fungal), Arbuscular Mycorrhizal Fungi (AMF) and PGPR associated with the roots of Citrus reticulata from the major orchards of Sikkim
2. Raising monosporal (single spore) cultures of dominant AMF isolates and their morphological characterization and multiplication, alongwith In vitro functional characterization of PGPR of Citrus reticulata
3. Development of cost-effective microbial consortia of potential endophytes, AM fungi and PGPR
4. Validation of potential microbial consortia in plant and soil health management of Citrus reticulata

Sikkim University **Annexure -I**

Details of the Equipment sanctioned for the project entitled **"Harnessing Endophytes and Arbuscular Mycorrhizal (AM) fungi from Citrus Microbiome for Plant and Soil Health Management in North East India":**

Sikkim University

SNo.	Name of Equipment	No.	Cost(Rs.)
1.	Mini Glass house for plants with exhaust fan and coolers (dimension 15'x15'x15')	1	299720.00
2.	Hot air oven (digital controller) size 18''x18''x18''	1	29937.00
3.	Cooling centrifuge with voltage stabilizer	1	410340.00
4.	Trinocular Microscope with HD camera	1	236708.00
		Total	**976705.00**

Balurghat Mahila Mahavidyalaya

SNo.	Name of Equipment	No.	Cost(Rs.)
1.	Small scale poly house (Dimension 10 x10 x15').	1	135700.00
2.	Cooling centrifuge with voltage stabilizer	1	410340.00
3.	Trinocular Microscope with HD camera	1	236708.00
4.	Biological safety cabinet (as per class II) Working size- 3x2x2 fit)	1	122531.00
		Total	**905279.00**

Asaam Agricultural University

SNo.	Name of Equipment	No.	Cost(Rs.)
1.	Plant Growth Chamber	1	525000.00
		Total	**525000.00**

2.4 Time Schedule:

The duration of the project is 3 Year 0 Month from the date of this sanction order.

2.5 Project Cost:

The total cost of the project is Rs. **11335464**/-(Rupees One Crores Thirteen Lakhs Thirty Five Thousand Four Hundred and Sixty Four Only) as per details given below :

SNo.	Institute	Year I	Year II	Year III	Total Cost(Rs.)
1.	1. Asaam Agricultural University	1351760	826760	778600	**2957120**
	2. Balurghat Mahila Mahavidyalaya	1632039	726760	728600	**3087399**
	3. Sikkim University	1703465	726760	778600	**3208825**
	4. Tezpur University	676760	726760	678600	**2082120**
	Total (Rs.)	**5364024**	**3007040**	**2964400**	**11335464**

(Dr. Vaishali Panjabi)
Scientist 'E'

Institute wise details are:

Budget Head	Year I	Year II	Year III	**Total**(Rs.)
5. Sikkim University				
Equipment	976705.00			**976705.00**
Manpower	401760.00	401760.00	453600.00	**1257120.00**
Overhead	50000.00	50000.00	50000.00	**150000.00**
Travel	50000.00	50000.00	50000.00	**150000.00**
Consumables	150000.00	150000.00	150000.00	**450000.00**
Contingency	50000.00	50000.00	50000.00	**150000.00**

Miscellaneous	25000.00	25000.00	25000.00	**75000.00**
Total (Rs.)	**1703465.00**	**726760.00**	**778600.00**	**3208825.00**
6. Balurghat Mahila Mahavidyalaya				
Equipment	905279.00			**905279.00**
Manpower	401760.00	401760.00	453600.00	**1257120.00**
Overhead	50000.00	50000.00	50000.00	**150000.00**
Travel	50000.00	50000.00	50000.00	**150000.00**
Consumables	150000.00	150000.00	100000.00	**400000.00**
Contingency	50000.00	50000.00	50000.00	**150000.00**
Miscellaneous	25000.00	25000.00	25000.00	**75000.00**
Total (Rs.)	**1632039.00**	**726760.00**	**728600.00**	**3087399.00**
7. Asaam Agricultural University				
Equipment	525000.00			**525000.00**
Manpower	401760.00	401760.00	453600.00	**1257120.00**
Consumables	250000.00	300000.00	150000.00	**700000.00**
Contingency	50000.00	50000.00	50000.00	**150000.00**
Overhead	50000.00	50000.00	50000.00	**150000.00**
Travel	50000.00		50000.00	**100000.00**
Miscellaneous	25000.00	25000.00	25000.00	**75000.00**
Total (Rs.)	**1351760.00**	**826760.00**	**778600.00**	**2957120.00**
8. Tezpur University				
Manpower	401760.00	401760.00	453600.00	**1257120.00**
Consumables	100000.00	150000.00	100000.00	**350000.00**
Contingency	50000.00	50000.00	50000.00	**150000.00**
Overhead	50000.00	50000.00	50000.00	**150000.00**
Travel	50000.00	50000.00		**100000.00**
Miscellaneous	25000.00	25000.00	25000.00	**75000.00**
Total (Rs.)	**676760.00**	**726760.00**	**678600.00**	**2082120.00**

2.6 Equipment:

The details of the equipment sanctioned for the implementation of the project at **Annexure-I**

2.7 Manpower:

The details of the manpower sanctioned for the implementation of the project at **Annexure-II**

3. Head of Account:

The **Non-Recurring** expenditure involved is debitable to:

Demand No. 88	Department of Biotechnology
3425	Other Scientific Research 2020-2021
3425.60	Others (Sub Major Head)
3425.60.200	Assistance to other Scientific Bodies (Minor Head)
3425.60.200.29	Biotechnology Research and Development
3425.60.200.29.17	Assistance for Research and Development
3425.60.200.29.17.35	Grants for creation of capital assets

The **Recurring** expenditure involved is debitable to:

Demand No. 88	Department of Biotechnology
3425	Other Scientific Research 2020-2021
3425.60	Others (Sub Major Head)
3425.60.200	Assistance to other Scientific Bodies (Minor Head)
3425.60.200.29	Biotechnology Research and Development
3425.60.200.29.17	Assistance for Research and Development
3425.60.200.29.17.31	Grants -in-Aid General

4. Terms & Conditions:

Additional Terms and Conditions, specific for Twinning R and D program for NER
a. Both NER and Rest of India (RoI), Institutions scientists should work together for the objectives stated in the sanction of the project and any deviation from this would attract closure of the project at any point of time.

b. In the project review meetings, both the PIs from NER and RoI Institutions should participate and make presentation.

c. The outcomes of the project such as research papers, patents, copy rights etc. should be made jointly.

d. The NER Scientists are to be trained at the collaborating institute appropriately to empower the NER Scientists.

e. The project personal such as Research Associate, JRF or SRF, Research Assistant are also to be trained at least once in the collaborating national institute.

f. The collaborating institute scientist should visit NER institutions more frequently to guide NER scientists in design and conduct of experiments.

g. The Non-Recurring items must be procured and installed within 18-months of the sanction of the project, failing which the PIs have to return the remaining/unutilized Non Recurring grant with 10% of Interest.
h. In case the amount of grant-in-aid is refunded, the whole or a part amount of the grant, with an interest at 10% per annum there on shall be recovered.
Additional Terms and Conditions, specific for Twinning R and D program for NER
a. Both NER and Rest of India (RoI), Institutions scientists should work together for the objectives stated in the sanction of the project and any deviation from this would attract closure of the project at any point of time.

b. In the project review meetings, both the PIs from NER and RoI Institutions should participate and make presentation.

c. The outcomes of the project such as research papers, patents, copy rights etc. should be made jointly.

d. The NER Scientists are to be trained at the collaborating institute appropriately to empower the NER Scientists.

e. The project personal such as Research Associate, JRF or SRF, Research Assistant are also to be trained at least once in the collaborating national institute.

f. The collaborating institute scientist should visit NER institutions more frequently to guide NER scientists in design and conduct of experiments.

g. The Non-Recurring items must be procured and installed within 18-months of the sanction of the project, failing which the PIs have to return the remaining/unutilized Non Recurring grant with 10% of Interest.
h. In case the amount of grant-in-aid is refunded, the whole or a part amount of the grant, with an interest at 10% per annum there on shall be recovered.

4.1 The other terms and conditions governing this sanction are attached at Annexure- III.

4.2 A Memorandum of Agreement (MoA) will be signed between the Department of Biotechnology and the grantee institution on Non-Judicial stamp paper Rs. 100/- in the enclosed format and the second release/installment will be made only after signing of MoA between the grantee institutions and DBT. In case of NGO's and Private Institution's, execution of MOA is mandatory before first release. A format of the MoA is enclosed in Annexure-IV

4.3 The Institute/Agency will keep the whole of the grant in a Bank Account earning interest, and the interest so earned should be reported to DBT in the Utilisation Certificate and Statement of Expenditure. The Interest so earned will be treated as created to the institute/Agency and shall be adjusted towards further installment of the grant and or at the time of Final Settlement of Accounts.

5. No International Travel will be undertaken from the sanctioned project grant unless specified otherwise.

6. The Director, Asaam Agricultural University, Jorhat, Assam and The Dr Biman Chakraborty, Balurghat Mahila Mahavidyalaya, Malda, West Bengal and The Registrar, Sikkim University, Gangtok, Sikkim and The Registrar, Tezpur University, Tezpur, Assam would be responsible for submission of Statements of Expenditure (SoE), utilization certificates (UC), Assets Certificates, Manpower staffing & expenditure details in prescribed DBT formats to DBT in respect of grants released in this project from time to time.

7. PI's of DBT sponsored projects can consider appointment of JRF from Category-II merit list of DBT-BET exam so that candidates can be paid fellowships at par with NET/GATE/BET qualified candidates as per DST OM No. A.SR/S9/Z-05/2019 dated on 30 Jan 2019. However, there is no compulsion on PI's to select candidates for JRF in their projects from Category-II of DBT-BET.

8. As per Rule 236 (1) of GFR 2017, the accounts of all Grantee Institutions or Organisations shall be open to inspection by the sanctioning authority and audit, both by the Comptroller and Auditor General of India under the provision of CAG(DPC) Act 1971 and internal audit by the Principal Accounts Office of the Ministry or Department, whenever the Institution or Organisation is called upon to do so.

9. If the Research Project involves biological resources, the obligations under the Biological Diversity Act 2002 as applicable shall be complied with by the Project Investigator, the details of such obligations can be accessed at www.nbaindia.org

10. If this project works requires any Statutory Clearance (such as Ethical Committee Clearance, Informed Consent, Animal Ethics Committee Clarence, Institutional Bio-safety Committee Clearance, National Biodiversity Authority approval etc.), Project investigator(s) and Host Institution(s) shall compulsory comply the same before undertaking such activities.

11. This issues under the power delegated to this Department and with the concurrence of IFD vide their SAN No.**102/IFD/SAN/2350/2020-2021** dated **February, 09 2021.**

12. This sanction order has been noted at serial no._____ in the Register of Grants.

(Dr. Vaishali Panjabi)
Scientist 'E'

To,
The Pay & Accounts Officer,
Department of Biotechnology,
New Delhi – 110 003.

Copy to:

1 The Principal Director of Audit (Scientific Departments), DACR Building, New Delhi- 110 002.
2 Dr. Anjuma Gayan(Project Co-ordinator), College of Sericulture,Assam Agricultural University,Jorhat:785013, Assam
3 The Director, Asaam Agricultural University, Assam Agricultural University
Jorhat-785013
Assam, Jorhat - 785013, Assam
4 The Dr Biman Chakraborty, Balurghat Mahila Mahavidyalaya, Balurghat, Dakshin Dinajpur, West Bengal, Malda - 733101, West Bengal
5 The Registrar, Sikkim University, 6TH MILE, SAMDUR,P. O. : TADONG
GANGTOK, SIKKIM, Gangtok - 737102, Sikkim
6 The Registrar, Tezpur University, Tezpur University
Napaam, Tezpur
Sonitpur, Tezpur - 784028, Assam
7 Dr. Anjuma Gayan, Assistant Professor, Soil Science, Asaam Agricultural University, College of Sericulture,Soil Science department,
Assam Agricultural University,Jorhat-13, Jorhat - 785013, Assam
8 Dr. Arun Kumar Rai, Assistant Professor, Botany, Sikkim University, Department of Botany, Sikkim University (Central University), 6th mile, Samdur,
P.O.Tadong,Gangtok,Sikkim, Gangtok - 737102, Sikkim
9 Dr. Dhruba Jyoti Nath, Principal Scientist, Soil Science, Assam Agricultural University, Department of Soil Science
Assam Agricultural University
Jorhat, Jorhat - 785013, Assam
10 Dr. Kiran Sunar, Assistant Professor, Botany, Balurghat Mahila Mahavidyalaya, Balurghat Mahila Mahavidyalaya (Govt. aided college), Balurghat, P.O.Balurghat, District: Dakshin Dinjapur, West Bengal, 733101, Islampur West Bangal - 733101, West Bengal
11 Dr. Nirmali Gogoi, Assistant Professor, Environmental Science, Tezpur University, Department of Environmental Science Tezpur University
Napaam, Tezpur District: Sonitpur, Assam, Tezpur - 784028, Assam
12 Mr. Himadri Shekhar Datta, Assistant Professor, Horticulture, Asaam Agricultural University, College of Horticulture
Assam Agricultural University
Jorhat, Jorhat - 785013, Assam
13 Cash Section, DBT (2 copies).
14 Sanction Folder.
15 File Copy.

(Dr. Vaishali Panjabi)
Scientist 'E'

Annexure -II

Details of the manpower sanctioned for the implementation of the project entitled **"Harnessing Endophytes and Arbuscular Mycorrhizal Fungi from Citrus Microbiome for Plant and Soil Health Management in North East India":**

Head	No. of Position	Year I	Year II	Year III	Total (Rs.)
5. Asaam Agricultural University					
Junior Research Fellow To carry out laboratory and field works	1	401760	401760	453600	**1257120**
Total(Rs.)		**401760**	**401760**	**453600**	**1257120**
6. Balurghat Mahila Mahavidyalaya					
Junior Research Fellow Working with AMF and PGPR related laboratory and field works	1	401760	401760	0	**803520**
Senior Research Fellow Working with AMF and PGPR related laboratory and field works	1	0	0	453600	**453600**
Total(Rs.)		**401760**	**401760**	**453600**	**1257120**
7. Sikkim University					
Junior Research Fellow Working with Endophytes (bacteria and fungi) and PGPR related laboratory and field works	1	401760	401760	0	**803520**
Senior Research Fellow Working with Endophytes (bacteria and fungi) and PGPR related laboratory and field works	1	0	0	453600	**453600**
Total(Rs.)		**401760**	**401760**	**453600**	**1257120**
8. Tezpur University					
Junior Research Fellow To carry out laboratory and field works	1	401760	401760	0	**803520**
Senior Research Fellow To carry out laboratory and field works	1	0	0	453600	**453600**
Total(Rs.)		**401760**	**401760**	**453600**	**1257120**

Emoluments detail of research personal(s) mentioned in table(s) of Annexure-II shall be applicable only if candidate(s) met educational qualification and eligibility criteria as per DST

OM No. SR/S9/Z-05/2019 dated 10.07.2020.

(Dr. Vaishali Panjabi)
Scientist 'E'

कार्यालय
अयोध्या शोध संस्थान
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54/अयो०शो०सं०/2017-18

पत्र सं0 .....दिनांक 29/08/2017

सेवामें,

डॉ0 अनुशब्द,
सहायक प्राध्यापक (हिन्दी विभाग)
केन्द्रीय विश्वविद्यालय, तेजपुर, आसाम।

विषय- त्रि-दिवसीय अन्तर्राष्ट्रीय संगोष्ठी के सम्बन्ध में।

कृपया ''एशियाई जीवन कला में राम (पूर्वोत्तर भारत के विशेष सन्दर्भ में)'' आयोजित त्रि-दिवसीय अन्तर्राष्ट्रीय संगोष्ठी के सन्दर्भ में आपके प्रेषित प्रस्ताव का सन्दर्भ ग्रहण करने का कष्ट करें, जो तेजपुर विश्वविद्यालय में 09 से 11 नवम्बर, 2017 तक हिन्दी विभाग, तेजपुर विश्वविद्यालय के सहयोग से आयोजित की जानी है।

आयोजन हेतु रु0 6.50 लाख की प्राथमिक स्वीकृति प्रदान की गयी है, कृपया आवश्यक व्यवस्थाएं सुनिश्चित करें।

आयोजन हेतु आवश्यकतानुसार 50 प्रतिशत की धनराशि अग्रिम प्रदान की जायेगी। भुगतान के सम्बन्ध में आवश्यक वित्तीय नियमों का पालन किया जाना होगा।

आपसे अनुरोध है कि आमंत्रित महानुभावों के नाम से शीघ्र अवगत कराने का कष्ट करें। असम की ''माजुली कला'' को भी विशेष स्थान दिया जाना अपेक्षित है।

भवदीय

(डॉ0 योगेन्द्र प्रताप सिंह) 29/8/17
निदेशक

Tulsi Smarak Bhawan Raiganj, Ayodhya, Faizabad, U.P., Pin Code: 224-123 INDIA
9th Floor, Room No-938, Jawahar Bhawan, Lucknow, U.P., Pin Code : 226001 INDIA

ayodhyaresearch1986@gmail.com, www-ayodhya.co, ayodhyashodh

Tezpur University

Intellectual Property Rights Cell

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Activity 2016-2017

Colloquium on World IP Day celebration on 26 April, 2016

The 16th World Intellectual Property Day was celebrated in the Tezpur University Campus on 26th of April, 2016. The theme of this year's celebrations as declared by the World Intellectual Property Organisation (WIPO) was "Digital Creativity: Culture Reimagined". This annual event in the campus was aimed at enhancing IPR Awareness and encouraging the University fraternity to involve themselves in creative activities for betterment of our world. The celebrations included IP Day Quiz, IP Day Talk, Best Innovative Idea Competition and Best Slogan Writing Competition. The Best Innovative Idea Competition and Best Slogan Writing Competition were conducted for the first time in the Campus. The IP day Talk was delivered by Professor Dhruba Kumar Bhattacharya, Dean Academics, Tezpur University, who enlightened the audience with a very informative and interesting talk on "Internet Security".

Patent Search and Analysis workshop on 25th February, 2017

The IPR Cell, Tezpur University, organised a Patent Search and Analysis Workshop for faculty/researchers to familiarise and train them on search of patent databases and also to initiate patent analysis so that the researchers can develop a base-level competence to conduct patent searches, analyse the patents for their technical content, prepare technical landscapes and design meaningful research proposals and approaches to their research prior to embarking on their research path. The workshop on February 25, 2017. This was done to ensure better quality of research in Tezpur University, enhance the possibility of patentable output and improve the prospects of commercialisation of the output of researchers from Tezpur University. During this program, each participant with their laptop computers did hands-on experience on accessing and manipulating various free databases such as the USPTO, EPO, SIPO (Chinese), Indian Patent Office, WIPO (for PCT Publications). The participants were exposed to the power of these databases and how they could search and access patents in a targeted manner.

Geographical Indications (GI) Camp at Dhakuakhana on 30th April, 2017

The Tezpur University Intellectual Property Rights (TUIPR) Cell initiated a project to identify the key reasons for such low number of authorised users of GI Muga Silk of Assam in the State and initiate immediate corrective measures at the grassroots level and across the MUGA Value Chain. Targeted field work in the Muga Hub in North Assam helped to identify the critical factors that needed to be addressed. Low awareness of the benefits of GI, uncontrolled quality of MUGA production, insignificant marketing infrastructure, non-existence of a regulatory framework for enforcement of GI, lack of technology infusion for enhancement of productivity and quality including insignificant organisational support were some of key weaknesses identified. The follow-up activities of the TUIPR Cell in the MUGA Hub in North Assam finally culminated in a major GI Camp with the MUGA Stakeholders in the Lakhimpur Region. These efforts formed the basis of the GI Camp on April 30, 2016 organised by the TUIPR Cell jointly with IPR Cell of Dibrugarh University and ASTEC (the owners of GI Muga Silk of Assam) at the Dhakuakhana College, Lakhimpur. More than 200 MUGA Stake Holders from 12 villages of Lakhimpur District across the MUGA Value Chain, including students and faculty of Dhakuakhana College and prominent local personalities actively participated in this GI Camp. This Camp therefore established a new trend of how "train the trainers" sessions at the grassroots level can lead to a chain reaction for facilitation of such processes in the village clusters. The new applicants were appreciative of this gesture by their village peers and are now motivated play their role in the promotion of the GI movement in their villages.

Quick Links

Annual Report (annual-report.php)
Newsletters (newsletter.php)
WIPO booklets on IPR (booklates.php)
IPR Policy of Tezpur University (quick_links_files/IPR_Policy_Feb_2019.pdf)
Invention Disclosure Form (quick_links_files/Invention_Disclosure_Form.pdf)

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Tezpur University

Intellectual Property Rights Cell

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- National Webinar On Intellectual Property as a Strategic Tool for Regional Development organized at Dhakuakhana College, Assam (http://www.tezu.ernet.in/tuipr_cell/pdf/Report_Dhakuakhana_College.pdf)

Activity 2017-2018

Workshop on Information Access and Analysis in the Innovation and Research Value Chain on 10 May, 2017

A day long workshop was organised by the Tezpur University Intellectual Property Rights Cell in collaboration with the Central Library, Tezpur University on 10 May, 2017 for the students and researchers in the University. The workshop was themed on Information Access and Analysis in the Innovation and Research Value Chain. The main aim behind organising such a workshop, in addition to the annual Patent Search and Analysis workshop, was to give a hands-on exposure to the diverse information sources/databases that may be accessed in various subjects of interest in Science, Technology, Humanities and Languages etc. Shri N.V. Sathyanarayana, who is one of pioneers and experts working in the field of information access and technologies, and Shri. Sambhunath Sahoo, Information Scientist in Central Library, Tezpur University deliberated upon various topics. The topics of discussion ranged from the Art of Searching: Non-Patent Literature which covered broadly the issue of conducting targeted search of databases for contextual information with special emphasis on approaches, strategies and methods, to Access of E-Resources at Tezpur University. 37 participants from within and outside the university participated in the same.

Colloquium on World IP Day celebration on 26 April, 2017

With the theme “Innovation – Improving Lives”, Tezpur University Intellectual Property Rights Cell celebrated the World Intellectual Property Day, on 26 April, 2017. The celebrations included a half day programme comprising of the IP Day Quiz, Best Innovative Idea Competition and Slogan writing competition relevant to Intellectual Property Rights. The event was organised in the Council Hall of Tezpur University with a gathering of students, research scholars and faculty members from various departments. This year 23 groups, consisting of 46 students participated in the quiz competition which lasted for 4 long hours. Other events also witnesses encouraging participation. IP Day talk was delivered by an eminent scientist in the field of Chemistry, Prof Niranjan Karak, who is also a Professor in the Advanced Polymer and Nanomaterial Laboratory Center for Polymer Science and Technology, Department of Chemical Sciences & Head, SAIC, Tezpur University. Keeping track with theme of this year’s IP Day celebration, Prof Karak presented the developments and innovations that has been taking place in the world of Material and Material sciences. 57 students and faculty members participated in the event.

Quick Links

Annual Report (annual-report.php)
Newsletters (newsletter.php)
WIPO booklets on IPR (booklates.php)
IPR Policy of Tezpur University (quick_links_files/IPR_Policy_Feb_2019.pdf)
Invention Disclosure Form (quick_links_files/Invention_Disclosure_Form.pdf)

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Tezpur University

Intellectual Property Rights Cell

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- National Webinar On Intellectual Property as a Strategic Tool for Regional Development organized at Dhakuakhana College, Assam

Activity 2018-2019

2nd Industry Academia Conclave on 30th December 2018

As part of celebration of its silver jubilee, a regional workshop on “Translational Research and Innovation-……making use of IPR across value chain” was organized on 30th of Dec (Sunday), 2018. This workshop cum training programme, jointly organised by TUIPR Cell with NRDC, was aimed at providing solutions to technological demands through deliberations from top technocrats belonging to various sectors like Oil India Ltd, Duliajan, M/s Medlis Healthcare Pvt Ltd. , Ahmedabad, Bhogali Food Products Pvt. Ltd., Guwahati, Williamson Magor Group etc. and professionals from organisations like LS Davar and Co. (IP attorney), Indian Patent Office, Kolkata and NEIST Jorhat. Prof. Pritam Deb, Coordinator, TUIPR Cell, shared with the gathering the idea behind organizing such a workshop for the innovators in his welcome address. Inaugurating the workshop, Dr. H. Purushotham, CMD, NRDC stressed “inventors must focus on Technology Readiness Level (TRLs) and work more aggressively on bridging the gap between technology and market requirement”. Resident Chief Executive, OIL, Shri B. P. Sarma, the Guest of Honour, while discussing the research and development activities of Oil India Limited, expressed that technology infusion and collaboration seems possible in terms certain areas like Enhanced Oil Recovery (EOR), reducing wastage of crude oil while exploration etc. The workshop was attended with 30 select participants from various fields of science and technology, who had one or more technologies in hand and are getting ready for translation in the next level. Besides, having dedicated sessions on sectors like Pharmaceuticals, Oil Industry, Food Industry, Tea Industry, Industry-Academic Business Incubation etc., the workshop included presentations from 14 innovators which were showcased to the industry representative and IPR professionals for comments. The daylong programme ended with a vote of thanks from the Research Officer, IPR Cell Dr. Juri B. Saikia.

Workshop on IPR for school children on 1st October, 2018

As part of the Silver Jubilee Celebration of Tezpur University and Decadal Existence of the IPR Cell, a special IPR workshop was conducted for school children in Tezpur University campus on 1st October, 2018. Schools from Tezpur town and villages in and around Tezpur University participated. As many as 256 students and school teachers attended the programme that was organized in collaboration with CIPAM, a professional body under the aegis of Department of Industrial Policy and Promotion (DIPP), Govt. of India. It is noteworthy to mention here that this programme was a maiden event in this region of the country at school level to facilitate, students, teachers and educators realize the importance of Intellectual Property Rights and to orient them towards an IP conscious ecosystem. With National Intellectual Property Rights (IPR) Policy in place from May 2016, the Government of India has been undertaking several IPR awareness programmes across the country, under the flagship of CIPAM, for schools. The key focus of every such initiative is to nurture creativity and the ability to innovate from a young age and to become conscious of its protection through various IPR tools.

The gathering was welcomed by Prof. P. Deb, the Coordinator, TUIPR Cell and inaugurated by the Vice Chancellor of Tezpur University, Prof. V.K. Jain. Ms. Divya Srinivasan, CIPAM, deliberated on “IPR Awareness for School Children”. A quick summary of the proceedings was provided in Assamese by Dr. Juri B. Saikia, Research Officer, IPR Cell followed by a vote of thanks.

Industry-Academia conclave organized on 26th April, 2018

A national symposium was organized on 26th April, 2018 in Tezpur University, coinciding with celebration of 10th year of establishment of the TUIPR Cell, and Silver Jubilee Celebration of the University’s establishment. This symposium on “Role of Innovation and Technology for Sustainable Development : Integrating Transformation with Sustainability (RITES-2018)” was a maiden attempt in this region to bring together top management professionals from industries, academicians, researchers & students of diverse fields together to discuss various dimensions of academia-industry partnership today and exchange ideas, practice and experiences on different aspects of industry-academia integration. Invited experts from industries like Hindustan Unilever, IBM, Permacel Ltd., ONGC, etc. deliberated upon the latest innovations in their respective sectors. They also participated in discussions on facilities that are expected for working in collaboration with universities. Entire focus of these deliberations was concentrated around facilitating pro-industry research in the region. Experts from government bodies like Indian Patent Office, NRDC and MSME apprised the gathering about the schemes that are available for supporting such industry-academia collaborations and entrepreneurship. A special session was held under the aegis of Indian Patent Office for apprising the gathering about one of the less talked about IP tool “Integrated Layout and Design” and its benefits. The symposium was attended by 145 researchers, students and faculty members of Tezpur University and nearby institutions. A local exhibition of promising innovations by University researchers was held in the symposium venue that exposed the invited guests from the industry to the different directions of research undertaken in the campus.

World IP Day Celebration in the campus

Like every year, this year too, the World Intellectual Property Day was celebrated in the campus with much enthusiasm and creative engagements. However, the competitive events for IP Day 2018, had to be organised on the eve 26th April, to accommodate events for RITES-2018. The much-awaited quizzing event, the IP Quiz, was conducted by Mr. Ashutosh Das, a TU alumnus, that was based on issues related to intellectual property and its associated rights. The exercise exposed the participants not only to the theme of this year “Powering change: Women in innovation and creativity”, but also to small nitty-gritties of intellectual property in our lives, that we come across regularly, but often fail to recognise due to ignorance. 26 teams participated in the event and two teams were adjudged winner and runners up respectively.

The day was also marked with participation in the Innovative Idea Competition and slogan writing competition where the innovation based on a **‘mobile robot capable of sensing hazardous gas for military application’** was adjudged the winner.

Signing of MoA with NRDC

In a landmark initiative for commercializing the homegrown technologies, Tezpur University has come in agreement with National Research Development Corporation (NRDC), an enterprise of the Department of Scientific & Industrial Research, Ministry of Science & Technology, Govt. of India, to develop fruitful collaborations with suitable industry partners and commercialise promising technologies developed in the campus. A Memorandum of Agreement has been signed in this regard on 02.06.2018 in the University between Tezpur University and NRDC in presence of key functionaries. Dr. Biren Das, Registrar, TU and Dr. H. Purushotham, CMD, NRDC signed the documents on behalf of their respective organisations. This initiative is believed to accelerate the process of rolling out of the home-grown technologies to the market, that will make these available to the society, beyond the university campus.

IPR Policy adopted for the University

With Tezpur University making way to various national and international rankings, it has become well known how IPRs owned by any institution add weightage to such evaluations. Therefore, it is strategically important that certain guidelines are laid down for the fraternity to follow while dealing with Intellectual Properties created within the University.

Understanding this, the IPR policy of Tezpur University has been approved and adopted in Tezpur University. Henceforth, the university has joined the rare league of institutions within the country, which has an institutional IPR policy.

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Tezpur University

Intellectual Property Rights Cell

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Activity 2019-20

IPR workshop for the Muga weavers and cultivators

A workshop was conducted at Harhi College on 24 the October, 2019 on '**Effective use of Geographical Indication and Trademark in the market for handloom industry**'. This workshop was the culminating workshop in the series of the earlier two IPR workshops held in and around the Muga Cultivation Hub of Assam, i.e. Lakhimpur. These workshops were meant for empowering the Muga Cultivators to face competitive market forces with renewed confidence by using various IPR tools. This time, going a step ahead, the workshop was organised jointly with the MSMEDi Guwahati to acquaint the cultivators and weavers about various Govt. schemes to help them grow and establish as independent entrepreneurs. The workshop was also supported by GI Registry Chennai for its effort to bring this unorganised sector and the workers into a more organised form. The participants, mostly the Muga weavers and cultivators from Gobindapur-Dhokuakhana areas, applauded the persistent efforts put in by the TUIPR Cell to make the intellectual property of the community... Muga silk ...unadulterated entity in the market and increase the visibility for the interested buyers across the country and beyond.

Officials from MSMEDi Guwahati and TUIPR Cell, Tezpur University participated in the workshop as resource persons. These IPR workshops have resulted in more than 300 Authorised Users for the GI- Muga Silk so far. The authorised users from Assam are now empowered to certify Muga Silk's purity and authenticity by using the GI mark and logo on the finished products. Chief Guest of the programme, the SDO, Gobindapur, applauded the initiative of the TUIPR Cell to reach out and support the grassroot people. In his concluding remarks he reiterated the need to protect the intellectual properties associated with the traditional goods.

IP Day 2019

Intellectual Property Day was celebrated in Tezpur University on April 26 under the flagship of TUIPR Cell with an array of programmes that were meant to focus on encouraging innovation and creativity. This year the theme of the celebrations was centred around "Reach for Gold: IP and Sports". Together with the annual IP competitions like quiz, best innovative idea competition and slogan writing competition, the IP Day was celebrated with an IP lecture series where deliberations were held on the latest innovations in various fields from the invited speakers. Deliberations from experienced industry executives like Asian Paints, Playin App Technologies limited, Panacea Medical Technologies Pvt. Ltd. and officials from institutions like Guwahati Medical College & Hospital, highlighted on how some of the home grown technologies are finding acceptability and making their way not only in the Indian market replacing highly competitive MNC products, but are also being exported to other countries. One of the heart-warming examples was discussed after the presentation on physics of applying radiation therapy to cancer patients, which was on indigenous technologies developed in the form of "Bhabhatron -II" which is a Cobalt Therapy Machine made by BARC in collaboration with Board of Radiation and Isotope Technology & SAMEER and licensed to Panacea Medical Technologies, which shall replace the earlier imported Radiation Therapy Machines for treating Cancer Patients in many Govt. Hospitals due to its precision and effectiveness. This is believed to bring down the price of the treatment of cancer patients as these are manufactured with less than one-tenth of the price of the imported machine. Similarly, the evolution of the Asian Paints from manufacturing mere exterior paints to creating paints that are weather proof, purifies indoor air, kills bacteria etc. took the participants through an interesting journey of self-evolution fuelled by the zeal to remain market leaders. During the interactive sessions, the participants realized that remaining articulated and self-motivated is key to innovate constantly. One presentation on the perspective towards Sports and Robotics at Tezpur University by one of the inhouse faculty member showcased the AI capabilities of the University to the invited guests and the participants. Another presentation that was dedicated to the theme of the celebrations was from the app developer enterprise, Playin App Technologies, which showcased an integrated platform to connect all the sports related services and utilities through one mobile based application.

120 students from University and nearby colleges/institutes participated in the same.

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- National Webinar On Intellectual Property as a Strategic Tool for Regional Development organized at Dhakuakhana College, Assam (http://www.tezu.ernet.in/tuipr_cell/pdf/Report_Dhakuakhana_College.pdf)
- State level Webinar on Intellectual Property Rights in the context of Sustainable Development in collaboration with IQAC, Mangaldai College, Assam on 5th

Activity 2021-2022

IP Day 2021 on 26th April, 2021

Human endeavor to struggle and survive unfavorable situations makes our race superior and stronger. Thus, Intellectual Property Day-2021 was organized amidst the second wave of Covid-19 pandemic to celebrate the spirit of innovation for survival of human beings on 26th April, 2021. This time the entire programme was organized on online mode. Even with limitations, the Cell organized the IP Day lectures, IP Quiz and Innovative Idea competition with a good number of participants from Tezpur University and nearby institutions. The IP Day lectures were delivered by two eminent experts of the field Dr. A. K. Garg, Senior Director, Ministry of Electronics and Information Technology, Govt. of India and Shri Sanjay Bhattacharya, Deputy Controller of Patent and Design, Indian Patent Office, Kolkata, whose lectures on the event had inspired the participants of the program. The theme of this year's celebration was IP & SMEs: Taking your ideas to market. This year entire discussion was oriented on the critical role of small and medium-sized enterprises (SMEs) in the economy and how they can use intellectual property (IP) rights to build stronger, more competitive and resilient businesses. 50 participants from within and outside the university participated in the programme.

National Webinar on Intellectual Property as a Strategic Tool for Regional Development at Dhakuakhana College on 18th June, 2021

'I have an Innovation... How do I Protect it in the market???', on this note, Department of Political Science & IQAC Dhakuakhana College in collaboration with the DPIIT IPR Chair, Intellectual Property Right Cell, Tezpur University, organized a National Level Webinar on a a contemporary issue, 'Intellectual Property As A Strategic Tool For Regional Development'. 'Why is IP important to us?' on this note the Key Note Speaker of the Webinar, Prof. Pritam Deb, DPIIT Chair Professor, IPR Cell, Tezpur University, highlighted how in today's globally competitive environment, Intellectual Property (IP) has placed itself on a pedestal in the context of economic growth and is becoming increasingly important. It is also tinted how the increasing significance of intangible assets in the global economy is forcing business organizations to actively manage their IPR as a key driver for building and sustaining their competitive advantage and achieving superior performance and development. Citing many instances from local to global, the Speaker also focused on how Intellectual Property Rights (IPR) can become the fuel that powers the engine of prosperity, fostering invention and innovation not only in Assam but also in the entire North Eastern region. 156 participants from within and outside the state joined in the programme.

State Level Webinar on Intellectual Property Right in the context of sustainable development at Mangaldai College on 5th July, 2021

On 5th July 2021, a state level webinar was organized by IQAC, Mangaldai College in collaboration with the DPIIT IPR Chair, Intellectual Property Right Cell, Tezpur University through the Google Meet Platform. The topic of the webinar was 'Intellectual Property Rights in the context of sustainable development'. The main aim of this webinar was to sensitize the students, faculties, academicians and scholars about IPRs, especially patent and copyrights and to guide in respect of patenting their inventions. The Resource Person, Prof. Pritam Deb, gives a detailed account of the issues related to IPR, such as industrial property related and copyright related aspects including patents, designs, trademarks and GIs. He highlighted on the need of IP protection by saying that it is important for new and exciting inventions to strengthen the economy, rewards for someone's work's etc. The resource person also highlighted on the debate of IPR in Covid-19 treatment regarding inventions of vaccines like Covishield by Bharat Biotech and Serum Institute of India. In his view IP rights are generally for the benefit of the larger society but there is a debate regarding parenting and its applicability for elite and poor sections of society. 56 participants from within and outside the college participated in the programme.

National Webinar on Intellectual Property Rights in Contemporary World, C.K.B. College, Teok on 5th July, 2021

The Dept. of Commerce and Dept. of Economics, C.K.B. College jointly organised a National Webinar in collaboration with Intellectual Property Rights Cell, Tezpur University & IQAC, C.K.B. College, Teok on "INTELLECTUAL PROPERTY RIGHTS IN CONTEMPORARY WORLD" on 5th July, 2021. The Resource Person for the webinar was Dr. Juri Borbora Saikia, Research Assistant, IPR Cell, Tezpur University. The webinar was specially targeted to the students, scholars, academicians and others. The aim was to impart knowledge about intellectual property rights (IPR) and what role do they play in the contemporary world. 37 participants from within and outside the university participated in the same.

National Level Webinar on Need of Intellectual Property Rights for sustainable development at LOKD College, Dhekiajuli on 2nd August, 2021

A National Level Webinar on the topic "Need of Intellectual Property Rights for sustainable development" was organized on 2nd August, 2021 by LOKD College, Dhekiajuli, Assam in collaboration with the DPIIT IPR Chair, Intellectual Property Right Cell, Tezpur University through the Google Meet Platform. The Resource Person, Prof. P. Deb, delivered lecture on this emergent contemporary topic. He elaborated almost all the horizon of Intellectual Property in the contemporary world for a sustainable societal development. The participants of the program were mostly Faculty members, who are expected to become the ambassador of IP knowledge for disseminating among their students and fellow colleagues.

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GENERAL AGREEMENT
of ACADEMIC and CULTURAL COOPERATION
between

AVANS UNIVERSITY of APPLIED SCIENCES (Breda, Tilburg & 's-Hertogenbosch, THE NETHERLANDS)

and

DEPARTMENT OF BUSINESS ADMINISTRATION, TEZPUR UNIVERSITY (Assam, INDIA)

This Agreement is made between Avans University of Applied Sciences, The Netherlands, hereafter referred to as AVANS and the Tezpur University, hereafter referred to as TU. This Agreement specifically involves the academic cooperation between AVANS' School of International Studies, hereafter referred to as ASIS and TU's Department of Business Administration, hereafter referred to as DBA.

RECITALS

A. Cordial relations exist between AVANS and TU;

And

B. AVANS and TU, hereafter referred to as "both parties" aim to **continue certain erasmus programmes** useful to the respective educational institutions. The exchange programmes will primarily promote and facilitate the exchange of students and staff members and may also encourage the development of other educational programmes of mutual interest. This Agreement will expand learning opportunities for **students from AVANS and TU**.

TERMS

In continuing the relationship already established and for valuable consideration, both parties agree as follows:

1.0 Purpose of the Agreement Definitions.

1.1 The primary objective of this Agreement is to continue cooperative efforts between AVANS and TU to effect the exchange of students and (academic) staff and the exchange of academic and research information between the two Institutions.

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Registrar
Tezpur University
Tezpur-784028, Assam

1.2 "Home Institution" is a party to this Agreement that sends its students and/or staff to the other party for the purposes of study, teaching or research.

1.3 "Host Institution" is a party to this Agreement that accepts the visiting students and/or staff from the other party for study, teaching or research.

1.4 The academic programmes of AVANS offered by its University enterprise AVANS-plus are not included in this Agreement.

1.5 The academic programmes of other departments/centres/cells offered by TU are not included in this Agreement.

2.0 Term, Renewal, and Termination.

2.1 This Agreement shall take effect on the first (1st) day of January, 2017, and shall continue until the last (31st) day of December, 2026, unless terminated as provided below.

2.2 If either party fails to follow the terms and conditions of this Agreement, the other party has the right to terminate this Agreement. Termination shall be made only after consultation between the legal representatives of both parties and/or the deans of the involved departments/schools/faculties of both parties as described in paragraph 14.0 below.

2.3 In the event that termination occurs during but before completion of student(s) or staff exchanges, the involved exchange student(s) and/or staff shall be allowed to complete the exchange in accordance with the provisions of this Agreement.

3.0 Exchange of Information and Assistance

3.1 To facilitate compliance with local accreditation processes and to assess the validity of waivers of credits obtained at the Home or Host Institution, both parties will mutually comply with any request for information regarding content, quality and delivery of the involved study programmes.

3.2 Both parties agree to provide to each other all relevant current catalogues, brochures and other information on a regular basis in order to provide reliable and comprehensive information to potential visiting students and staff members.

3.3 Both parties will assist visiting students and/or staff with registration, living arrangements, and other needs, within reasonable limits.

3.4 Both parties are to be responsible for the certification of academic standing for potential exchange students and/or staff to our respective institutions.

3.5 Within four weeks after completion of the exchange, comprehensive transcripts of records and if applicable other assessments of student work will be sent to the student's Home Institution at the student's request, unless a student has failed to

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settle his/her account, in which case the transcript may be withheld at the discretion of the Host Institution.

4.0 Exchange of (Academic) Staff Members.

4.1 Prior to any exchange of staff, academic officials of both parties shall annually agree upon the specific staff member(s) to be exchanged, the specific courses to be taught and other duties to be performed by the exchanged staff, the specific dates of the exchange(s), ranging from short-term visit(s) to one year visit(s), and the financial considerations pertinent to the exchange(s), if any. This is the business of the involved academic departments/schools/faculties and will be communicated by their respective representatives as part of the intended cooperation process.

4.2 The Home Institution shall pay all salaries, benefits, insurance coverage, travel expenses, all withholdings required for tax and other purposes and any other remuneration and expenses to which the exchanged staff may be subject or entitled. All such exchanged staff members shall during all times of the exchange, remain employees only of the Home Institution. Nevertheless, the personnel of each party while on the premises of the other shall at all times be subject to the rules and regulations adopted by the Host Institution with respect to the conduct of its own employees and be accorded such privileges as staff members of the Host Institution.

4.3 The Host Institution will assist in arranging housing for the exchanged staff and if applicable their families within reasonable limits.

5.0 Enrolment of AVANS and TU Exchange Students

5.1 Prior to any exchange of students, both parties shall annually agree in writing upon specified general conditions and the number of students to be exchanged. This is the business of the involved academic departments/schools/faculties concerned and will be communicated by their respective representatives as part of the intended cooperation process. The annual maximum of exchange students in total is ten (10) per institution.
For a complete list of English taught 30 ECTS exchange programmes available at the respective Avans Schools it refers to:
http://www.avans.nl/international/programs/programfinder/erasmus-exchange.[1]
The courses offered in a particular semester of the MBA program will be available for the exchange students.

5.2 Exchange students must only pay tuition fees at the Home Institution but must apply for admission to both AVANS and TU. Both parties will advise and support

[1] Please note that this URL links to all available Exchange programmes at AVANS. Under this Agreement, only the Exchange Programmes of ASIS are available for the TU students.

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the exchange students in the respective admission procedures. It is the prerogative of each institution to make final decisions on admission to its own institution.

5.3 Both parties agree that imbalances of student exchanges will be compensated by exchanges in the consecutive years or may be approved and accepted in writing, prior to the imbalanced exchanges, by the Host Institution.

5.4 Both parties agree that exchange students are only eligible to receive academic credit at the Home Institution for, by the Home Institution prior approved course units that have been successfully completed and for which pass marks have been obtained at the Host Institution. Solely on the basis of an exchange period, exchange students are not eligible to pursue a degree at the Host Institution.

5.5 Year 3 & 4 undergraduate students of Avans and MBA 2nd semester students of DBA, TU are considered for the exchange program. The exchange program will offered during both semesters by both universities and participating students will chose their courses as per what is available during the relevant semester.

5.6 ASIS will offer the exchange students of DBA, TU the opportunity to complete a minimum of 20 credits and as per TU regulations, the visiting students will not be obliged to register for more than a maximum of 24 credits during the exchange semester. Once the students complete all the academic requirements of the visiting university the credits earned will be transferred to the parent university of the student. TU will offer ASIS students all credit courses as would be available to a local student of the MBA programme in TU. ASIS will offer TU students level 4 courses.

5.7 If applicable, it is the responsibility of the exchange students to acquire their own internships and, concerning the academic standards, comply with the applicable exam regulations of the degree awarding institution. Both parties mutually agree to certify the academic standing of the students involved in internships in each other's geographic area and assist internship students in acquiring the appropriate work and visa documents within reasonable limits. Decisions regarding the awarding of academic credit for student internships will be the responsibility of the degree awarding institution.

6.0 Housing and Financial Matters for Exchange Students

6.1 Financial responsibility including, but not limited to, tuition fees *(to be paid in the Home institution),* insurances, board, room, books, travel and all other incidental fees and expenses shall be the responsibility of the individual student. Each institution shall provide advisory and counselling assistance on these matters.

6.2 Exchange students are eligible for university housing under the same conditions as other students at the Host Institution.

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7.0 Insurances for Exchange Students and Staff.

Both parties will ensure that their exchange students and staff have, at their own cost or if applicable at the Home Institutions' expense, an applicable basic health insurance, personal accident insurance and private third-party insurance, for personal bodily injury and wrongful death claims and for property damage liability. As secondary cover, each party has concluded a group third-party liability for its students and staff members. Evidence of such insurance or the acceptable alternative shall be provided to the other party upon request.

8.0 Language

8.1 The working language for this Agreement and all academic activities that result from this Agreement is English. This includes all teaching activities and exams. Each specific exchange student and staff member will be properly informed by both parties about the applicable language of instruction. Besides from specific other language courses, the working language for international exchange students at AVANS is English and at TU is English too.

8.2 Language requirements for international exchange students at AVANS: To maintain high standard levels of English language competences in the international classrooms and in compliance with Dutch national guidelines, AVANS only allows foreign students to exchange if they master English at, at least, an IELTS overall academic band score of 6.0 or equivalent. In addition to this prerequisite, Avans School of International Studies only allows foreign students to exchange if they master English at, at least, an IELTS overall academic band score of 6.5 or equivalent: (TOEFL 630 [PBT] or 100[IBT], TOEIC 720 or European Framework B1+/B2). It is the responsible of the Home Institution to only allow students to exchange who meet these language proficiency requirements. Students with an undergraduate and/or graduate curriculum that is entirely taught in English automatically qualify for an exchange.

8.3 Language requirements for international exchange students at TU is same as above.

9.0 National Laws, Academic and Disciplinary Rules and Regulations Compliance.

Both parties agree to comply to all applicable national laws. The academic requirements of both institutions shall also be mutually respected and all current, applicable policies covering matters of academic responsibility and social discipline, including institutional rules, regulations and code of conducts, will carefully be followed. Exchange staff and students at the Host Institution are required to comply to all national laws and active rules and regulations of the Host Institution.

10.0 Non-Discrimination.

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Tezpur-784028, Assam

The parties agree to comply with all national and local rules, regulations, executive orders, and laws forbidding unlawful discrimination which AVANS and TU promulgates and to which AVANS and TU are subject.

11.0 Dutch Code of Conduct.

AVANS is by Dutch law committed to act according the Dutch ***Code of Conduct with respect to International Students in Dutch Higher Education.*** AVANS expects all his international partners to also act according to this Code of Conduct. This Agreement will be dissolved immediately if the exchange programme does not comply with the active Code of Conduct. The active Dutch Code of Conduct is published at: http://www.internationalstudy.nl/en/.

12.0 Relationship of the Parties.

This Agreement shall not be construed to create a relationship of partners, brokers, employees, servants or agents as between the parties. The parties to this Agreement are acting as independent contractors.

13.0 Use of Institutions' Name: Advertising and Publicity.

Neither party shall use the other institution's name, or any name that is likely to suggest that it is related to the other institution, in any advertising, promotion or sales literature without first obtaining the written consent of the other institution.

14.0 Disputes & Termination of this Agreement.

If any controversy or dispute should arise between the parties with respect to the Agreement or performance thereunder, such controversy or dispute shall be submitted to the legal representatives of both parties or the involved deans of AVANS and TU who shall endeavour to find an amicable settlement of such dispute within thirty (30) days (or such longer period as may be mutually agreed upon) of submission of the matter to them. Within thirty (30) days following such consultation the withdrawing party may, by notice in writing to the other, terminate the Agreement effective on the 60th day following the receipt of such notice. If, and only if, the dispute relates to non-payment of monies aimed to be due, does a party have the right to bring an action therefore in a competent court of law.

15.0 Joint Research Projects.

Before any activity may commence on any joint research project, it shall be mutually agreed in writing that any publication resulting therefrom shall be credited to previously agreed-upon author(s), provided that each of the person(s) participate in the project.

Intellectual Property and Research Protocol: should any cooperation result in any potential for intellectual property, the parties shall immediately meet through designated representatives and seek an equitable and fair understanding as to ownership and other property of interest that may arise. Any such discussions shall at all times strive to preserve a harmonious and continuing relationship between the parties.

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Tezpur-784028, Assam

All research and other activities conducted under this Agreement must be conducted in accordance with the laws, rules, and regulations applicable to each institution.

Before commencement of any mutual research cooperation, both parties will agree in detailed writing on all content, legal and financial aspects of such cooperation.

16.0 Waiver.

A waiver of any breach of any provision of this Agreement shall not be construed as a continuing waiver of said breach or a waiver of any other breaches of the same or other provisions of this Agreement.

17.0 Non-Assignment.

This Agreement may not be assigned or changed by either party without the advance written consent of the other. This Agreement shall be binding upon the heirs, personal representatives, successors, and permitted assigns of both parties.

18.0 Notices.

Administration of this Agreement shall be the responsibility of the Office of the Executive Board at Avans University of Applied Sciences and the Office of the Registrar at Tezpur University. Any additions, changes or deletions must be approved by the legal representatives at both universities. All notices shall be in writing and shall be directed to these individuals as follows:

To Avans University of Applied sciences: Avans University of Applied Sciences, Executive Board, P.O. Box 1310, NL-5004 BH TILBURG, The Netherlands; T: +(31) 88 525 8484; E: collegevanbestuur@avans.nl

To Tezpur University : Tezpur University, Registrar, P.O. Napaam, 784028, Tezpur, India , T: + (91) 3712 267003 E: registrartu@tezu.ernet.in

Both parties will assign contact person(s) and/or administrative office(s) that are responsible for the execution of this Agreement:

Contact person(s) and/or administrative office(s):

Contact for AVANS: Backoffice ASIS, Avans University of Applied Sciences, Avans School of International Studies, P.O. Box 90.116, NL-4800RA BREDA, The Netherlands. T: +31 88 525 7630; E: backoffice.ASIS@avans.nl www.avans.nl

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Warranty: Each of the parties warrants it has the legal power to enter into this Agreement.

In accordance we sign four (4) copies of this General Understanding Agreement of Academic and Cultural Cooperation, in English, all contents being identical and valid.

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drs. P.L.A. (Paul) Rüpp, President

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Place and date:

Tilburg, 9 november 2017

Tezpur University
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GOVERNMENT OF INDIA
MINISTRY OF SCIENCE & TECHNOLOGY
DEPARTMENT OF BIOTECHNOLOGY

Block 2, 6-8th Floors
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Dated:31/03/2021

ORDER

Sanction of the President is hereby accorded, under Rule 18 of the Delegation of Financial Powers Rules ,1978 , for the implementation of the project entitled: "**Exploiting chemical ecology for IPM: Deciphering the phyto-semiochemicals involved in Insect-Plant interactions of major crop pests of North East Region-India**" for a period of 3 Year 0 Month at a total cost of Rs. **47579782** (Rupees Four Crores Seventy Five Lakhs Seventy Nine Thousand Seven Hundred and Eighty Two Only) on the terms and conditions detailed here under:-

2 The Project :

2.1 Title : **"Exploiting chemical ecology for IPM: Deciphering the phyto-semiochemicals involved in Insect-Plant interactions of major crop pests of North East Region-India"**

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PROOFS OF SOME CONJECTURES OF Z. -H. SUN ON RELATIONS BETWEEN SUMS OF SQUARES AND SUMS OF TRIANGULAR NUMBERS

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Let $N(a,b,c,d;n)$ be the number of representations of n as $ax^2+by^2+cz^2+dw^2$ and $T(a,b,c,d;n)$ be the number of representations of n as $a\frac{X(X+1)}{2}+b\frac{Y(Y+1)}{2}+c\frac{Z(Z+1)}{2}+d\frac{W(W+1)}{2}$, where a,b,c,d are positive integers, n,X,Y,Z,W are nonnegative integers, and x,y,z,w are integers. Recently, Z.-H. Sun found many relations between $N(a,b,c,d;n)$ and $T(a,b,c,d;n)$ and conjectured 23 more relations. Yao proved five of Sun's conjectures by using (p,k)-parametrization of theta functions and stated that six more could be proved by using the same method. More recently, Sun himself confirmed two more conjectures by proving a general result whereas Xia and Zhong proved three more conjectures of Sun by employing theta function identities. In this paper, we prove the remaining seven conjectures. Six are proved by employing Ramanujan's theta function identities and one is proved by elementary techniques.

Key words : Sum of squares; sum of triangular numbers; Ramanujan's theta function; representation of quaternary quadratic forms.

2010 Mathematics Subject Classification : 11D85, 11E20, 11E25, 33E20

1. INTRODUCTION

Let $\mathbb{N}^+$, $\mathbb{N}$ and $\mathbb{Z}$ denote the set of positive integers, the set of nonnegative integers, and the set of integers, respectively. Let $\mathbb{N}^4=\mathbb{N}\times\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and $\mathbb{Z}^4=\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$. For $a,b,c,d\in\mathbb{N}^+$ and

$n \in \mathbb{N}$, define

$$N(a,b,c,d;n) := \left|\left\{(x,y,z,w) \in \mathbb{Z}^4 : ax^2+by^2+cz^2+dw^2 = n\right\}\right|$$

and

$$T(a,b,c,d;n) := \left|\left\{(x,y,z,w) \in \mathbb{N}^4 : a\frac{x(x+1)}{2}+b\frac{y(y+1)}{2}+c\frac{z(z+1)}{2}+d\frac{w(w+1)}{2} = n\right\}\right|,$$

where we take $N(a,b,c,d;0) = T(a,b,c,d;0) = 1$.

Jacobi and Legendre proved that

$$N(1,1,1,1;n) = 8\sum_{d|n,4\nmid d} d$$

and

$$T(1,1,1,1;n) = \sigma(2n+1),$$

respectively, where $\sigma(n) = \sum_{d|n} d$.

For further formulas for $N(a,b,c,d;n)$ and $T(a,b,c,d;n)$ for certain values of $a,b,c,d \in \mathbb{N}^+$, we refer to Dickson's historical comments [11], Cooper's papers [12, 13], Alaca's papers [2, 3], papers [4-8] by Alaca, Alaca, Lemire and Williams, Williams' papers [20, 21] and book [22], and papers [18, 19] by Wang and Sun.

Finding relations between $N(a,b,c,d;n)$ and $T(a,b,c,d;n)$ is another interesting area of research. For $a,b,c,d \in \mathbb{N}^+$ with $5 \le a+b+c+d \le 8$, let

$$C(a,b,c,d) = 16 + 4i_1(i_1-1)i_2 + 8i_1i_3,$$

where i_j is the number of elements in $\{a,b,c,d\}$ which are equal to j. When $5 \le a+b+c+d \le 7$, Adiga, Cooper and Han [1] proved that

$$C(a,b,c,d)T(a,b,c,d;n) = N(a,b,c,d;8n+a+b+c+d).$$

When $a+b+c+d = 8$, Baruah, Cooper and Hirschhorn [9] proved that

$$C(a,b,c,d)T(a,b,c,d;n) = N(a,b,c,d;8n+8) - N(a,b,c,d;2n+2).$$

Wang and Sun [18, 19] and Sun [16] discovered several new relations between $N(a,b,c,d;n)$ and $T(a,b,c,d;n)$. In particular, in [16], Sun posed 23 conjectures (Conjecture 2.1-Conjecture 2.23) stating some relations between $N(a,b,c,d;n)$ and $T(a,b,c,d;n)$. Five of the conjectures (Conjectures

2.2, 2.3, 2.4, 2.7, and 2.11) were proved by Yao [24] by utilizing (p,k)-parametrization of theta functions. In Section 8 of [24], Yao also remarked that Conjectures 2.1, 2.14, 2.15, 2.19, 2.20, and 2.21 can also be proved in a similar way. In fact, in another recent paper, Yao [25] proved some general relations from which Conjectures 2.1, 2.15, 2.19, 2.20, and 2.21 follow as special cases. Recently, Sun [17] himself confirmed Conjectures 2.2 and 2.6-2.8 by proving the following general result.

Let $m\equiv 1 \pmod 4$ or $m\equiv 4 \pmod 8$. Suppose that there is an odd prime divisor p of m such that $\left(\frac{4n+5}{p}\right)=-1$, where $\left(\frac{a}{p}\right)$ is the Legendre symbol. Then

$$32T(1,1,8,m;n)=N(1,1,8,m;8n+10+m).$$

Most recently, Xia and Zhong [23] proved Conjectures 2.18, 2.22, and 2.23 by using theta function identities. In this paper, we prove the remaining seven conjectures, namely, Conjectures 2.5, 2.9, 2.10, 2.12, 2.13, 2.16, and 2.17, of Sun [16]. The main results are presented in the following seven theorems.

***Theorem* 1.1** — *(Conjecture 2.9 in Sun [16]). Let $n\in\mathbb{N}^+$ with $n\equiv 0,3,5,6,7 \pmod{11}$. Then*

$$48T(1,1,4,11;n)=N(1,1,4,11;8n+17). \tag{1.1}$$

***Theorem* 1.2** — *(Conjecture 2.10 in Sun [16]). Let $n\in\mathbb{N}^+$ with $n\equiv 0,1,2,4,7 \pmod{11}$. Then*

$$48T(1,1,2,22;n)=N(1,1,2,22;8n+26). \tag{1.2}$$

***Theorem* 1.3** — *(Conjecture 2.12 in Sun [16]). Let $n\in\mathbb{N}^+$ with $n\equiv 1 \pmod 4$. Then*

$$32T(3,5,20,32;n)=N(3,5,20,32;8n+60)-4N(3,5,20,32;2n+15). \tag{1.3}$$

***Theorem* 1.4** — *(Conjecture 2.13 in Sun [16]). Let $n\in\mathbb{N}^+$ with $n\equiv 1 \pmod 4$. Then*

$$24T(1,6,15,18;n)=N(1,6,15,18;8n+40)-3N(1,6,15,18;2n+10). \tag{1.4}$$

***Theorem* 1.5** — *(Conjecture 2.16 in Sun [16]). Let $n\in\mathbb{N}^+$ with $n\equiv 0 \pmod 4$. Then*

$$24T(1,7,10,30;n)=N(1,7,10,30;8n+48)-3N(1,7,10,30;2n+12). \tag{1.5}$$

***Theorem* 1.6** — *(Conjecture 2.17 in Sun [16]). Let $n\in\mathbb{N}^+$ with $n\equiv 3 \pmod 4$. Then*

$$24T(1,10,15,30;n)=N(1,10,15,30;8n+56)-3N(1,10,15,30;2n+14). \tag{1.6}$$

***Theorem* 1.7** — *(Conjecture 2.5 in Sun [16]). Let $n \in \mathbb{N}^+$ with $n \equiv 0, 2 \pmod 8$. Then*

$$4T(1,2,4,17;n) = N(1,2,4,17;n+3), \tag{1.7}$$

We employ elementary dissections of Ramanujan's theta functions to prove the first six theorems. We could not effectively use that method to prove the last theorem. So we prove that theorem by employing some elementary techniques. It is to be noted that in [17], Sun also listed seven more conjectures. Five are on ternary quadratic forms and two are on quaternary quadratic forms. The five conjectures on the ternary case are proved in [15] by an elementary method. All the seven conjectures in [17] are proved in [10] by using Ramanujan's theta functions.

We organize the paper in the following way. In the next section, we present the background material on Ramanujan's theta functions and some useful lemmas. In Sections 3-9, we prove Theorems 1.1-1.7, respectively.

2. Ramanujan's Theta Functions and Some Useful Lemmas

Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \quad |ab| < 1.$$

Three special cases of $f(a,b)$ are

$$\phi(q) := f(q,q) = \sum_{k=-\infty}^{\infty} q^{k^2},$$

$$\psi(q) := f(q,q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2},$$

$$f(-q) := f(-q,-q^2) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2}.$$

It is clear from the definitions of ϕ and ψ that

$$\sum_{n=0}^{\infty} N(a,b,c,d;n) q^n = \phi(q^a)\phi(q^b)\phi(q^c)\phi(q^d)$$

and

$$\sum_{n=0}^{\infty} T(a,b,c,d;n) q^n = \psi(q^a)\psi(q^b)\psi(q^c)\psi(q^d).$$

In the following lemma, we record some well-known 2-dissections and identities from Berndt's book [11, pp. 39, 40, 49, 114, and 115]. The Identities (2.1), (2.2), (2.5)-(2.7) can also be found in a recent book by Hirschhorn [14, Eqs. (1.9.4), (1.10.1), (1.7.1), (10.7.3), and (10.7.6)], which contains many other interesting results.

Lemma 2.1 — We have

$$\phi(q) = \phi(q^4) + 2q\psi(q^8), \tag{2.1}$$

$$\phi(q)^2 = \phi(q^2)^2 + 4q\psi(q^4)^2, \tag{2.2}$$

$$\phi(q)\psi(q^2) = \psi(q)^2, \tag{2.3}$$

$$\psi(q) = f(q^6, q^{10}) + qf(q^2, q^{14}), \tag{2.4}$$

$$f(-q)^3 = \phi(-q)^2\psi(q) = \sum_{k=0}^{\infty}(-1)^k(2k+1)q^{k(k+1)/2}, \tag{2.5}$$

$$\phi(-q)^2 f(-q) = \sum_{k=-\infty}^{\infty}(6k+1)q^{k(3k+1)/2}, \tag{2.6}$$

and

$$\psi(q^2)f(-q)^2 = \sum_{k=-\infty}^{\infty}(3k+1)q^{k(3k+2)}. \tag{2.7}$$

Some more useful 2-dissections are given in the following lemma.

Lemma 2.3 — We have

$$\phi(q)\phi(q^3) = \phi(q^4)\phi(q^{12}) + 2q\psi(q^2)\psi(q^6) + 4q^4\psi(q^8)\psi(q^{24}), \tag{2.8}$$

$$\psi(q)\psi(q^3) = \psi(q^4)\phi(q^6) + q\phi(q^2)\psi(q^{12}), \tag{2.9}$$

$$\psi(q)\psi(q^7) = \psi(q^8)\phi(q^{28}) + q\psi(q^2)\psi(q^{14}) + q^6\phi(q^4)\psi(q^{56}), \tag{2.10}$$

$$\psi(q^3)\psi(q^5) = \psi(q^8)\phi(q^{60}) + q^3\psi(q^2)\psi(q^{30}) + q^{14}\phi(q^4)\psi(q^{120}), \tag{2.11}$$

$$\phi(q^3)\phi(q^5) = \phi(-q^2)\phi(-q^{30}) + 2q^2\psi(q)\psi(q^{15}), \tag{2.12}$$

$$\begin{aligned}\phi(q)\phi(q^{15}) = {} & \phi(-q^6)\phi(-q^{10}) + 2q\psi(q^8)\phi(q^{60}) + 2q^4\psi(q^2)\psi(q^{30}) \\ & + 2q^{15}\phi(q^4)\psi(q^{120}).\end{aligned} \tag{2.13}$$

PROOF : Identity (2.8) follows by setting $(\mu, \nu) = (2, 1)$ in [11, p. 68, eq. (36.2)], and then employing (2.1), identities (2.9), (2.10) and (2.11) follow from (36.8) of [11, p. 69] by setting $(\mu, \nu) = (2, 1)$, $(\mu, \nu) = (4, 3)$ and $(\mu, \nu) = (4, 1)$, respectively. Identity (2.12) is in [11, p. 377, Entry 9(ii)]. Finally, identity (2.13) follows from [11, p. 377, Entry 9(ii)] and (2.11). □

Lemma 2.3 — We have

$$\phi(q^{22})\psi(q^4)+q^5\phi(q^2)\psi(q^{44})=f(-q)f(-q^{11})+q\psi(q)\psi(q^{11}). \tag{2.14}$$

PROOF : With the aid of (2.1) and the identity (see Berndt's book [11, p. 365, eq. (7.5)])

$$\phi(q)\phi(q^{11})-\phi(-q)\phi(-q^{11})=4qf(-q^2)f(-q^{22})+4q^3\psi(q^2)\psi(q^{22}),$$

we have

$$\begin{aligned}\phi(q^{44})\psi(q^8)+q^{10}\phi(q^4)\psi(q^{88}) &= \left(\frac{\phi(q^{11})+\phi(-q^{11})}{2}\right)\left(\frac{\phi(q)-\phi(-q)}{4q}\right)\\ &\quad+q^{10}\left(\frac{\phi(q)+\phi(-q)}{4q}\right)\left(\frac{\phi(q^{11})-\phi(-q^{11})}{2}\right)\\ &=\frac{1}{4q}\left(\phi(q)\phi(q^{11})-\phi(-q)\phi(-q^{11})\right)\\ &=\frac{1}{4q}\left(4qf(-q^2)f(-q^{22})+4q^3\psi(q^2)\psi(q^{22})\right)\\ &=f(-q^2)f(-q^{22})+q^2\psi(q^2)\psi(q^{22}).\end{aligned}$$

Replacing q^2 by q we arrive at (2.14). □

Lemma 2.4 — We have

$$\phi(q^{10})\psi(q^{12})+q\phi(q^6)\psi(q^{20})-f(q,q^7)f(q^{45},q^{75})-q^7f(q^3,q^5)f(q^{15},q^{105})=0. \tag{2.15}$$

PROOF : With the help of (2.4), we find that

$$\begin{aligned}&f(q^2,q^{14})f(q^{90},q^{150})+q^{14}f(q^6,q^{10})f(q^{30},q^{210})\\ &=\frac{1}{4q}\left(\psi(q)-\psi(-q)\right)\left(\psi(q^{15})+\psi(-q^{15})\right)+\frac{1}{2q}\left(\psi(q)+\psi(-q)\right)\left(\psi(q^{15})-\psi(-q^{15})\right)\\ &=\frac{1}{2q}\left(\psi(q)\psi(q^{15})-\psi(-q)\psi(-q^{15})\right).\end{aligned} \tag{2.16}$$

On the other hand, with the aid of (2.1) and (2.12), we find that

$$\begin{aligned}&\phi(q^{20})\psi(q^{24})+q^2\phi(q^{12})\psi(q^{40})\\ &=\frac{1}{8q^3}\left(\phi(q^5)+\phi(-q^5)\right)\left(\phi(q^3)-\phi(-q^3)\right)+\frac{1}{8q^3}\left(\phi(q^3)+\phi(-q^3)\right)\left(\phi(q^5)-\phi(-q^5)\right)\\ &=\frac{1}{4q^3}\left(\phi(q^3)\phi(q^5)-\phi(-q^3)\phi(-q^5)\right)\\ &=\frac{1}{2q}\left(\psi(q)\psi(q^{15})-\psi(-q)\psi(-q^{15})\right).\end{aligned} \tag{2.17}$$

From (2.16) and (2.17) , we arrive at

$$f(q^2,q^{14})f(q^{90},q^{150})+q^{14}f(q^6,q^{10})f(q^{30},q^{210}) =\phi(q^{20})\psi(q^{24})+q^2\phi(q^{12})\psi(q^{40}),$$

which is clearly equivalent to (2.15) with q^2 replaced by q. □

Lemma 2.5 —

$$\psi(q)\psi(q^{15})=\psi(q^6)\psi(q^{10})+q\phi(q^{20})\psi(q^{24})+q^3\phi(q^{12})\psi(q^{40}). \tag{2.18}$$

PROOF : Identity (2.18) easily follows from (2.17) and the identity

$$\psi(q)\psi(q^{15})+\psi(-q)\psi(-q^{15})=2\psi(q^6)\psi(q^{10}),$$

in [11, p. 377, Entry 9]. □

3. PROOF OF THEOREM 1.1

We have

$$\begin{aligned}\sum_{n=0}^{\infty}N(1,1,4,11;n)q^n &= \phi(q)^2\phi(q^4)\phi(q^{11})\\ &= \phi(q^4)\left(\phi(q^2)^2+4q\psi(q^4)^2\right)\left(\phi(q^{44})+2q^{11}\psi(q^{88})\right),\end{aligned}$$

from which we extract

$$\begin{aligned}\sum_{n=0}^{\infty}N(1,1,4,11;2n+1)q^n &= 4\phi(q^2)\phi(q^{22})\psi(q^2)^2+2q^5\phi(q^2)\phi(q)^2\psi(q^{44})\\ &= 4\phi(q^2)\phi(q^{22})\psi(q^2)^2+2q^5\phi(q^2)\psi(q^{44})\left(\phi(q^2)^2+4q\psi(q^4)^2\right),\end{aligned}$$

from which we further extract

$$\begin{aligned}\sum_{n=0}^{\infty}N(1,1,4,11;4n+1)q^n &= 4\psi(q)^2\phi(q)\phi(q^{11})+8q^3\psi(q^2)^2\phi(q)\psi(q^{22})\\ &= 4\phi(q)^2\phi(q^{11})\psi(q^2)+8q^3\phi(q)\psi(q^2)^2\psi(q^{22})\\ &= 4\left(\phi(q^2)^2+4q\psi(q^4)^2\right)\left(\phi(q^{44})+2q^{11}\psi(q^{88})\right)\psi(q^2)\\ &\quad+8q^3\left(\phi(q^4)+2q\psi(q^8)\right)\psi(q^2)^2\psi(q^{22}),\end{aligned}$$

where the second equality is due (2.3).

It follows that

$$\sum_{n=0}^{\infty} N(1,1,4,11;8n+1)q^n = 4\psi(q)\phi(q)^2\phi(q^{22}) + 32q^6\psi(q)\psi(q^2)^2\psi(q^{44}) + 16q^2\psi(q)^2\psi(q^4)\psi(q^{11}),$$

which becomes

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,1,4,11;8n+1)q^n - 16\sum_{n=0}^{\infty} T(1,1,4,11;n)q^{n+2}\\
&= 4\phi(q)^2\phi(q^{22})\psi(q) + 32q^6\psi(q)\psi(q^2)^2\psi(q^{44})\\
&= 4\left(\phi(-q)^2 + 8q\psi(q^4)^2\right)\phi(q^{22})\psi(q) + 32q^6\psi(q)\psi(q^2)^2\psi(q^{44})\\
&= 4\phi(-q)^2\phi(q^{22})\psi(q) + 32q\left(\phi(q^{22})\psi(q)\psi(q^4)^2 + q^5\psi(q)\psi(q^2)^2\psi(q^{44})\right)\\
&= 4\phi(-q)^2\phi(q^{22})\psi(q) + 32q\left(\phi(q^{22})\psi(q^4)^2 + q^5\phi(q^2)\psi(q^4)\psi(q^{44}))\right)\psi(q)\\
&= 4\phi(-q)^2\phi(q^{22})\psi(q) + 32q\left(\phi(q^{22})\psi(q^4) + q^5\phi(q^2)\psi(q^{44}))\right)\psi(q)\psi(q^4)\\
&= 4\phi(-q)^2\phi(q^{22})\psi(q) + 32q\left(f(-q)f(-q^{11}) + q\psi(q)\psi(q^{11})\right)\psi(q)\psi(q^4)\\
&= 4\phi(-q)^2\phi(q^{22})\psi(q) + 32q\psi(q^4)\psi(q)f(-q)f(-q^{11}) + 32q^2\psi(q)^2\psi(q^4)\psi(q^{11}),
\end{aligned}$$

and so,

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,1,4,11;8n+1)q^n - 48q^2\sum_{n=0}^{\infty} T(1,1,4,11;n)q^n\\
&= 4\phi(-q)^2\phi(q^{22})\psi(q) + 32q\psi(q^4)f(-q^2)^2f(-q^{11})\\
&= 4\phi(q^{22})f(-q)^3 + 32qf(-q^{11})\psi(q^4)f(-q^2)^2\\
&= 4\phi(q^{22})\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{(n^2+n)/2} + 32f(-q^{11})\sum_{n=-\infty}^{\infty}(3n+1)q^{6n^2+4n+1},
\end{aligned}$$

where the last equality is due to (2.5) and (2.7).

Now, it can be easily verified that $(n^2+n)/2 \equiv 0,1,3,4,6,$ or $10 \pmod{11}$ and $6n^2+4n+1 \equiv 0,1,3,4,6,$ or $10 \pmod{11}$. Therefore, extracting the terms involving q^n for $n \equiv 2,5,7,8,9 \pmod{11}$ in the above, we find that

$$\sum_{n=0}^{\infty} N(1,1,4,11;8n+1)q^n - 48\sum_{n=0}^{\infty} T(1,1,4,11;n)q^{n+2} = 0,$$

which readily implies that, for $n \equiv 0,3,5,6,7 \pmod{11}$,

$$N(1,1,4,11;8n+17) = 48T(1,1,4,11;n).$$

This completes the proof.

4. Proof of Theorem 1.2

We have

$$\sum_{n=0}^{\infty} N(1,1,2,22;n)q^n = \phi(q)^2\phi(q^2)\phi(q^{22})$$
$$= \left(\phi(q^2)^2 + 4q\psi(q^4)^2\right)\phi(q^2)\phi(q^{22}),$$

from which we extract

$$\sum_{n=0}^{\infty} N(1,1,2,22;2n)q^n = \phi(q)^3\phi(q^{11})$$
$$= \left(\phi(q^4) + 2q\psi(q^8)\right)^3\left(\phi(q^{44}) + 2q^{11}\psi(q^{88})\right).$$

It follows that

$$\sum_{n=0}^{\infty} N(1,1,2,22;8n+2)q^n = 6\phi(q)^2\psi(q^2)\phi(q^{11}) + 24q^3\phi(q)\psi(q^2)^2\psi(q^{22})$$
$$= 6\phi(q)\psi(q)^2\phi(q^{11}) + 24q^3\psi(q)^2\psi(q^2)\psi(q^{22}),$$

which implies that

$$\sum_{n=0}^{\infty} N(1,1,2,22;8n+2)q^n - 24\sum_{n=0}^{\infty} T(1,1,2,22;n)q^{n+3}$$
$$= 6\phi(q)\psi(q)^2\phi(q^{11})$$
$$= 6\psi(q)^2\phi(-q)\phi(-q^{11}) + 24q^3\psi(q)^2\psi(q^2)\psi(q^{22}) + 24q\psi(q)^2 f(-q^2)f(-q^{22}),$$

and so,

$$\sum_{n=0}^{\infty} N(1,1,2,22;8n+2)q^n - 48q^3\sum_{n=0}^{\infty} T(1,1,2,22;n)q^n$$
$$= 6\psi(q)^2\phi(-q)\phi(-q^{11}) + 24q\psi(q)^2 f(-q^2)f(-q^{22})$$
$$= 6f(-q^2)^3\phi(-q^{11}) + 24q\psi(q^2)f(q)^2 f(-q^{22})$$
$$= 6\phi(-q^{11})\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{n^2+n} + 24f(-q^{22})\sum_{n=-\infty}^{\infty}(3n+1)(-q)^{3n^2+2n+1}.$$

Since $n^2+n \equiv 0,1,2,6,8,9 \pmod{11}$ and $3n^2+2n+1 \equiv 0,1,2,6,8,9 \pmod{11}$, extracting the terms involving q^n for $n \equiv 3,4,5,7,10 \pmod{11}$ in the above, we find that

$$\sum_{n=0}^{\infty} N(1,1,2,22;8n+2)q^n - 48\sum_{n=0}^{\infty} T(1,1,2,22;n)q^{n+3} = 0.$$

Thus, for $n \equiv 0,1,2,4,7 \pmod{11}$, we have

$$N(1,1,2,22;8n+26) = 48T(1,1,2,22;n).$$

5. Proof of Theorem 1.3

We have

$$\begin{aligned}\sum_{n=0}^{\infty} N(3,5,20,32;n)q^n &= \phi(q^3)\phi(q^5)\phi(q^{20})\phi(q^{32}) \\ &= \left(\phi(q^{12})+2q^3\psi(q^{24})\right)\left(\phi(q^{20})+2q^5\psi(q^{40})\right)\phi(q^{20})\phi(q^{32}).\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{n=0}^{\infty} N(3,5,20,32;4n)q^n &= \left(\phi(q^3)\phi(q^5)+4q^2\psi(q^6)\psi(q^{10})\right)\phi(q^5)\phi(q^8) \\ &= \Bigg(\left(\phi(q^{12})+2q^3\psi(q^{24})\right)\left(\phi(q^{20})+2q^5\psi(q^{40})\right)+4q^2\psi(q^6)\psi(q^{10})\Bigg) \\ &\quad\times\left(\phi(q^{20})+2q^5\psi(q^{40})\right)\phi(q^8),\end{aligned}$$

from which we extract

$$\begin{aligned}&\sum_{n=0}^{\infty} N(3,5,20,32;8n+4)q^n \\ &= 2q\phi(q^4)\phi(q^{10})^2\psi(q^{12}) + 4q^2\phi(q^4)\phi(q^6)\phi(q^{10})\psi(q^{20}) + 8q^6\phi(q^4)\psi(q^{12})\psi(q^{20})^2 \\ &\quad + 8q^3\psi(q^3)\psi(q^5)\phi(q^4)\psi(q^{20}) \\ &= 2q\phi(q^4)\phi(q^{10})^2\psi(q^{12}) + 4q^2\phi(q^4)\phi(q^6)\phi(q^{10})\psi(q^{20}) + 8q^6\phi(q^4)\psi(q^{12})\psi(q^{20})^2 \\ &\quad + 8q^3\left(\psi(q^8)\phi(q^{60}) + q^3\psi(q^2)\psi(q^{30}) + q^{14}\phi(q^4)\psi(q^{120})\right)\phi(q^4)\psi(q^{20}).\end{aligned}$$

We further extract

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(3,5,20,32;16n+4)q^n\\
&= 4q\phi(q^2)\phi(q^3)\phi(q^5)\psi(q^{10}) + 8q^3\phi(q^2)\psi(q^6)\psi(q^{10})^2 + 8q^3\phi(q^2)\psi(q^{10})\psi(q)\psi(q^{15})\\
&= 4q\phi(q^2)\psi(q^{10})\left(\phi(q^{12}) + 2q^3\psi(q^{24})\right)\left(\phi(q^{20}) + 2q^5\psi(q^{40})\right) + 8q^3\phi(q^2)\psi(q^6)\psi(q^{10})^2\\
&\quad + 8q^3\phi(q^2)\psi(q^{10})\left(f(q^6,q^{10}) + qf(q^2,q^{14})\right)\left(f(q^{90},q^{150}) + q^{15}f(q^{30},q^{210})\right),
\end{aligned}$$

from which we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(3,5,20,32;32n+4)q^n\\
&= 8q^2\phi(q)\psi(q^5)\phi(q^{10})\psi(q^{12}) + 8q^3\phi(q)\psi(q^5)\phi(q^6)\psi(q^{20})\\
&\quad + 8q^2\phi(q)\psi(q^5)f(q,q^7)f(q^{45},q^{75}) + 8q^9\phi(q)\psi(q^5)f(q^3,q^5)f(q^{15},q^{105}).
\end{aligned} \tag{5.1}$$

Next, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} N(3,5,20,32;4n+1)q^n &= 2q\phi(q^3)\psi(q^{10})\phi(q^5)\phi(q^8)\\
&= 2q\left(\phi(q^{12}) + 2q^3\psi(q^{24})\right)\left(\phi(q^{20}) + 2q^5\psi(q^{40})\right)\phi(q^8)\psi(q^{10}),
\end{aligned}$$

from which it follows that

$$\sum_{n=0}^{\infty} N(3,5,20,32;8n+1)q^n = 4q^2\phi(q^4)\psi(q^5)\psi(q^{12})\phi(q^{10}) + 4q^3\phi(q^4)\psi(q^5)\phi(q^6)\psi(q^{20}). \tag{5.2}$$

Again,

$$\begin{aligned}
&\sum_{n=0}^{\infty} T(3,5,20,32;n)q^n\\
&= \psi(q^3)\psi(q^5)\psi(q^{20})\psi(q^{32})\\
&= \left(\psi(q^8)\phi(q^{60}) + q^3\psi(q^2)\psi(q^{30}) + q^{14}\phi(q^4)\psi(q^{120})\right)\psi(q^{20})\psi(q^{32}),
\end{aligned}$$

from which we extract

$$\begin{aligned}
\sum_{n=0}^{\infty} T(3,5,20,32;2n+1)q^n &= q\psi(q)\psi(q^{15})\psi(q^{10})\psi(q^{16})\\
&= q\big(\phi(-q^6)\phi(-q^{10}) + 2q\psi(q^8)\phi(q^{60}) + 2q^4\psi(q^2)\psi(q^{30})\\
&\quad + 2q^{15}\phi(q^4)\psi(q^{120})\big)\psi(q^{10})\psi(q^{16}),
\end{aligned}$$

from which we further extract

$$\sum_{n=0}^{\infty} T(3,5,20,32;4n+1)q^n = q\psi(q^5)\psi(q^8)f(q,q^7)f(q^{45},q^{75}) + q^8\psi(q^5)\psi(q^8)f(q^3,q^5)f(q^{15},q^{105}). \tag{5.3}$$

From (5.1), (5.2) and (5.3), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(3,5,20,32;32n+4)q^n - 4\sum_{n=0}^{\infty} N(3,5,20,32;8n+1)q^n \\
&- 32\sum_{n=0}^{\infty} T(3,5,20,32;4n+1)q^{n+2} \\
&= 8q^2\phi(q)\psi(q^5)\phi(q^{10})\psi(q^{12}) + 8q^3\phi(q)\psi(q^5)\phi(q^6)\psi(q^{20}) \\
&\quad + 8q^2\phi(q)\psi(q^5)f(q,q^7)f(q^{45},q^{75}) + 8q^9\phi(q)\psi(q^5)f(q^3,q^5)f(q^{15},q^{105}) \\
&\quad - 16q^2\phi(q^4)\psi(q^5)\psi(q^{12})\phi(q^{10}) - 16q^3\phi(q^4)\psi(q^5)\phi(q^6)\psi(q^{20}) \\
&\quad - 32q^3\psi(q^5)\psi(q^8)f(q,q^7)f(q^{45},q^{75}) - 32q^{10}\psi(q^5)\psi(q^8)f(q^3,q^5)f(q^{15},q^{105}) \\
&= 8q^2\psi(q^5)\left(\phi(q)-2\phi(q^4)\right)\left(\phi(q^{10})\psi(q^{12})+q\phi(q^6)\psi(q^{20})\right) \\
&\quad + 8q^2\psi(q^5)\left(\phi(q^4)-2q\psi(q^8)\right)\left(f(q,q^7)f(q^{45},q^{75})+q^7f(q^3,q^5)f(q^{15},q^{105})\right) \\
&= -8q^2\psi(q^5)\phi(-q)\left(\phi(q^{10})\psi(q^{12})+q\phi(q^6)\psi(q^{20})\right) \\
&\quad + 8q^2\psi(q^5)\phi(-q)\left(f(q,q^7)f(q^{45},q^{75})+q^7f(q^3,q^5)f(q^{15},q^{105})\right) \\
&= -8q^2\psi(q^5)\phi(-q)\big(\phi(q^{10})\psi(q^{12})+q\phi(q^6)\psi(q^{20}) - f(q,q^7)f(q^{45},q^{75}) \\
&\quad - q^7f(q^3,q^5)f(q^{15},q^{105})\big) \\
&= 0,
\end{aligned}$$

where the last equality is due to (2.15).

It follows that

$$N(3,5,20,32;32n+68) - 4N(3,5,20,32;8n+17) = 32T(3,5,20,32;4n+1).$$

This completes the proof.

6. Proof of Theorem 1.4

We have

$$\sum_{n=0}^{\infty} N(1,6,15,18;n)q^n = \phi(q)\cdot\phi(q^{15})\cdot\phi(q^6)\phi(q^{18})$$
$$= \left(\phi(q^4)+2q\psi(q^8)\right)\left(\phi(q^{60})+2q^{15}\psi(q^{120})\right)$$
$$\times\left(\phi(q^{24})\phi(q^{72})+2q^6\psi(q^{12})\psi(q^{36})+4q^{24}\psi(q^{48})\psi(q^{144})\right),$$

from which it follows that

$$\sum_{n=0}^{\infty} N(1,6,15,18;4n)q^n$$
$$= \left(\left(\phi(q^4)+2q\psi(q^8)\right)\left(\phi(q^{60})+2q^{15}\psi(q^{120})\right)+4q^4\psi(q^2)\psi(q^{30})\right)$$
$$\times\left(\phi(q^6)\phi(q^{18})+4q^6\psi(q^{12})\psi(q^{36})\right). \tag{6.1}$$

We extract from here that

$$\sum_{n=0}^{\infty} N(1,6,15,18;8n)q^n$$
$$= \left(\phi(q^2)\phi(q^{30})+4q^8\psi(q^4)\psi(q^{60})+4q^2\psi(q)\psi(q^{15})\right)\left(\phi(q^3)\phi(q^9)+4q^3\psi(q^6)\psi(q^{18})\right)$$
$$= \left(\phi(q^2)\phi(q^{30})+4q^8\psi(q^4)\psi(q^{60})+4q^2\big(\psi(q^6)\psi(q^{10})+q\phi(q^{20})\psi(q^{24})\right.$$
$$\left.+q^3\phi(q^{12})\psi(q^{40})\big)\right)\left(\phi(q^{12})\phi(q^{36})+6q^3\psi(q^6)\psi(q^{18})+4q^{12}\psi(q^{24})\psi(q^{72})\right),$$

from which we extract

$$\sum_{n=0}^{\infty} N(1,6,15,18;16n)q^n$$
$$= \left(\phi(q^6)\phi(q^{18})+4q^6\psi(q^{12})\psi(q^{36})\right)\left(\phi(q)\phi(q^{15})+4q^4\psi(q^2)\psi(q^{30})+4q\psi(q^3)\psi(q^5)\right)$$
$$+24q^3\psi(q^3)\psi(q^9)\left(\phi(q^{10})\psi(q^{12})+q\phi(q^6)\psi(q^{20})\right)$$
$$= \left(\phi(q^6)\phi(q^{18})+4q^6\psi(q^{12})\psi(q^{36})\right)\Bigg(\left(\phi(q^4)+2q\psi(q^8)\right)\left(\phi(q^{60})+2q^{15}\psi(q^{120})\right)$$
$$+4q^4\psi(q^2)\psi(q^{30})+4q\left(\psi(q^8)\phi(q^{60})+q^3\psi(q^2)\psi(q^{30})+q^{14}\phi(q^4)\psi(q^{120})\right)\Bigg)$$
$$+24q^3\left(\psi(q^{12})\phi(q^{18})+q^3\phi(q^6)\psi(q^{36})\right)\left(\phi(q^{10})\psi(q^{12})+q\phi(q^6)\psi(q^{20})\right),$$

from which we further extract

$$\sum_{n=0}^{\infty} N(1,6,15,18;32n+16)q^n$$
$$= 6\phi(q^3)\phi(q^9)\psi(q^4)\psi(q^{30})+6q^7\phi(q^3)\phi(q^9)\phi(q^2)\psi(q^{60})+24q^3\psi(q^6)\psi(q^{18})\psi(q^4)\psi(q^{30})$$
$$+24q^{10}\psi(q^6)\psi(q^{18})\phi(q^2)\psi(q^{60})+24q\phi(q^5)\psi(q^6)^2\phi(q^9)+24q^3\phi(q^3)^2\psi(q^{10})\psi(q^{18}). \tag{6.2}$$

From (6.1), we also extract

$$\begin{aligned}&\sum_{n=0}^{\infty}N(1,6,15,18;8n+4)q^n\\&=2\phi(q^3)\phi(q^9)\psi(q^4)\psi(q^{30})+2q^7\phi(q^3)\phi(q^9)\phi(q^2)\psi(q^{60})\\&\quad+8q^3\psi(q^6)\psi(q^{18})\psi(q^4)\psi(q^{30})+8q^{10}\psi(q^6)\psi(q^{18})\phi(q^2)\psi(q^{60}).\end{aligned}\tag{6.3}$$

Again,

$$\begin{aligned}&\sum_{n=0}^{\infty}T(1,6,15,18;n)q^n\\&=\psi(q)\psi(q^{15})\cdot\psi(q^6)\psi(q^{18})\\&=\left(\psi(q^6)\psi(q^{10})+q\phi(q^{20})\psi(q^{24})+q^3\phi(q^{12})\psi(q^{40})\right)\psi(q^6)\psi(q^{18}),\end{aligned}$$

from which we extract

$$\begin{aligned}&\sum_{n=0}^{\infty}T(1,6,15,18;2n+1)q^n\\&=\psi(q^3)\psi(q^9)\left(\phi(q^{10})\psi(q^{12})+q\phi(q^6)\psi(q^{20})\right)\\&=\left(\psi(q^{12})\phi(q^{18})+q^3\phi(q^6)\psi(q^{36})\right)\left(\phi(q^{10})\psi(q^{12})+q\phi(q^6)\psi(q^{20})\right),\end{aligned}$$

from which it follows that

$$\sum_{n=0}^{\infty}T(1,6,15,18;4n+1)q^n=\psi(q^6)^2\phi(q^9)\phi(q^5)+q^2\phi(q^3)^2\psi(q^{18})\psi(q^{10}).\tag{6.4}$$

From (6.2), (6.3) and (6.4), we have

$$\begin{aligned}&2\sum_{n=0}^{\infty}N(1,6,15,18;32n+16)q^n-6\sum_{n=0}^{\infty}N(1,6,15,18;8n+4)q^n\\&=48\sum_{n=0}^{\infty}T(1,6,15,18;4n+1)q^{n+1},\end{aligned}$$

and hence,

$$N(1,6,15,18;32n+48)-3N(1,6,15,18;8n+12)=24T(1,6,15,18;4n+1).$$

Thus we complete the proof.

7. Proof of Theorem 1.5

We have

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,7,10,30;n)q^n\\
&=\phi(q)\phi(q^7)\phi(q^{10})\phi(q^{30})\\
&=\left(\phi(q^4)+2q\psi(q^8)\right)\left(\phi(q^{28})+2q^7\psi(q^{56})\right)\\
&\quad\times\left(\phi(q^{40})\phi(q^{120})+2q^{10}\psi(q^{20})\psi(q^{60})+4q^{40}\psi(q^{80})\psi(q^{240})\right),
\end{aligned}$$

from which it follows that

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,7,10,30;4n)q^n\\
&=\left(\phi(q^{10})\phi(q^{30})+4q^{10}\psi(q^{20})\psi(q^{60})\right)\left(\phi(q)\phi(q^7)+4q^2\psi(q^2)\psi(q^{14})\right)\\
&=\left(\phi(q^{10})\phi(q^{30})+4q^{10}\psi(q^{20})\psi(q^{60})\right)\\
&\quad\times\left(\left(\phi(q^4)+2q\psi(q^8)\right)\left(\phi(q^{28})+2q^7\psi(q^{56})\right)+4q^2\psi(q^2)\psi(q^{14})\right),
\end{aligned}\tag{7.1}$$

from which we extract

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,7,10,30;8n)q^n\\
&=\left(\phi(q^5)\phi(q^{15})+4q^5\psi(q^{10})\psi(q^{30})\right)\left(\phi(q^2)\phi(q^{14})+4q^4\psi(q^4)\psi(q^{28})+4q\psi(q)\psi(q^7)\right)\\
&=\left(\phi(q^{20})\phi(q^{60})+6q^5\psi(q^{10})\psi(q^{30})+4q^{20}\psi(q^{40})\psi(q^{120})\right)\\
&\quad\times\left(\phi(q^2)\phi(q^{14})+4q^4\psi(q^4)\psi(q^{28})+4q\psi(q^8)\phi(q^{28})+4q^2\psi(q^2)\psi(q^{14})\right.\\
&\quad\left.+4q^7\phi(q^4)\psi(q^{56})\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,7,10,30;16n)q^n\\
&=\left(\phi(q^{10})\phi(q^{30})+4q^{10}\psi(q^{20})\psi(q^{60})\right)\left(\phi(q)\phi(q^7)+4q^2\psi(q^2)\psi(q^{14})+4q\psi(q)\psi(q^7)\right)\\
&\quad+6\psi(q^5)\psi(q^{15})\left(4q^3\psi(q^4)\phi(q^{14})+4q^6\phi(q^2)\psi(q^{28})\right)\\
&=\left(\phi(q^{10})\phi(q^{30})+4q^{10}\psi(q^{20})\psi(q^{60})\right)\left(\phi(q^4)\phi(q^{28})+6q\psi(q^8)\phi(q^{28})\right.\\
&\quad\left.+6q^7\phi(q^4)\psi(q^{56})+4q^8\psi(q^8)\psi(q^{56})+8q^2\psi(q^2)\psi(q^{14})\right)\\
&\quad+24q^3\left(\psi(q^{20})\phi(q^{30})+q^5\phi(q^{10})\psi(q^{60})\right)\left(\psi(q^4)\phi(q^{14})+q^3\phi(q^2)\psi(q^{28})\right),
\end{aligned}$$

from which we extract

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,7,10,30;32n+16)q^n\\
&= 6\phi(q^5)\psi(q^4)\phi(q^{14})\phi(q^{15}) + 6q^3\phi(q^2)\phi(q^5)\phi(q^{15})\psi(q^{28})\\
&\quad + 24q^5\psi(q^4)\psi(q^{10})\phi(q^{14})\psi(q^{30}) + 24q^8\phi(q^2)\psi(q^{10})\psi(q^{28})\psi(q^{30})\\
&\quad + 24q\psi(q^2)\phi(q^7)\psi(q^{10})\phi(q^{15}) + 24q^5\phi(q)\phi(q^5)\psi(q^{14})\psi(q^{30}). \qquad (7.2)
\end{aligned}$$

From (7.1), we also extract

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,7,10,30;8n+4)q^n\\
&= 2\psi(q^4)\phi(q^{14})\phi(q^5)\phi(q^{15}) + 2q^3\phi(q^2)\phi(q^5)\phi(q^{15})\psi(q^{28})\\
&\quad + 8q^5\psi(q^4)\psi(q^{10})\phi(q^{14})\psi(q^{30}) + 8q^8\phi(q^2)\psi(q^{10})\psi(q^{28})\psi(q^{30}). \qquad (7.3)
\end{aligned}$$

We have

$$\begin{aligned}
&\sum_{n=0}^{\infty} T(1,7,10,30;n)q^n\\
&= \psi(q)\psi(q^7)\psi(q^{10})\psi(q^{30})\\
&= \left(\psi(q^8)\phi(q^{28}) + q\psi(q^2)\psi(q^{14}) + q^6\phi(q^4)\psi(q^{56})\right)\left(\psi(q^4)\phi(q^6) + q\phi(q^2)\psi(q^{12})\right),
\end{aligned}$$

from which it follows that

$$\begin{aligned}
&\sum_{n=0}^{\infty} T(1,7,10,30;4n)q^n\\
&= \psi(q^2)\phi(q^7)\psi(q^{10})\phi(q^{15}) + q^4\phi(q)\phi(q^5)\psi(q^{14})\psi(q^{30}). \qquad (7.4)
\end{aligned}$$

From (7.2), (7.3) and (7.4), we have

$$\begin{aligned}
&2\sum_{n=0}^{\infty} N(1,7,10,30;32n+16)q^n - 6\sum_{n=0}^{\infty} N(1,7,10,30;8n+4)q^n\\
&= 48\sum_{n=0}^{\infty} T(1,7,10,30;4n)q^{n+1},
\end{aligned}$$

which implies that

$$N(1,7,10,30;32n+48) - 3N(1,7,10,30;8n+12) = 24T(1,7,10,30;4n).$$

Thus we finish the proof.

8. Proof of Theorem 1.6

We have

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,10,15,30;n)q^n\\
&=\phi(q)\phi(q^{15})\cdot\phi(q^{10})\phi(q^{30})\\
&=\left(\phi(q^4)+2q\psi(q^8)\right)\left(\phi(q^{60})+2q^{15}\psi(q^{120})\right)\\
&\quad\times\left(\phi(q^{40})\phi(q^{120})+2q^{10}\psi(q^{20})\psi(q^{60})+4q^{40}\psi(q^{80})\psi(q^{240})\right),
\end{aligned}$$

from which we extract

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,10,15,30;4n)q^n\\
&=\left(\phi(q)\phi(q^{15})+4q^4\psi(q^2)\psi(q^{30})\right)\left(\phi(q^{10})\phi(q^{30})+4q^{10}\psi(q^{20})\psi(q^{60})\right)\\
&=\left(\left(\phi(q^4)+2q\psi(q^8)\right)\left(\phi(q^{60})+2q^{15}\psi(q^{120})\right)+4q^4\psi(q^2)\psi(q^{30})\right)\\
&\quad\times\left(\phi(q^{10})\phi(q^{30})+4q^{10}\psi(q^{20})\psi(q^{60})\right),
\end{aligned}\tag{8.1}$$

from which it follows that

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,10,15,30;8n)q^n\\
&=\left(\phi(q^2)\phi(q^{30})+4q^8\psi(q^4)\psi(q^{60})+4q^2\psi(q)\psi(q^{15})\right)\\
&\quad\times\left(\phi(q^5)\phi(q^{15})+4q^5\psi(q^{10})\psi(q^{30})\right)\\
&=\left(\phi(q^2)\phi(q^{30})+4q^8\psi(q^4)\psi(q^{60})+4q^2\psi(q^6)\psi(q^{10})\right.\\
&\quad\left.+4q^3\phi(q^{20})\psi(q^{24})+4q^5\phi(q^{12})\psi(q^{40})\right)\\
&\quad\times\left(\phi(q^{20})\phi(q^{60})+6q^5\psi(q^{10})\psi(q^{30})+4q^{20}\psi(q^{40})\psi(q^{120})\right).
\end{aligned}$$

From the above we extract

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,10,15,30;16n)q^n\\
&=\left(\phi(q^{10})\phi(q^{30})+4q^{10}\psi(q^{20})\psi(q^{60})\right)\\
&\quad\times\left(\phi(q)\phi(q^{15})+4q^4\psi(q^2)\psi(q^{30})+4q\psi(q^3)\psi(q^5)\right)
\end{aligned}$$

$$
\begin{aligned}
&+ 6q^2\psi(q^5)\psi(q^{15})\left(4q^2\phi(q^{10})\psi(q^{12}) + 4q^3\phi(q^6)\psi(q^{20})\right)\\
&= \left(\phi(q^{10})\phi(q^{30}) + 4q^{10}\psi(q^{20})\psi(q^{60})\right)\\
&\quad\times\left(\phi(-q^6)\phi(-q^{10}) + 6q\psi(q^8)\phi(q^{60}) + 6q^4\psi(q^2)\psi(q^{30}) + 6q^{15}\phi(q^4)\psi(q^{120})\right.\\
&\quad\left.+ 4q^4\psi(q^2)\psi(q^{30})\right) + 6q^2\left(\psi(q^{20})\phi(q^{30}) + q^5\phi(q^{10})\psi(q^{60})\right)\\
&\quad\times\left(4q^2\phi(q^{10})\psi(q^{12}) + 4q^3\phi(q^6)\psi(q^{20})\right),
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(1,10,15,30;32n+16)q^n\\
&= 6\psi(q^4)\phi(q^5)\phi(q^{15})\phi(q^{30}) + 6q^7\phi(q^2)\phi(q^5)\phi(q^{15})\psi(q^{60})\\
&\quad + 24q^5\psi(q^4)\psi(q^{10})\phi(q^{30})\psi(q^{30}) + 24q^{12}\phi(q^2)\psi(q^{10})\psi(q^{30})\psi(q^{60})\\
&\quad + 24q^2\phi(q^3)\psi(q^{10})^2\phi(q^{15}) + 24q^4\phi(q^5)^2\psi(q^6)\psi(q^{30}).
\end{aligned}
\tag{8.2}
$$

Now, from (8.1) we also extract

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(1,10,15,30;8n+4)q^n\\
&= \left(2\psi(q^4)\phi(q^{30}) + 2q^7\phi(q^2)\psi(q^{60})\right)\left(\phi(q^5)\phi(q^{15}) + 4q^5\psi(q^{10})\psi(q^{30})\right)\\
&= 2\psi(q^4)\phi(q^5)\phi(q^{15})\phi(q^{30}) + 2q^7\phi(q^2)\phi(q^5)\phi(q^{15})\psi(q^{60})\\
&\quad + 8q^5\psi(q^4)\psi(q^{10})\phi(q^{30})\psi(q^{30}) + 8q^{12}\phi(q^2)\psi(q^{10})\psi(q^{30})\psi(q^{60}).
\end{aligned}
\tag{8.3}
$$

On the other hand, we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} T(1,10,15,30;n)q^n &= \psi(q)\psi(q^{15})\cdot\psi(q^{10})\psi(q^{30})\\
&= \left(\psi(q^6)\psi(q^{10}) + q\phi(q^{20})\psi(q^{24}) + q^3\phi(q^{12})\psi(q^{40})\right)\\
&\quad\times\left(\psi(q^{40})\phi(q^{60}) + q^{10}\phi(q^{20})\psi(q^{120})\right),
\end{aligned}
$$

from which we extract

$$
\sum_{n=0}^{\infty} T(1,10,15,30;4n+3)q^n = \phi(q^3)\psi(q^{10})^2\phi(q^{15}) + q^2\phi(q^5)^2\psi(q^6)\psi(q^{30}).
\tag{8.4}
$$

From (8.2), (8.3) and (8.4), we have

$$2\sum_{n=0}^{\infty} N(1,10,15,30;32n+16)q^n - 6\sum_{n=0}^{\infty} N(1,10,15,30;8n+4)q^n$$
$$= 48\sum_{n=0}^{\infty} T(1,10,15,30;4n+3)q^{n+2},$$

which readily implies that

$$N(1,7,10,30;32n+80) - 3N(1,7,10,30;8n+20) = 24T(1,7,10,30;4n+3).$$

This completes the proof of Theorem 1.6.

9. Proof of Theorem 1.7

Our proof of Theorem 1.7 is quite different from the proofs of Theorems 1.1-1.6. In fact, the method is quite similar to that of [15], which considers the ternary case of Sun's conjectures in [17].

For a quaternary quadratic form $f(x,y,z,w)$ and a positive integer n, we define

$$R(f,n) = \{(x,y,z,w) \in \mathbb{Z}^4 : f(x,y,z,w) = n\} \quad \text{and} \quad r(f,n) = |R(f,n)|.$$

At first, we state and prove a proposition.

Proposition 9.1 — For any positive integer $n \equiv 3,5 \pmod 8$, we have

$$r(x^2+2y^2+4z^2+17w^2,n) = r(2x^2+3y^2+4z^2+8w^2+2xy+2yz+2yw,n).$$

Proof of Proposition 9.1 : Let

$$f = f(x,y,z,w) = x^2+2y^2+4z^2+17w^2,$$
$$g = g(x,y,z,w) = 2x^2+3y^2+4z^2+8w^2+2xy+2yz+2yw.$$

First, we consider the case when n is a positive integer congruent to $3 \pmod 8$. Note that if $(x,y,z,w) \in R(f,n)$, then $x \not\equiv w \pmod 2$. Furthermore, one may easily show that if $(x,y,z,w) \in R(f,n)$ and $x \equiv 1 \pmod 2$, then $2x+2y-2z-3w \equiv 0 \pmod 4$. Since $f(-x,y,z,w) = f(x,y,z,w)$, the map $\eta_1 : R(f,n) \to R(f,n)$ defined by

$$\eta_1(x,y,z,w) = (-x,y,z,w),$$

is a well defined bijective map. Hence we have

$$\begin{aligned}&|\{(x,y,z,w)\in R(f,n): x\equiv 1 \pmod 2, 2x+2y-2z-3w\equiv 0 \pmod 8\}|\\&=|\{(x,y,z,w)\in R(f,n): x\equiv 1 \pmod 2, 2x+2y-2z-3w\equiv 4 \pmod 8\}|,\end{aligned}$$

which implies that

$$\begin{aligned}&|\{(x,y,z,w)\in R(f,n): x\equiv 1 \pmod 2\}|\\&=2|\{(x,y,z,w)\in R(f,n): x\equiv 1 \pmod 2, 2x+2y-2z-3w\equiv 0 \pmod 8\}|.\end{aligned}$$

Note that if $(x,y,z,w)\in R(f,n)$ and $x\equiv 0 \pmod 2$, then $x+6y-6z+6w\equiv 0 \pmod 4$. Since $f(x,y,z,-w)=f(x,y,z,w)$, the map $\eta_2: R(f,n)\to R(f,n)$ defined by

$$\eta_2(x,y,z,w)=(x,y,z,-w),$$

is a well defined bijective map. Hence we have

$$\begin{aligned}&|\{(x,y,z,w)\in R(f,n): x\equiv 0 \pmod 2, x+6y-6z+6w\equiv 0 \pmod 8\}|\\&=|\{(x,y,z,w)\in R(f,n): x\equiv 0 \pmod 2, x+6y-6z+6w\equiv 4 \pmod 8\}|,\end{aligned}$$

which implies that

$$\begin{aligned}&|\{(x,y,z,w)\in R(f,n): x\equiv 0 \pmod 2\}|\\&=2|\{(x,y,z,w)\in R(f,n): x\equiv 0 \pmod 2, x+6y-6z+6w\equiv 0 \pmod 8\}|.\end{aligned}$$

Now, if we define

$$\begin{aligned}F_1&=\{(x,y,z,w)\in R(f,n): x\equiv 1 \pmod 2, 2x+2y-2z-3w\equiv 0 \pmod 8\},\\F_2&=\{(x,y,z,w)\in R(f,n): x\equiv 0 \pmod 2, x+6y-6z+6w\equiv 0 \pmod{16}\},\\F_3&=\{(x,y,z,w)\in R(f,n): x\equiv 0 \pmod 2, x+6y-6z+6w\equiv 8 \pmod{16}\},\end{aligned}$$

then we have

$$r(f,n)=2(|F_1|+|F_2|+|F_3|).$$

Now, we analyze the set $R(g,n)$. First, we note that $y\equiv 1 \pmod 2$ for any $(x,y,z,w)\in R(g,n)$. Since $g(x+y,-y,-z,-w)=g(x,y,z,w)$, the map $\eta_3: R(g,n)\to R(g,n)$ defined by

$$\eta_3(x,y,z,w)=(x+y,-y,-z,-w)$$

is a well defined bijective map. Therefore, we have

$$r(g,n) = 2|\{(x,y,z,w) \in R(g,n) : x \equiv 0 \pmod 2\}|.$$

One may easily check that for $(x,y,z,w) \in R(g,n)$, if $x \equiv 0 \pmod 2$, then $x - z + w \equiv 0 \pmod 4$. Furthermore, if $x - z + w \equiv 4 \pmod 8$, then $7x - 4y + 9z - w \equiv 0 \pmod 8$. Thus if we define

$$\begin{aligned}
G_1 &= \{(x,y,z,w) \in R(g,n) : x \equiv 0 \pmod 2,\ \ x - z + w \equiv 0 \pmod 8\},\\
G_2 &= \left\{(x,y,z,w) \in R(g,n) : \begin{array}{r} x \equiv 0 \pmod 2,\ \ x - z + w \equiv 4 \pmod 8,\\ 7x - 4y + 9z - w \equiv 8 \pmod{16} \end{array}\right\},\\
G_3 &= \left\{(x,y,z,w) \in R(g,n) : \begin{array}{r} x \equiv 0 \pmod 2,\ \ x - z + w \equiv 4 \pmod 8,\\ 7x - 4y + 9z - w \equiv 0 \pmod{16} \end{array}\right\},
\end{aligned}$$

then the set $\{(x,y,z,w) \in R(g,n) : x \equiv 0 \pmod 2\}$ is a disjoint union of G_1, G_2 and G_3. Hence we have

$$r(g,n) = 2(|G_1| + |G_2| + |G_3|).$$

Now, for $j = 1, 2, 3$, we define maps $\phi_j : G_j \to F_j$ by

$$\begin{aligned}
\phi_1(x,y,z,w) &= \frac{1}{8}\begin{pmatrix} 4 & 8 & -4 & -12\\ -2 & -8 & -6 & -10\\ -3 & 0 & -5 & 5\\ -2 & 0 & 2 & -2 \end{pmatrix}\begin{pmatrix} x\\ y\\ z\\ w \end{pmatrix},\\
\phi_2(x,y,z,w) &= \frac{1}{16}\begin{pmatrix} 2 & 24 & -2 & 18\\ -10 & -8 & -6 & 22\\ -3 & -4 & -13 & -11\\ -4 & 0 & 4 & -4 \end{pmatrix}\begin{pmatrix} x\\ y\\ z\\ w \end{pmatrix},\\
\phi_3(x,y,z,w) &= \frac{1}{16}\begin{pmatrix} 6 & -8 & -6 & -42\\ 2 & -8 & 14 & 2\\ 7 & 12 & 9 & -1\\ 4 & 0 & -4 & 4 \end{pmatrix}\begin{pmatrix} x\\ y\\ z\\ w \end{pmatrix}.
\end{aligned}$$

It is easy to check that all of them are well defined bijective maps. Therefore, we have

$$r(f,n) = 2(|F_1| + |F_2| + |F_3|) = 2(|G_1| + |G_2| + |G_3|) = r(g,n).$$

Next, we consider the case when n is a positive integer congruent to 5 (mod 8). Note that if $(x,y,z,w) \in R(f,n)$ and $x \equiv 1 \pmod 2$, then $2x+2y-2z+5w \equiv 0 \pmod 4$. Since $f(-x,y,z,w) = f(x,y,z,w)$, we have

$$\begin{aligned}&|\{(x,y,z,w) \in R(f,n) : x \equiv 1 \pmod 2,\ \ 2x+2y-2z+5w \equiv 0 \pmod 8\}| \\ &= |\{(x,y,z,w) \in R(f,n) : x \equiv 1 \pmod 2,\ \ 2x+2y-2z+5w \equiv 4 \pmod 8\}|,\end{aligned}$$

which implies that

$$\begin{aligned}&|\{(x,y,z,w) \in R(f,n) : x \equiv 1 \pmod 2\}| \\ &= 2|\{(x,y,z,w) \in R(f,n) : x \equiv 1 \pmod 2,\ 2x+2y-2z+5w \equiv 0 \pmod 8\}|.\end{aligned}$$

For $(x,y,z,w) \in R(f,n)$, if $x \equiv 0 \pmod 2$, then we have $w \equiv 1 \pmod 2$ and $x+6y-6z+6w \equiv 0 \pmod 4$. Since $f(x,y,z,-w) = f(x,y,z,w)$, we have

$$\begin{aligned}&|\{(x,y,z,w) \in R(f,n) : x \equiv 0 \pmod 2,\ x+6y-6z+6w \equiv 0 \pmod 8\}| \\ &= |\{(x,y,z,w) \in R(f,n) : x \equiv 0 \pmod 2,\ x+6y-6z+6w \equiv 4 \pmod 8\}|,\end{aligned}$$

which implies that

$$\begin{aligned}&|\{(x,y,z,w) \in R(f,n) : x \equiv 0 \pmod 2\}| \\ &= 2|\{(x,y,z,w) \in R(f,n) : x \equiv 0 \pmod 2,\ x+6y-6z+6w \equiv 0 \pmod 8\}|.\end{aligned}$$

Thus, if we define

$$\begin{aligned}X_1 &= \{(x,y,z,w) \in R(f,n) : x \equiv 1 \pmod 2,\ 2x+2y-2z+5w \equiv 0 \pmod 8\}, \\ X_2 &= \{(x,y,z,w) \in R(f,n) : x \equiv 0 \pmod 2,\ x+6y-6z+6w \equiv 0 \pmod{16}\}, \\ X_3 &= \{(x,y,z,w) \in R(f,n) : x \equiv 0 \pmod 2,\ x+6y-6z+6w \equiv 8 \pmod{16}\},\end{aligned}$$

then we have

$$r(f,n) = 2(|X_1| + |X_2| + |X_3|).$$

Now, we analyze the set $R(g,n)$. One may check the followings;

(i) if $(x,y,z,w) \in R(g,n)$ and $x \equiv 0 \pmod 2$, then $x+y+z-w \equiv 0 \pmod 4$;

(ii) if $(x,y,z,w) \in R(g,n)$ and $x \equiv 1 \pmod 2$, then $x-z+w \equiv 0 \pmod 4$.

Since $g(x+y,-y,-z,-w)=g(x,y,z,w)$, we have

$$\begin{aligned}&|\{(x,y,z,w)\in R(g,n): x\equiv 0 \pmod 2,\ x+y+z-w\equiv 0 \pmod 8\}|\\&=|\{(x,y,z,w)\in R(g,n): x\equiv 1 \pmod 2,\ x-z+w\equiv 0 \pmod 8\}|\end{aligned}$$

and

$$\begin{aligned}&|\{(x,y,z,w)\in R(g,n): x\equiv 0 \pmod 2,\ x+y+z-w\equiv 4 \pmod 8\}|\\&=|\{(x,y,z,w)\in R(g,n): x\equiv 1 \pmod 2,\ x-z+w\equiv 4 \pmod 8\}|.\end{aligned}$$

Therefore, we have

$$\begin{aligned}r(g,n)=&2|\{(x,y,z,w)\in R(g,n): x\equiv 0 \pmod 2\}|\\=&2|\{(x,y,z,w)\in R(g,n): x\equiv 0 \pmod 2,\ x+y+z-w\equiv 0 \pmod 8\}|\\&+2|\{(x,y,z,w)\in R(g,n): x\equiv 0 \pmod 2,\ x+y+z-w\equiv 4 \pmod 8\}|\\=&2|\{(x,y,z,w)\in R(g,n): x\equiv 0 \pmod 2,\ x+y+z-w\equiv 0 \pmod 8\}|\\&+2|\{(x,y,z,w)\in R(g,n): x\equiv 1 \pmod 2,\ x-z+w\equiv 4 \pmod 8\}|.\end{aligned}$$

One may easily show that for $(x,y,z,w)\in R(g,n)$, if $x\equiv 1 \pmod 2$ and $x-z+w\equiv 4 \pmod 8$, then $7x-4y+9z-w\equiv 0 \pmod 8$. Thus if we define

$$\begin{aligned}Y_1&=\{(x,y,z,w)\in R(g,n): x\equiv 0 \pmod 2,\ \ x+y+z-w\equiv 0 \pmod 8\},\\Y_2&=\left\{(x,y,z,w)\in R(g,n): \begin{array}{r} x\equiv 1 \pmod 2,\ \ x-z+w\equiv 4 \pmod 8,\\ 7x-4y+9z-w\equiv 8 \pmod{16}\end{array}\right\},\\Y_3&=\left\{(x,y,z,w)\in R(g,n): \begin{array}{r} x\equiv 1 \pmod 2,\ \ x-z+w\equiv 4 \pmod 8,\\ 7x-4y+9z-w\equiv 0 \pmod{16}\end{array}\right\},\end{aligned}$$

then, we have

$$r(g,n)=2(|Y_1|+|Y_2|+|Y_3|).$$

For $j=1,2,3$, if we define maps $\psi_j: Y_j\to X_j$ by

$$\psi_1(x,y,z,w)=\frac{1}{8}\begin{pmatrix}4&-4&4&12\\-2&6&6&10\\-3&-3&5&-5\\-2&-2&-2&2\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix},$$

$$\psi_2(x,y,z,w) = \frac{1}{16}\begin{pmatrix} 2 & 24 & -2 & 18 \\ -10 & -8 & -6 & 22 \\ -3 & -4 & -13 & -11 \\ -4 & 0 & 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix},$$

$$\psi_3(x,y,z,w) = \frac{1}{16}\begin{pmatrix} 6 & -8 & -6 & -42 \\ 2 & -8 & 14 & 2 \\ 7 & 12 & 9 & -1 \\ 4 & 0 & -4 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix},$$

then one may check that they are all bijective. Therefore, we have

$$r(f,n) = 2(|X_1| + |X_2| + |X_3|) = 2(|Y_1| + |Y_2| + |Y_3|) = r(g,n),$$

which completes the proof. □

Now we are in a position to prove Theorem 1.7.

PROOF OF THEOREM 17 : Let

$$\begin{aligned} f &= f(x,y,z,w) = x^2 + 2y^2 + 4z^2 + 17w^2, \\ g &= g(x,y,z,w) = 2x^2 + 3y^2 + 4z^2 + 8w^2 + 2xy + 2yz + 2yw, \\ h_1 &= h_1(x,y,z,w) = 2x^2 + 4y^2 + 4z^2 + 6w^2 + 2xw + 2yw + 4zw, \\ h_2 &= h_2(x,y,z,w) = x^2 + 2y^2 + 2z^2 + 9w^2 + 2zw. \end{aligned}$$

First, note that

$$\begin{aligned} &|\{(x,y,z,w) \in R(h_1, 2n+6) : w \equiv 0 \pmod 2\}| \\ &= |\{(x,y,z,w) \in \mathbb{Z}^4 : h_1(x,y,z,2w) = 2n+6\}| \\ &= |\{(x,y,z,w) \in \mathbb{Z}^4 : 2 \cdot h_2(x+w, z+w, y, w) = 2n+6\}| = r(h_2, n+3). \end{aligned}$$

Next, for $(x,y,z,w) \in R(h_1, 2n+6)$, if $w \equiv 1 \pmod 2$, then $y \equiv 0 \pmod 2$. Hence we have

$$\begin{aligned} &|\{(x,y,z,w) \in R(h_1, 2n+6) : x \equiv 0 \pmod 2,\ y \equiv 0 \pmod 2,\ w \equiv 1 \pmod 2\}| \\ &= |\{(x,y,z,w) \in \mathbb{Z}^4 : h_1(2x, 2y, z, w) = 2n+6\}| \\ &= |\{(x,y,z,w) \in \mathbb{Z}^4 : 2 \cdot g(z,w,x,y) = 2n+6\}| = r(g, n+3). \end{aligned}$$

Finally, since $h_1(w-2x,2y,z,w)=2\cdot g(z,w,x-w,y)$, we have,

$$\begin{aligned}&|\{(x,y,z,w)\in R(h_1,2n+6): x\equiv 1\ (\mathrm{mod}\ 2),\ y\equiv 0\ (\mathrm{mod}\ 2),\ w\equiv 1\ (\mathrm{mod}\ 2)\}|\\&=\left|\left\{(x,y,z,w)\in\mathbb{Z}^4: h_1(w-2x,2y,z,w)=2n+6\right\}\right|\\&=r(g,n+3).\end{aligned}$$

Therefore, we have

$$r(h_1,2n+6)=r(h_2,n+3)+2r(g,n+3), \tag{9.1}$$

for any nonnegative even integer n.

By Proposition 9.1 and (9.1), we arrive at

$$2\cdot r(f,n+3)=r(h_1,2n+6)-r(h_2,n+3)\ \text{ for any } n\equiv 0,2\ (\mathrm{mod}\ 8). \tag{9.2}$$

Now, if $8x^2+y^2+2z^2+9w^2-4xw\equiv 0\ (\mathrm{mod}\ 4)$, then $y\equiv z\equiv w\ (\mathrm{mod}\ 2)$. Since $h_1(y,x,z,-w)=4x^2+2y^2+4z^2+6w^2-2xw-2yw-4zw$, we have

$$\begin{aligned}&|\{(x,y,z,w)\in R(f,8n+24): x\equiv w\ (\mathrm{mod}\ 4)\}|\\&=r((w-4x)^2+2y^2+4z^2+17w^2,8n+24)\\&=r(8x^2+y^2+2z^2+9w^2-4xw,4n+12)\\&=r(8x^2+(w-2y)^2+2(w-2z)^2+9w^2-4xw,4n+12)\\&=r(4x^2+2y^2+4z^2+6w^2-2xw-2yw-4zw,2n+6)\\&=r(h_1,2n+6).\end{aligned}$$

Now, if $x^2+8y^2+16z^2+17w^2=8n+24$, then $x\equiv w\equiv 0\ (\mathrm{mod}\ 2)$. Since $2\cdot h_2(y,z,x,-w)=(w-2x)^2+2y^2+4z^2+17w^2$, we see that

$$\begin{aligned}&|\left\{(x,y,z,w)\in\mathbb{Z}^4: x^2+8y^2+16z^2+17w^2=8n+24\right\}|\\&=|\left\{(x,y,z,w)\in\mathbb{Z}^4: 4x^2+8y^2+16z^2+68w^2=8n+24\right\}|\\&=r(x^2+2y^2+4z^2+17w^2,2n+6)=r((w-2x)^2+2y^2+4z^2+17w^2,2n+6)\\&=r(h_2,n+3).\end{aligned}$$

From these equalities and (9.2), we have

$$\begin{aligned}2\cdot r(f,n+3)=&|\{(x,y,z,w)\in R(f,8n+24): x\equiv w\ (\mathrm{mod}\ 4)\}|\\&-|\left\{(x,y,z,w)\in\mathbb{Z}^4: x^2+8y^2+16z^2+17w^2=8n+24\right\}|,\end{aligned} \tag{9.3}$$

for any $n \equiv 0, 2 \pmod 8$.

Now, as $\dfrac{x(x-1)}{2} = \dfrac{(-x+1)(-x)}{2}$, we see that

$$\begin{aligned}
&16T(1,2,4,17;n)\\
&= 16\left|\left\{(x,y,z,w) \in \mathbb{N}^4 : \frac{x(x+1)}{2} + 2\frac{y(y+1)}{2} + 4\frac{z(z+1)}{2} + 17\frac{w(w+1)}{2} = n\right\}\right|\\
&= \left|\left\{(x,y,z,w) \in \mathbb{Z}^4 : \frac{x(x+1)}{2} + 2\frac{y(y+1)}{2} + 4\frac{z(z+1)}{2} + 17\frac{w(w+1)}{2} = n\right\}\right|\\
&= |\{(x,y,z,w) \in \mathbb{Z}^4 : (2x+1)^2 + 2(2y+1)^2 + 4(2z+1)^2 + 17(2w+1)^2 = 8n+24\}|\\
&= |\{(x,y,z,w) \in R(f, 8n+24) : xyzw \equiv 1 \pmod 2\}|.
\end{aligned}$$

Note that if $x^2 + 2y^2 + 4z^2 + 17w^2 = 8n + 24$, then

$$(x^2, 2y^2, 4z^2, 17w^2) \equiv (1,2,4,1), (0,0,0,0), (4,0,0,4), (4,0,4,0) \text{ or } (0,0,4,4) \pmod 8.$$

From this and (9.3), we may easily deduce that

$$\begin{aligned}
8T(1,2,4,17;n) =& |\{(x,y,z,w) \in R(f,8n+24) : x \equiv w \pmod 4,\ y \equiv z \equiv 1 \pmod 2\}|\\
=& |\{(x,y,z,w) \in R(f,8n+24) : x \equiv w \pmod 4\}|\\
&- |\{(x,y,z,w) \in R(f,8n+24) : y \equiv z \equiv 0 \pmod 2\}|\\
=& |\{(x,y,z,w) \in R(f,8n+24) : x \equiv w \pmod 4\}|\\
&- |\left\{(x,y,z,w) \in \mathbb{Z}^4 : x^2 + 8y^2 + 16z^2 + 17w^2 = 8n+24\right\}|\\
=& 2 \cdot r(f, n+3),
\end{aligned}$$

which is equivalent to (1.7). This completes the proof. □

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Bounds on linear codes capable of detecting, locating and correcting of repeated burst errors prevailing in multiple sub-blocks

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Abstract

Linear codes are presented that can detect, locate and correct all repeated burst errors of length b or less which occur in multiple sub-blocks. We obtain lower and upper bounds on the number of check digits for the existence of these codes. Three examples, one for each type of code, are provided.

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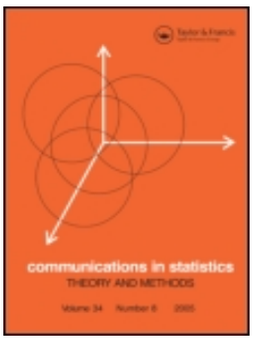
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Consistency of multivariate density estimators using random bandwidths

Santanu Dutta & Koushik Saha

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Consistency of multivariate density estimators using random bandwidths

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Asymptotic properties of a kernel density estimator using a random bandwidth are difficult to establish. Under some assumptions we prove the L_1 consistency of a class of multivariate kernel density estimators using different bandwidth vector selectors. The expected L_1 distance between such an estimator and the density is also shown to converge to zero. Our results hold even when the marginal densities are heavy-tailed. As a special case, we propose a simple estimator that depends on only one parameter, irrespective of the dimension. Its L_1 distance from the density goes to zero, exponentially. Simulations suggest that this estimator performs well in terms of the integrated squared error as well.

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1. Introduction

Let $X_1, X_2, \ldots, X_n$ be $\mathbb{R}^d$-valued independent random vectors with common density function f. A simple kernel based estimator of f is defined as follows:

$$f_{n,h}(x) = \frac{1}{n|h|} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right), \tag{1.1}$$

where $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, $\{h = (h_1, h_2, \ldots, h_d)\}_{n \geq 1}$ is a sequence of vectors with positive components, $|h| = \prod_{i=1}^{d} h_i$ and K is a d-variate density on $\mathbb{R}^d$, and

$$\left(\frac{x - X_i}{h}\right) = \left(\frac{x_1 - X_{i,1}}{h_1}, \frac{x_2 - X_{i,2}}{h_2}, \ldots, \frac{x_d - X_{i,d}}{h_d}\right).$$

$h_1, \ldots, h_d$ control the amount of smoothing along the d coordinate directions. h is referred to as the bandwidth vector, and its dependence on n is suppressed to make notation simple.

A special case of the above estimator is the *product* kernel density estimator (see Scott (1992) and Dutta (2011)), where $K(y) = \prod_{j=1}^{d} K^*(y_j)$, $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$, and K^* is a univariate density function. Scott (1992) obtained an asymptotic representation of the mean integrated squared error (MISE) of a product kernel density estimator. From Scott (1992) we see that the optimal values of $h_1, \ldots, h_d$ in terms of minimizing the MISE are of the order $O(n^{-1/(4+d)})$. However no general closed form expression exists for these optimal bandwidths which minimize the MISE, and they depend on unknown constants (see Scott (1992)).

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A practical problem in density estimation is to determine the appropriate values of $h_1, h_2, \ldots, h_d$ based on $X_1, X_2, \ldots, X_n$ (see Sain et al. (1994), Dutta (2011)). Sain et al. (1994) discussed a number of algorithms, such as the least squares or the unbiased cross-validation (UCV), the biased cross-validation (BCV), and the smooth bootstrap bandwidth selector (SCV) for data based choice of h. Duong and Hazelton (2005) and Dutta (2011) have proposed other methods of choosing the bandwidth vector h based on the given data. Let $\hat{h} = (\hat{h}_1, \ldots, \hat{h}_d)$ be a bandwidth vector obtained by any one of these bandwidth selectors. Such a $\hat{h}$ is a random vector, i.e., a vector valued function of $X_1, \ldots, X_n$. A natural problem is to investigate the asymptotic properties of the resulting estimator $f_{n,\hat{h}}$.

Devroye (1983) proved that for any density f and arbitrary $\epsilon > 0$, $P(\int |f_{n,h}(x) - f(x)|dx > \epsilon)$ goes to zero at an exponential rate. The author assumed that $h_1 = h_2 = \cdots = h_d$ and the common bandwidth h_1 is a *non random* positive sequence where $h_1 \to 0$ and $nh_1^d \to \infty$ as $n \to \infty$. Devroye and Györfi (1985) extended the result on L_1 convergence in Devroye (1983) to the case where common bandwidth h_1 is a function of $X_1, \ldots, X_n$ satisfying $h_1 + \frac{1}{nh_1^d} = o(1)$, in probability/almost surely/completely. However if $\{(h_1, \ldots, h_d)\}_n$ is a vector valued stochastic process (without the constraint $h_1 = \cdots = h_d$), determined by some bandwidth selection algorithm, the proof of L_1 convergence of the resulting density estimator does not seem to be known.

In this paper we prove the L_1 consistency of $f_{n,\hat{h}}$, where $\hat{h}$ is obtained by a number of different algorithms discussed in Sain et al. (1994), Duong and Hazelton (2005), and also by the plug-in rule in Scott (1992). We also show the convergence (to zero) of the expected L_1 distance or the mean integrated absolute error (MIAE) for these estimators. There are two theorems and two corollaries in this direction. In Theorem 1 and Corollary 1.1 we assume that the search for ideal bandwidths is confined to a sequence of suitable compact intervals, that depend on the data. In Theorem 2 this assumption is relaxed, and replaced by some other assumptions. While the L_1 distance possess many interesting properties (see Scott (1992)), a popular criterion in density estimation is the integrated squared error (ISE). We also compare the performance of some of these estimators in terms of the ISE criterion using detailed simulations. No estimator seems to be the best from all perspectives.

As a special case we propose to use $h_j = \lambda \eta_j^*, \ j = 1, 2, \ldots, d$, in the estimator (1.1) where $\lambda \in [c_1 n^{-1/(4+d)}, c_2 n^{-1/(4+d)}]$ and η_j^* is the sample interquartile range along the jth coordinate direction. This estimator depends only on one parameter λ irrespective of data dimension d. If the λ is chosen by a cross-validation method, as described in this paper, the resulting estimator compares well with the estimators based on the UCV and SCV selectors in Sain et al. (1994) and Duong and Hazelton (2005). The convergence (to zero) of the L_1 distance and the MIAE of this estimator is proved in Corollary 2.1.

2. Main results

Let us introduce some notation and assumptions used in the sequel. Let $\{a_n\}, \{b_n\}$ be two positive sequences. $a_n = O(b_n)$ denotes that $\limsup_{n\to\infty} \frac{a_n}{b_n} < \infty$. A sequence of random variables $\{Z_n\}$ is said to converge to zero exponentially as $n \to \infty$, if for every $\epsilon > 0$ there exists positive constants r, n_0 such that $P(|Z_n| > \epsilon) \leq \exp(-nr\epsilon^2) \ \forall \ n > n_0$. Let X_{ij} denote the jth component of X_i where $j = 1, 2, \ldots, d$ and $i = 1, 2, \ldots, n$, and η_j^* be the sample interquartile range based on $X_{1j}, X_{2j}, \ldots, X_{nj}$. Further let f_j denote the density of X_{1j}.

Assumption 1. f_j is strictly positive on some open neighborhood of Q_{1j} and Q_{3j}, where Q_{1j} and Q_{3j} are the 1st and the 3rd quartiles of the distribution of X_{1j} for $j = 1, \ldots, d$.

Assumption 2. $\hat{h} \in I_n = \prod_{j=1}^{d} [c_{1j}\eta_j^* n^{-1/(4+d)},\ c_{2j}\eta_j^* n^{-1/(4+d)}]$, where $c_{1j} < c_{2j}, j = 1, \ldots, d$ are positive constants.

Remark 1. 1. A number of well-known bandwidth selectors such as the UCV, BCV, SCV, and the different versions of the smooth bootstrap based bandwidth selectors in Sain et al. (1994) and Dutta (2011) involve optimization of some criterion F, say with respect to $h_1, \ldots, h_d$. One can ensure that Assumption 2 holds for the resulting bandwidth vector by optimizing the data based bandwidth selection criterion on I_n. Also I_n covers a wide range of reasonable values of $h_1, \ldots, h_d$.

Theorem 1. *Let f be a density function on $\mathbb{R}^d$, such that Assumption 1 holds for $f_j, j = 1, \ldots, d$. Then under Assumption 2, there exists a positive number r such that*

$$P\left(\int \left|f_{n,\hat{h}}(x) - f(x)\right| dx > \epsilon\right) = O\left(n^d \exp(-nr\epsilon^2)\right).$$

As mentioned above, a number of different bandwidth selectors involve optimization of some criterion F, say. Theorem 1 holds for any such bandwidth selector, irrespective of the choice of F, as long as F is optimized on I_n.

Corollary 1.1. *Under Assumptions 1 and 2, $\int |f_{n,\hat{h}}(x) - f(x)|dx = o(1)$ almost surely and hence*

$$\mathrm{E}\left[\int |f_{n,\hat{h}}(x) - f(x)|dx\right] \to 0 \text{ as } n \to \infty.$$

For a plug-in type bandwidth selector such as the Scott's rule (see Scott (1992), page 152), the condition $\hat{h} \in I_n$ may not hold for every sample. However for such a random bandwidth vector, we have some knowledge about the rate at which its components go to zero in probability or almost surely as n goes to infinity. In the next theorem we replace Assumption 2 by some assumptions on the asymptotic behavior of the random vector $\hat{h}$.

Theorem 2. *Let f be a density function on $\mathbb{R}^d$, such that Assumption 1 holds for $f_j, j = 1, \ldots, d$. Suppose the random bandwidth $\hat{h} = (\hat{h}_1, \ldots, \hat{h}_d)$ satisfies $\hat{h}_j + \frac{1}{n\hat{h}_j^d} \to 0$ in probability/almost surely/exponentially as $n \to \infty$, for each $j = 1, \ldots, d$, then $\int |f_{n,\hat{h}}(x) - f(x)|dx \to 0$, in probability/almost surely/exponentially, and hence $\mathrm{E}[\int |f_{n,\hat{h}}(x) - f(x)|dx] \to 0$ as $n \to \infty$.*

The above-mentioned theorem is an extension of Theorem 1 on page 148 of Chapter 6 in Devroye and Györfi (1985). Under Scott's rule, each $\hat{h}_j = \hat{\sigma}_j n^{-1/(d+4)},\ j = 1, \ldots, d$, where $\hat{\sigma}_j$ is the sample standard deviation along the jth coordinate direction. Clearly the Scott's random bandwidth satisfies the assumption in Theorem 2, almost surely. So Theorem 2 implies the L_1 consistency of the estimator using Scott's random bandwidth. This result seems to be new.

2.1. *A Simple Density Estimator*

For larger d the estimator (1.1) is computationally more expensive, as the number of parameters increase with dimension d. In order to reduce the computational burden, we consider the following parametrization:

$$\hat{h}_j = \eta_j^* \lambda,\ j = 1, \ldots, d,$$

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where η_j^* is the sample interquartile range based on $X_{1j}, \ldots, X_{nj}$. The reviewers have rightly pointed out that the above parametrization of the bandwidth vector is similar to the $\mathcal{C}_2$ parametrization class introduced by Wand and Jones (1993). In the $\mathcal{C}_2$ parametrization, $\hat{h}_j$ is chosen to be proportional to the marginal sample standard deviation along the jth coordinate direction. Unfortunately this parametrization does not seem to work well (see pages 525 and 526 in Wand and Jones (1993)). Wand and Jones (1993) have pointed out that if the underlying density is multi-modal and the modes are sufficiently far apart, the marginal variance/standard deviation can be a very poor surrogate for measuring the optimal amount of smoothing. Consequently the density estimator using Wand and Jones's $\mathcal{C}_2$ class parametrization can perform rather poorly for certain multi-modal densities (such as the density f in Wand and Jones (1993)). The authors mention that ".....one could make $\mathcal{C}_2$ perform arbitrarily poorly by taking the modes to be sufficiently far apart, because this increases the variance but not the optimal amount of smoothing" (see page 526 in Wand and Jones (1993)). Further, we observe that if the marginal densities are heavy tailed (such as the log-normal density), the Wand and Jones's $\mathcal{C}_2$ class bandwidths exhibit high sampling fluctuation. We replace the marginal sample standard deviations in the Wand and Jones's $\mathcal{C}_2$ class by more robust (against extreme observations) marginal sample interquartile range.

For the above choice of $\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_d$ and with a product kernel in (1.1), we get the following estimator:

$$f_{n,\lambda}(x) = \frac{1}{n\lambda^d} \sum_{i=1}^{n} \prod_{j=1}^{d} \frac{1}{\eta_j^*} K^* \left(\frac{x_j - X_{ij}}{\eta_j^* \lambda} \right). \tag{2.1}$$

This estimator depends only on one parameter λ, irrespective of d. Consequently the bandwidth selection is much faster to compute (especially for high dimensional data) in comparison to $f_{n,\hat{h}}$ in (1.1), based on d bandwidths. For h_j equal to constant multiple of λ, $j = 1, \ldots, d$, the asymptotic MISE of the estimator $f_{n,h}$ is minimized for λ equal to multiple of $n^{-1/(4+d)}$. Therefore while implementing $f_{n,\lambda}$ we restrict the search for λ in an interval $[c_1 n^{-1/(4+d)}, c_2 n^{-1/(4+d)}]$, where $c_1 < c_2$. Let $\hat{\lambda}$ denote the bandwidth λ chosen by a data based algorithm. The following corollary is a direct consequence of Theorem 1.

Corollary 2.1. *Let f be a density function on $\mathbb{R}^d$, such that Assumption 1 holds for f_j, $j = 1, \ldots, d$. If $\hat{\lambda} \in [c_1 n^{-1/(4+d)}, c_2 n^{-1/(4+d)}]$, then $\int |f_{n,\hat{\lambda}}(x) - f(x)|dx \to 0$ exponentially as $n \to \infty$ and as a consequence $\mathrm{E}\left[\int |f_{n,\hat{\lambda}}(x) - f(x)|dx\right] \to 0$ as $n \to \infty$.*

3. Simulation and data analysis

We compare the performances of the estimators $f_{n,\hat{\lambda}}$ in (2.1) and $f_{n,\hat{h}}$ in (1.1), where the bandwidths are chosen by cross-validation method. The reason for using least square cross-validation to choose bandwidths is motivated by the fact that under the constraint $h_1 = h_2 = \cdots = h_d = h$, the relative and absolute performance of the cross-validation bandwidth selector is known to improve with increase in dimension (see Sain et al. (1994)).

Let $\delta_{ijk} = (X_{ik} - X_{jk})/h_k, k = 1, \ldots, d$. The multivariate least square cross-validation method (see Sain et al. (1994)) involves minimization of the function $UCV(h_1, \ldots, h_d)$ defined as follows:

$$UCV(h_1, \ldots, h_d) = \frac{1}{n(2\sqrt{\pi})^d \prod_{i=1}^{d} h_i}$$

$$+\frac{1}{n^2(2\sqrt{\pi})^d \prod_{i=1}^d h_i} \sum_{i=1}^d \sum_{j\neq i}^d \times \left[\exp\left\{ -\sum_{k=1}^d \delta_{ijk}^2/4 \right\} - 2^{1+d/2} \exp\left\{ -\sum_{k=1}^d \delta_{ijk}^2/2 \right\} \right].$$

The bandwidth vector $\hat{h}$ is chosen by minimizing $UCV(h_1, h_2, \ldots, h_d)$ with respect to h_1, h_2, $\ldots$, h_d on d-fold product of $(0, \infty)$. To choose $\hat{\lambda}$, we use $h_j = \eta_j^* \lambda$, $j = 1, 2, \ldots, d$ in UCV and minimize it with respect to λ on $[c_1 n^{-1/(4+d)}, c_2 n^{-1/(4+d)}]$. We observe that $\hat{\lambda}$ is sensitive to f. We use $c_1 = 0.001$ and $c_2 = 1000$. For these choice of c_1, c_2, the interval $[c_1 n^{-1/(4+d)}, c_2 n^{-1/(4+d)}]$ covers a wide range of possible values of $\hat{\lambda}$ which may be suitable for widely varying choices of f.

Two modifications, viz, the biased and the smooth cross-validation, of the above-mentioned multivariate least square cross-validation (UCV) selector were proposed by Duong and Hazelton (2005). The smooth cross-validation bandwidth (SCV) selector was recommended by the authors. We also compare the performance of $f_{n,\hat{\lambda}}$ with the estimator based on SCV bandwidth.

One of the reviewers pointed out that when $d = 2$, there exists closed form expressions for the bandwidths h_1, h_2 minimizing the asymptotic MISE (AMISE) of the estimator in (1.1) (see Wand and Jones (1995)). The expressions of these optimal h_1, h_2 depend on unknown constants, such as the integrated squared partial derivatives of f. While these constants are unknown in practice, in a simulation study they can be approximated numerically given a specific choice of f. In the bivariate case, we compare the ISE values of the simple estimator $f_{n,\hat{\lambda}}$ with the same for the estimator in (1.1), using the optimal choice of (h_1, h_2) as mentioned in Wand and Jones (1995) (we are thankful to the anonymous referee for this suggestion).

Let $\sum_{i=1}^k p_i N(m_{i1}, m_{i2}; \sigma_{i1}^2, \sigma_{i2}^2, \rho_i \sigma_{i1} \sigma_{i2})$ denote the density of a bivariate mixed normal distribution, with k mixing components. $p_1, p_2, \ldots, p_k$ denote the mixing proportions; (m_{i1}, m_{i2}) is the mean vector of the ith mixing component; σ_{i1}, σ_{i2} are the standard deviations along the two coordinate directions; and ρ_i is the correlation coefficient of the ith mixing component for $i = 1, 2, \ldots, k$. We consider the following bivariate test distributions, viz:

A. $N(0, 0; 1, 1, 0.0)$;

B. $0.9N(0, 0; 1, 1, 0.0) + 0.1N(2, 2; 1/2, 1/2, 0.0)$ (outlier);

C. $N(0, 0; 1, 1, 0.9)$ (correlated normal);

D. $\frac{4}{11}N(-2, 2; 1, 1, 0.0) + \frac{3}{11}N(0, 0; 0.8, 0.8, -0.72) + \frac{4}{11}N(2, -2; 1, 1, 0.0)$ (dumbbell).

Let $N_d(0, I)$ denote a d-dimensional normal distribution with zero mean and $d \times d$ identity matrix as the dispersion matrix. Also let $N_d(0, \Sigma)$ denote a d-dimensional normal distribution with zero mean and the dispersion matrix Σ, where the diagonal elements are 1 and off diagonal elements are 0.9. To study the effect of higher dimensions we consider the following test distributions:

E. $N_d(0, I)$, $d = 3$; F. $N_d(0, I)$, $d = 6$; G. $N_d(0, \Sigma)$, $d = 3$, and H. $N_d(0, \Sigma)$, $d = 6$.

We also consider test distributions such that the marginal densities are heavy-tailed. Let I, J, and K denote the product of two, three, and six standard log normal densities, respectively.

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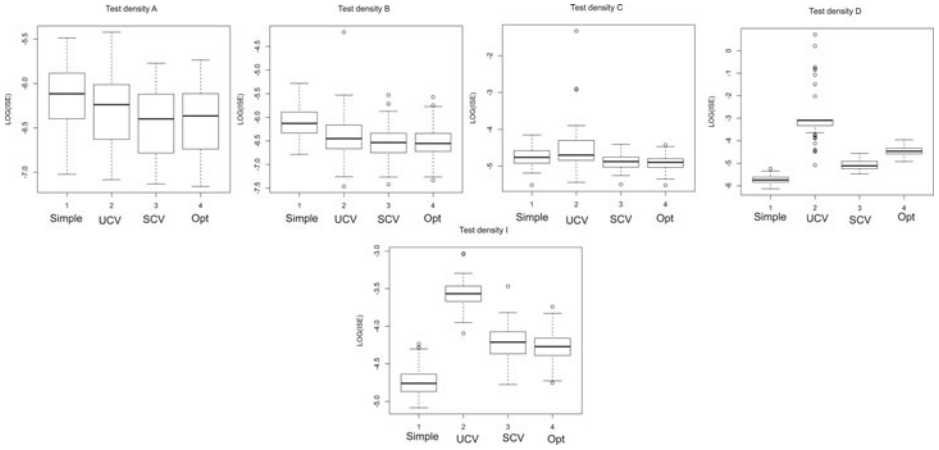


Figure 1. Logarithm of the ISE values of the four bivariate density estimators based on 100 samples from the test densities *A*–*D* and *I*, respectively, from left to right.

From each test density we draw 100 samples each of size $n = 500$. The ISE of each density estimate is approximated using Monte-Carlo integration. In Fig. 1, we plot the box plots of the logarithm of the 100 ISE values of the bivariate density estimators for the five bivariate test densities, viz, *A*–*D* and *I*. In each figure, the box plots numbered 1, 2, 3 and 4 represent the logarithm of the ISE values of the simple density estimator $f_{n,\hat{\lambda}}$ and the estimators using the UCV (by Sain et al. (1994)), SCV (by Duong and Hazelton (2005)), and the optimal bandwidths in Wand and Jones (1995), respectively. We call these four estimators as Simple, UCV, SCV, and Opt, respectively.

Figure 2 contains the box plots of the logarithm of the 100 ISE values of the three-dimensional density estimators for the test densities *E*, *G*, and *J*. In Fig. 3 we plot the same for the six-dimensional test densities *F*, *H*, and *K*. In Figs. 2 and 3 we plot the logarithm of the ISE values of $f_{n,\hat{\lambda}}$, the UCV, and the SCV based estimators, numbered as 1, 2, and 3, respectively. For $d > 2$, no closed form expression for $h_1, \ldots, h_d$ minimizing the AMISE seems to be known.

From Figs. 1–3 we have the following observations:

1. None of the estimators can be claimed to be uniformly the best in all the examples. For *f* equal to a multivariate normal density, the SCV based estimator seems to perform better than the other estimators. However, if the shape of the density is markedly

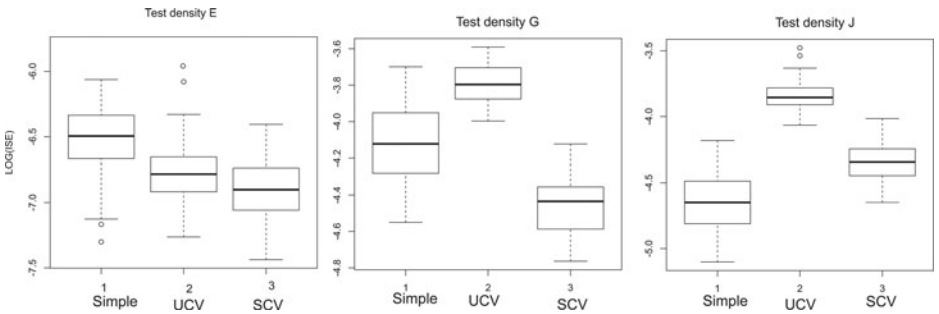


Figure 2. Logarithm of the ISE values of the three-dimensional density estimators based on 100 samples (of size 500) from the test densities *E*, *G*, and *J*.

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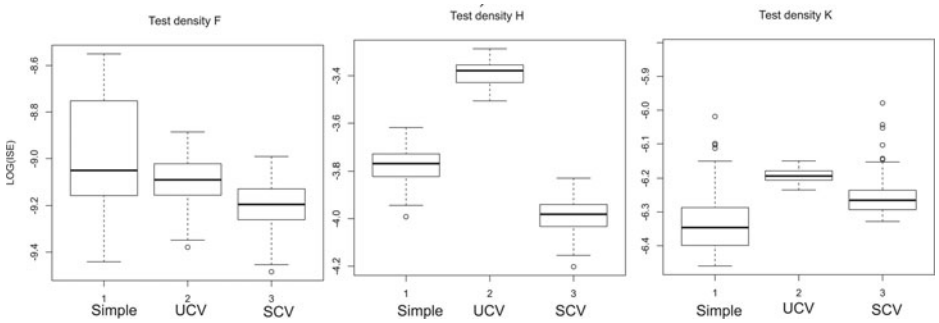


Figure 3. Logarithm of the ISE values of the six-dimensional density estimators based on 100 samples (of size 500) from the test densities *F*, *H*, and *K*.

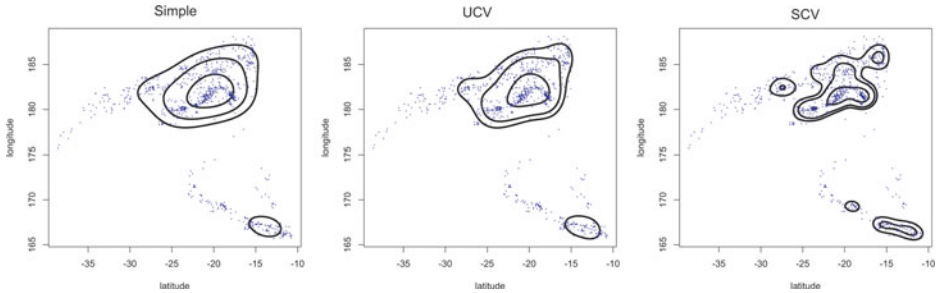


Figure 4. Contour plots of the marginal density estimates of the (latitude, longitude) distribution, using the proposed simple, UCV, and SCV bandwidths (left to right).

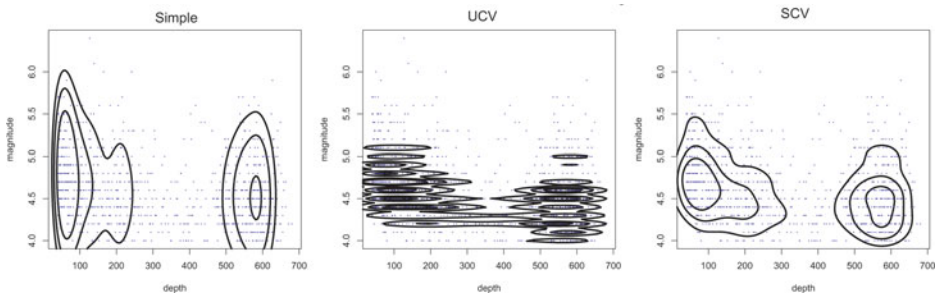


Figure 5. Contour plots of the marginal density estimates of the (depth, magnitude) distribution, using the proposed simple, UCV, and SCV bandwidths, respectively.

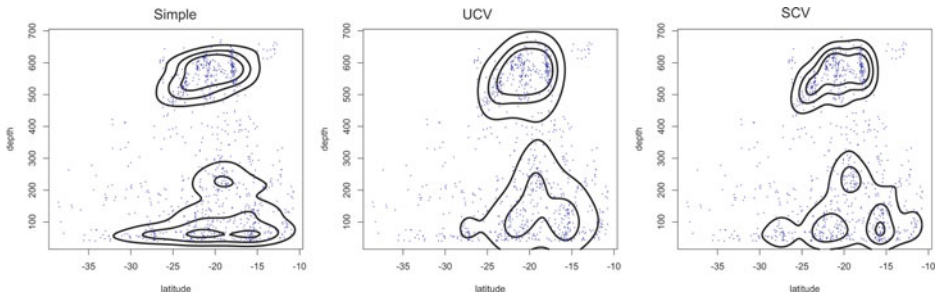


Figure 6. Contour plots of the marginal density estimates of the (latitude, depth) distribution, using the proposed simple, UCV, and SCV bandwidths, respectively.

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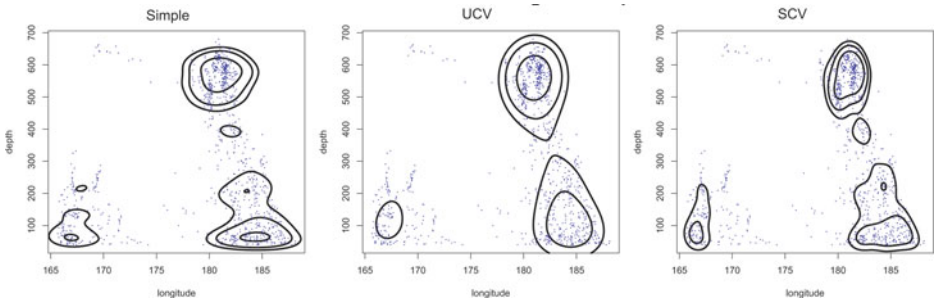


Figure 7. Contour plots of the marginal density estimates of the (longitude, depth) distribution, using the proposed simple, UCV, and SCV bandwidths, respectively.

different from a normal density or if the marginal densities are heavy-tailed, the proposed estimator $f_{n,\hat{\lambda}}$ appears to perform better than the other estimators, including the bivariate estimator using the ideal AMISE optimal bandwidths. See, for instance, the box plots for the densities *D*, *I*, *J*, and *K*.

2. In the presence of high correlation among $X_{1i}, X_{1j}, i, j = 1, \ldots, d$, the proposed simple estimator seems to perform better than the unconstrained UCV based estimator. This pattern remains the same irrespective of the dimension d (see the box plots for densities *C*, *G*, and *H* in Figs. 1–3, respectively).
3. One of the main criticisms against the UCV method is the high sampling fluctuation exhibited by the resulting estimator. For all the two dimensional test densities the simple estimator $f_{n,\hat{\lambda}}$ exhibits much less sampling fluctuation than the original UCV based estimator (see Fig. 1). However, we observe that the sampling fluctuation in the UCV based estimator seems to reduce substantially with increase in the data dimension d (see Figs. 2 and 3).
4. The SCV based selector is computationally much more expensive than the UCV selector, especially for large n and $d \geq 3$. For $d = 3$ and $n = 500$, it requires more than 10 minutes to implement the SCV selector, using the "ks" package in R and 2.60 GHz Pentium (R) dual core processor. In contrast under the same setup, it requires less than 3 seconds to implement the UCV based estimator for $d = 3$ and $n = 100$. This drastic difference in computation time in the two bandwidth selectors increases with increase in d, especially for large n. The simple estimator in (2.1) can be implemented quickly by choosing λ by least square cross-validation method, as explained above.

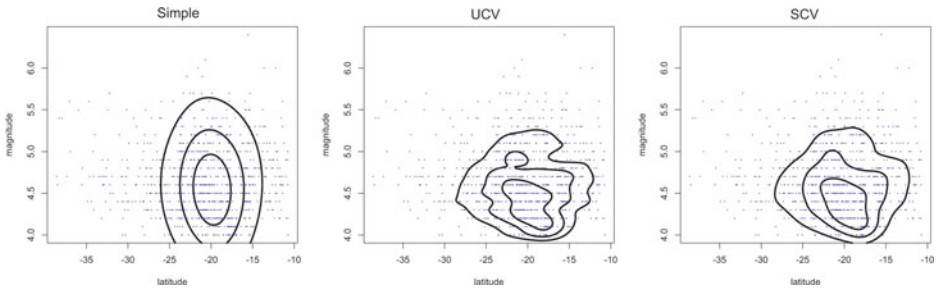


Figure 8. Contour plots of the marginal density estimates of the (latitude, magnitude) distribution, using the proposed simple, UCV, and SCV bandwidths, respectively.

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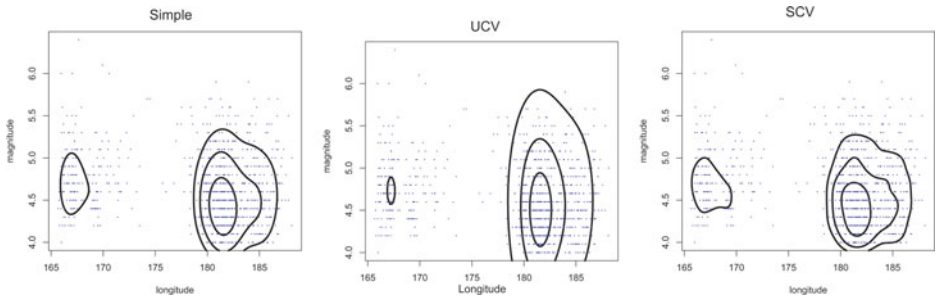


Figure 9. Contour plots of the marginal density estimates of the (longitude, magnitude) distribution, using the proposed simple, UCV, and SCV bandwidths, respectively.

3.1. *Fiji Earthquake data*

A well-known data set is the data on latitude, longitude, depth, and magnitude of 1,000 earthquakes in Fiji since 1964 (see the data set "quakes" in the software R). Here $n = 1000$ and $d = 4$. There are six extreme quakes of magnitude exceeding 5.5 in Richter scale and occurring at a depth less than 70 km from the surface. The above simulations suggest that in the presence of extreme observations in the data, the proposed simple estimator appears to be quite useful. In Figs. 4–9, we plot the contours of the bivariate marginal density estimates using the proposed simple, the UCV, and the SCV bandwidth vectors. In this case there are six bivariate marginal distributions, viz, the (latitude, longitude), the (depth, magnitude), the (latitude, depth), the (longitude, depth), the (latitude, magnitude), and the (longitude, magnitude) distributions. In each plot, the proposed simple estimate, the UCV, and the SCV based estimates are drawn from left to right. Some observations are mentioned below.

1. There seems to be two distinct peaks/modes present in the (latitude, longitude) distribution of the Fiji quakes (see first two contour plots in Fig. 4). The bigger peak represents the location of quakes with latitude and longitude varying in the range $[-25,-20]$ and [180, 185], respectively. The smaller peak represents the quakes occurring at latitude varying from -15 to -10 and longitude varying in the range [165, 170]. The SCV bandwidth based plot reveals several other smaller peaks, which appear to be sampling artifacts rather than important features of this data.
2. The UCV based estimator seems to perform poorly in estimating the density of the (depth, magnitude) distribution of the Fiji quakes (see the center panel in Fig. 5). The other two estimates in Fig. 5 of the (depth, magnitude) density seem to reveal two modes. The two peaks represent quakes with depths varying in the intervals [0, 100] and [500, 600], respectively.
3. The marginal densities of (latitude, depth) and (longitude, depth) appear to have multiple peaks. In both these distributions there seems to be a peak with depth varying in the interval [500, 600]. See Figs. 6 and 7. In the estimates of the (longitude, depth) density a small peak is observed with longitude in the range [165, 170] and depth less than 100 (see Fig. 7). The (latitude, depth) density also seems to exhibit a small peak with latitude varying between -20 to -15 and depth less than 100 (see Fig. 6).
4. The marginal density of (latitude, magnitude) seems to be uni-modal, and the peak represents quakes with latitude in $[-25, -15]$ and magnitude in [4, 5.5] (see Fig. 8). The marginal density of (longitude, magnitude) appears to be bimodal (see Fig. 9). The bigger peak represents quakes with longitude varying from 180 to 185 and magnitude

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varying from 4 to 5. The smaller peak represents quakes with longitude varying from 165 to 170 and magnitude from 4.4 to 5 (see Fig. 9).

The above observations suggest that in Fiji there is a high probability of observing quakes with latitude varying from −25 to −20, longitude varying from 180 to 185, and depth varying from 500 to 600. There is also a significant chance of observing quakes with latitude varying between −15 to −10, longitude in the range [165, 170], at depths less than 100 km and of magnitude above five. The quakes of magnitude exceeding 5 and closer to the surface can cause substantial damage. An important difference between the proposed and the SCV based estimates is that $f_{n,\hat{\lambda}}$ seems to assign higher probability of observing a quake with magnitude exceeding 5 and depth less than 100 (see the left and the right panels in Fig. 5).

4. Proofs

The main ingredient of the proof of Theorem 1 and Theorem 2 is Lemma 3. To prove Lemma 3, we need two more lemmas, Lemma 1 and Lemma 2. Lemma 1 is a generalization of part (iv) of Theorem 1 in Devroye (1983). We start this section with a brief review of this result of Devroye (1983).

4.1. *A review of the proof of devroye's (1983) theorem 1 (iv) and its extension*

Theorem 1 (iv), in Devroye (1983), states that $\int \left|f_{n,h_1}(x) - f(x)\right| dx \to 0$, exponentially as $n \to \infty$, where f_{n,h_1} is the estimator in (1.1) with $h_1 = \cdots = h_d$ and where $nh_1^d \to \infty$ and $h_1 \to 0$ as $n \to \infty$. The proof of this theorem is divided into two lemmas, namely, Lemma 1 and Lemma 4 in Devroye (1983). Clearly

$$\int \left|f_{n,h_1}(x) - f(x)\right| dx \le \int \left|f_{n,h_1}(x) - \mathrm{E}[f_{n,h_1}(x)]\right| dx + \int \left|\mathrm{E}[f_{n,h_1}(x)] - f(x)\right| dx.$$

In Lemma 1, Devroye (1983) showed that if $h_1 \to 0$ then $\int |\mathrm{E}[f_{n,h_1}(x)] - f(x)|dx \to 0$ as $n \to \infty$. Now repeating similar arguments as given in the proof of Lemma 1 of Devroye (1983), one can show that as $h_j \to 0$, for $j = 1, 2, \ldots, d$, as $n \to \infty$,

$$\int |\mathrm{E}[f_{n,h}(x)] - f(x)|dx \to 0 \text{ as } n \to \infty, \tag{4.1}$$

where $f_{n,h}$ is the estimator as defined in (1.1), and h_j's are not necessarily equal.

In Lemma 4, Devroye (1983) proved that $\mathrm{P}\left(\int |f_{n,h_1}(x) - \mathrm{E}[f_{n,h_1}(x)]|dx > \epsilon\right)$ converges to zero at an exponential rate as $n \to \infty$.

The main part in the proof of the Lemma 4 in Devroye (1983) is that for $\lim_n nh_1^d = \infty$, $\lim_n h_1 = 0$ and for any finite rectangle A of $\mathbb{R}^d$,

$$\frac{1}{h_1^d} \int |\mu(x + h_1 A) - \mu_n(x + h_1 A)|\, dx \to 0 \text{ (exponentially)} \tag{4.2}$$

as $n \to \infty$, where μ_n is the empirical probability measure based on $X_1, X_2, \ldots, X_n$, μ is the probability measure corresponding to f and

$$x + h_1 A = \left\{(x + h_1 y_1, \ldots, x + h_1 y_d) : \text{where } (y_1, \ldots, y_d) \in A\right\}.$$

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We observe that with slight modification in the arguments used by Devroye, the above result can be extended to the following result. If $\lim_n h_i = 0$ and $\lim_n nh_i^d = \infty,\ i = 1, \ldots, d$, then

$$\frac{1}{\prod_{i=1}^d h_i} \int |\mu\,(x + hA) - \mu_n\,(x + hA)|\,dx \to 0 \text{ (exponentially)} \tag{4.3}$$

as $n \to \infty$, where $x + hA = \{(x + h_1 y_1, \ldots, x + h_d y_d) : \text{where } (y_1, \ldots, y_d) \in A\}$.

To prove (4.2), Devroye partitioned $\mathbb{R}^d$ into sets which are d-fold product of the intervals of the form $[(i-1)h_1/N, ih_1/N)$, where i is an integer and N is a appropriately chosen positive constant.

To prove (4.3), we partition $\mathbb{R}^d$ into rectangles of the form $\prod_{j=1}^d [(i_j - 1)h_j/N, i_j h_j/N)$, where $\{i_j, j = 1, \ldots, d\}$ are integers and then repeat the same arguments as on pages 899 and 900 in Devroye (1983) with appropriate adjustment.

Using (4.1) and (4.3), one can easily generalize Theorem 1(iv) of Devroye (1983) and have the following lemma.

4.2. *Some lemmas and their proofs*

Lemma 1. *Let K be a non negative Borel measurable function on $\mathbb{R}^d$ with $\int K(x)dx = 1$. If $\lim_{n\to\infty} h_i = 0$ and $\lim_{n\to\infty} nh_i^d = \infty,\ i = 1, \ldots, d$, then for all $\epsilon > 0$, there exist r, $n_0 > 0$ such that*

$$P\left(\int \left|f_{n,h}(x) - f(x)\right| dx > \epsilon\right) \le \exp(-rn\epsilon^2) \text{ for } n \ge n_0.$$

Now we state a lemma which is a simple generalization of Lemma 1 of Chapter 6 in Devroye and Györfi (1985). We omit its proof.

Lemma 2. *For any density K on $\mathbb{R}^d$,*

$$\int |K_h(x) - K(x)|dx \to 0, \quad \text{if } h_i \to 1 \text{ for all } 1 \le i \le d,$$

where $K_h(x) = (\prod_{i=1}^d h_i)^{-1} K(x_1/h_1, \ldots, x_d/h_d)$.

We use the above two lemmas to prove of following result and this result is the main ingredient of the proof of the theorems.

Lemma 3. *Consider a sequence of intervals $H_n = \prod_{l=1}^d [h_l', h_l'']$, where for all l, $h_l'' \to 0$, $nh_l'^d \to \infty$ as $n \to \infty$. Then, for each $\epsilon > 0$ there exist r_0, $N > 0$ such that*

$$P\left(\sup_{h\in H_n} \int \left|f_{n,h}(x) - f(x)\right| > \epsilon\right) < \exp(-r_0 n\epsilon^2), \text{ for all } n \ge N,$$

where $h = (h_1, h_2, \ldots, h_d)$.

Proof. By our assumption $\dfrac{h_l''^d}{nh_l'^d} \to 0$ as $n \to \infty$ and thus $1 \le \dfrac{h_l''}{h_l'} = a_{l,n} n^{1/d}$ where $\lim_{n\to\infty} a_{l,n} = 0$. Let $\delta_{l,n} \ge 0$ be a solution of the equation

$$(1 + \delta_{l,n})^n = a_{l,n} n^{1/d}.$$

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Then clearly $\delta_{l,n} \to 0$ as $n \to \infty$. Now introduce

$$h_{l,i} = (1 + \delta_{l,n})^i h'_l \text{ for } i = 0, 1, 2, \ldots, n \text{ and } 1 \le l \le d.$$

Thus $h_{l,0} = h'_l$ and $h_{l,n} = h''_l$, so that

$$\sup_{h \in H_n} \int |f_{n,h} - f| dx \le \sup_{1 \le i_1, \ldots, i_d \le n} \left(\int \left| f_{n,h_{(i_1-1,\ldots,i_d-1)}} - f \right| dx + \sup_{h_{j,i_j-1} \le h_j < h_{j,i_j}} \int \left| f_{n,h} - f_{n,h_{(i_1-1,\ldots,i_d-1)}} \right| dx \right), \tag{4.4}$$

where $h = (h_1, h_2, \ldots, h_d)$, $dx = dx_1 dx_2 \cdots dx_d$ and

$$f_{n,h_{(i_1-1,\ldots,i_d-1)}} = \frac{1}{\prod_{j=1}^d h_{j,i_j-1}} \sum_{i=1}^n K\left(\frac{x_1 - X_{i,1}}{h_{1,i_1-1}}, \ldots, \frac{x_d - X_{i,d}}{h_{d,i_d-1}} \right).$$

Now for $u = (u_1, u_2, \ldots, u_d)$ and $|u| = u_1 u_2 \ldots u_d$,

$$\int \left| f_{n,h} - f_{n,u} \right| dx \le \frac{1}{n} \sum_{i=1}^n \int \left| \frac{1}{|h|} K\left(\frac{x - X_i}{h} \right) - \frac{1}{|u|} K\left(\frac{x - X_i}{u} \right) \right| dx.$$

Replacing $\dfrac{x_j - X_{i,j}}{u_j} = y_j$ in the last integration we get

$$\begin{aligned} \int \left| f_{n,h} - f_{n,u} \right| dx &\le \frac{1}{n} \sum_{i=1}^n \int \left| \left(\prod_{j=1}^d \frac{u_j}{h_j} \right) K\left(\frac{u_1}{h_1} y_1, \ldots, \frac{u_d}{h_d} y_d \right) - K\left(y_1, \ldots, y_d \right) \right| dy \\ &= \int \left| K_{h/u}(y) - K(y) \right| dy, \end{aligned}$$

where $K(y) = K(y_1, \ldots, y_d)$, $K_h(y) = \frac{1}{|h|} K\left(\frac{y_1}{h_1}, \ldots, \frac{y_d}{h_d} \right)$, and $|h| = \prod_{i=1}^d h_i$.

Now denote

$$\phi(\delta_1, \ldots, \delta_d) = \sup_{\substack{1 - \delta_i \le h_i \le 1 + \delta_i \\ 1 \le i \le d}} \int |K_h(y) - K(y)| dy.$$

Then for fixed $i_1, i_2, \ldots, i_d$,

$$\begin{aligned} \sup_{\substack{h_{j,i_j-1} \le h_j < h_{j,i_j} \\ 1 \le j \le d}} \int \left| f_{n,h} - f_{n,h_{(i_1-1,\ldots,i_d-1)}} \right| &\le \sup_{\substack{h_{j,i_j-1} \le h_j < h_{j,i_j} \\ 1 \le j \le d}} \phi\left(\frac{h_1}{h_{1,i_1-1}} - 1, \ldots, \frac{h_d}{h_{d,i_d-1}} - 1 \right) \\ &\le \phi\left(\frac{h_{1,i_1}}{h_{1,i_1-1}} - 1, \ldots, \frac{h_{d,i_d}}{h_{d,i_d-1}} - 1 \right) \\ &\quad \text{(by monotonicity of } \phi). \\ &= \phi(\delta_{1n}, \ldots, \delta_{dn}). \end{aligned}$$

Now by Lemma 2, for $\epsilon > 0$ there exists $n_1 > 0$ such that

$$\phi(\delta_{1n}, \ldots, \delta_{dn}) \le \epsilon \text{ for all } n \ge n_1. \tag{4.5}$$

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Now from (4.4) and (4.5)

$$\mathrm{P}\left(\sup_{(h_1,\ldots,h_d)\in H_n}\int |f_{n,h}-f|dx>2\epsilon\right)\le\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}\mathrm{P}\left(\int |f_{n,h_{(i_1-1,\ldots,i_d-1)}}-f|dx>\epsilon\right). \tag{4.6}$$

Now from Lemma 1 it follows that for $\epsilon>0$ there exists n_0 such that

$$\mathrm{P}\left(\int |f_{n,h_{(i_1-1,\ldots,i_d-1)}}-f|dx>\epsilon\right)\le\exp(-nr\epsilon^2),\ \text{for } n\ge n_0$$

and hence there exists a sufficiently large positive integer N such that for $n\ge N$

$$\mathrm{P}\left(\sup_{(h_1,\ldots,h_d)\in H_n}\int |f_{n,h}-f|dx>2\epsilon\right)\le n^d\exp(-nr\epsilon^2)<\exp(-nr_0\epsilon^2),$$

where $0<r_0<r$. This concludes the proof of this lemma. □

4.3. *Proof of theorems*

Proof of Theorem 1. Let $H_n=\prod_{j=1}^{d}[a_jn^{-1/(4+d)},\ b_jn^{-1/(4+d)}]$, where $a_j=(\eta_j-\eta_j/2)c_{1j}$, $b_j=(\eta_j+\eta_j/2)c_{2j}$. Then $\{H_n\}$ is a sequence of intervals as in Lemma 3. Now

$$\begin{aligned}\mathrm{P}\left(\int\left|f_{n,\hat{h}}(x)-f(x)\right|dx>\epsilon\right)&\le\mathrm{P}\left(\hat{h}\notin H_n\right)+\mathrm{P}\left(\int\left|f_{n,\hat{h}}(x)-f(x)\right|dx>\epsilon,\ \hat{h}\in H_n\right)\\&\le\sum_{i=1}^{d}\left\{\mathrm{P}\left(\hat{h}_i<a_in^{-1/(4+d)}\right)+\mathrm{P}\left(\hat{h}_i>b_in^{-1/(4+d)}\right)\right\}\\&\quad+\mathrm{P}\left(\sup_{h\in H_n}\int\left|f_{n,h}(x)-f(x)\right|>\epsilon\right).\end{aligned} \tag{4.7}$$

Under Assumption 2, $\hat{h}\in I_n=\prod_{j=1}^{d}\left[c_{1j}\eta_j^*n^{-1/(4+d)},\ c_{2j}\eta_j^*n^{-1/(4+d)}\right]$. Now

$$\hat{h}_i<a_in^{-1/(4+d)}\Rightarrow c_{1i}\eta_i^*<a_i\Rightarrow\eta_i^*<(\eta_i-\eta_i/2)$$

and hence $\mathrm{P}\left(\hat{h}_i<a_in^{-1/(4+d)}\right)\le\mathrm{P}\left(|\eta_i-\eta_i^*|>\eta_i/2\right)$. Also

$$\begin{aligned}\mathrm{P}\left(|\eta_i-\eta_i^*|>\eta_i/2\right)&=\mathrm{P}\left(|(Q_{3i}-Q_{3i}^*)-(Q_{1i}-Q_{1i}^*)|>\eta_i/2\right)\\&\le\mathrm{P}\left(|(Q_{3i}-Q_{3i}^*)|+|Q_{1i}-Q_{1i}^*|>\eta_i/2\right)\\&\le\mathrm{P}\left(|(Q_{3i}-Q_{3i}^*)|>\eta_i/4\right)+\mathrm{P}\left(|(Q_{1i}-Q_{1i}^*)|>\eta_i/4\right).\end{aligned}$$

Hence for $i=1,2,\ldots,d$,

$$\mathrm{P}\left(\hat{h}_i<a_in^{-1/(4+d)}\right)\le\mathrm{P}(|Q_{1i}-Q_{1i}^*|>\eta_i/4)+\mathrm{P}(|Q_{3i}-Q_{3i}^*|>\eta_i/4).$$

Similarly one can verify that $\mathrm{P}\left(\hat{h}_i>b_in^{-1/(4+d)}\right)\le\mathrm{P}(|Q_{1i}-Q_{1i}^*|>\eta_i/4)+\mathrm{P}(|Q_{3i}-Q_{3i}^*|>\eta_i/4)$.

Now using Theorem 2.3.2 on page 74 in Serfling (1980) we get

$$\mathrm{P}\left(\hat{h}_i<a_in^{-1/(4+d)}\right),\ \mathrm{P}\left(\hat{h}_i>b_in^{-1/(4+d)}\right)\le4\exp(-2n\delta^2),\ i=1,\ldots,d, \tag{4.8}$$

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where $\delta = \min\{\delta_1, \ldots, \delta_d\}$, $\delta_i^2 = \min\{\delta_{1i}^2, \delta_{2i}^2\}$, and $\delta_{1i} = \min\{F_i(Q_{1i}^* + \eta_i/4) - 1/4,\ 1/4 - F_i(Q_{1i}^* - \eta_i/4)\}$ and $\delta_{2i} = \min\{F_i(Q_{3i}^* + \eta_i/4) - 3/4,\ 3/4 - F_i(Q_{3i}^* - \eta_i/4)\}$, where F_i is the ith marginal cumulative distribution function. Under Assumption 1, δ is a positive constant. Clearly δ is free of ϵ.

Substituting the above inequalities in the right-hand side of (4.7) we get

$$\mathrm{P}\left(\int \left|f_{n,\hat{h}}(x) - f(x)\right| dx > \epsilon\right) \leq 8d\exp(-2n\delta^2) + \mathrm{P}\left(\sup_{h\in H_n}\int \left|f_{n,h}(x) - f(x)\right| > \epsilon\right).$$

Finally using Lemma 3 for the second term in the right-hand side of the above inequality, we complete the proof of Theorem 1. □

Proof of Theorem 2. To prove this theorem we use similar argument as in the proof of Theorem 1 of Chapter 3 in Devroye and Györfi (1985). We prove the result for $\hat{h}_j + \frac{1}{n\hat{h}_j^d} \to 0$ in probability $j = 1, \ldots, d$. The proof for almost sure and exponential convergence can be obtained by similar argument.

The assumption $\hat{h}_j + \frac{1}{n\hat{h}_j^d} \to 0$ in probability is equivalent to the assumption that there exists a positive sequence $\{\epsilon'_{nj}\} \downarrow 0$ such that

$$\mathrm{P}\left(\hat{h}_j + \frac{1}{n\hat{h}_j^d} > \epsilon'_{nj}\right) \to 0 \text{ as } n \to \infty.$$

Now we define

$$\epsilon_{nj} = \max\{\epsilon'_{nj}, n^{-1/(d+1)}\},\ \text{for } j = 1, 2, \ldots, d.$$

It is easy to see that for each $j = 1, \ldots, d$, $\{\epsilon_{nj}\}\downarrow 0$ and

$$\mathrm{P}\left(\hat{h}_j + \frac{1}{n\hat{h}_j^d} > \epsilon_{nj}\right) \to 0 \text{ as } n \to \infty.$$

Next we define $H_n = \prod_{j=1}^d [h'_j, h''_j]$, where

$$h'_j = (n\epsilon_{nj})^{-1/d},\ h''_j = \epsilon_{nj},\ j = 1, \ldots, d.$$

Clearly, $h''_j \to 0$ and $n(h'_j)^d \to \infty,\ j = 1, \ldots, d$, as $n \to \infty$. Further

$$\begin{aligned}
\mathrm{P}\left(\hat{h} \notin H_n\right) &\leq \sum_{j=1}^d \left\{\mathrm{P}(\hat{h}_j < h'_j) + \mathrm{P}(\hat{h}_j > h''_j)\right\} \\
&\leq \sum_{j=1}^d \left\{\mathrm{P}\left(\frac{1}{n\hat{h}_j^d} > \epsilon_{nj}\right) + \mathrm{P}\left(\hat{h}_j > \epsilon_{nj}\right)\right\} \\
&\leq 2\sum_{j=1}^d \mathrm{P}\left(\hat{h}_j + \frac{1}{n\hat{h}_j^d} > \epsilon_{nj}\right) \to 0,\ \text{as } n \to \infty. \qquad (4.9)
\end{aligned}$$

Finally we see that

$$\mathrm{P}\left(\int \left|f_{n,\hat{h}}(x) - f(x)\right| dx > \epsilon\right) \leq \mathrm{P}\left(\hat{h} \notin H_n\right) + \mathrm{P}\left(\sup_{h\in H_n}\int \left|f_{n,h}(x) - f(x)\right| dx > \epsilon\right).$$

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Using Lemma 3 and (4.9), we see that the right side of the above inequality goes to zero as $n \to \infty$.

To prove the convergence of $\mathrm{E}\big[\int |f_{n,\hat{h}}(x) - f(x)|dx\big]$, we note that $0 < \int |f_{n,\hat{h}}(x) - f(x)|dx \leq 2$. Now the result is proved by observing that if $0 < Y_n \leq C$, where C is a constant free of n and Y_n converges to zero in probability, then $\mathrm{E}(Y_n) \to 0$ as $n \to \infty$. □

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POINTWISE AND UNIFORM CONVERGENCE OF MULTIVARIATE KERNEL DENSITY ESTIMATORS USING RANDOM BANDWIDTHS

SANTANU DUTTA * KOUSHIK SAHA †

Abstract

We obtain the rates of pointwise and uniform convergence of multivariate kernel density estimators using a random bandwidth vector obtained by some data based algorithm. We are able to obtain faster rate for pointwise convergence. The uniform convergence rate is obtained under some moment condition on the marginal distribution. The rates are obtained under i.i.d. and strongly mixing type dependence assumptions.

keywords. multivariate kernel density estimator, random bandwidth.

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1 INTRODUCTION

Let $X_1, \ldots, X_n$ be $\mathbb{R}^d$ valued random variables with density f. A kernel density estimator (KDE) with bandwidth vector $h = (h_1, \ldots, h_d)$ is defined as

$$\hat{f}_{n,h}(x) = \frac{1}{n \prod_{j=1}^{d} h_j} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right) = \frac{1}{n} \sum_{i=1}^{n} g_{x-X_i}(h), \tag{1.1}$$

where $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and K is a density function on $\mathbb{R}^d$, $X_i = (X_{i1}, X_{i2}, \ldots, X_{id})$ and

$$g_z(h) = \frac{1}{\prod_{i=1}^{d} h_i} K\left(\frac{z}{h}\right) = \frac{1}{\prod_{i=1}^{d} h_i} K\left(\frac{z_1}{h_1}, \ \ldots, \frac{z_d}{h_d}\right), \text{ for } z = (z_1, z_2, \ldots, z_d) \in \mathbb{R}^d.$$

From Scott (1992) we see that under i.i.d. assumption the optimal values of $h_1, \ldots, h_d$ in terms of minimizing the MISE are of the order $O(n^{-1/(4+d)})$.

The asymptotic properties of the KDE has been widely studied for nonrandom h. See for instance Rao (1983) for a detailed review. However, a practical problem in density estimation is to determine the appropriate values of $h_1, h_2, \ldots, h_d$ based on $X_1, X_2, \ldots, X_n$. Sain et al. (1994) discussed a number of algorithms, such as the least squares or the unbiased cross validation (UCV), the biased cross validation (BCV) and the smooth bootstrap bandwidth selector (SCV) for data based choice of h. Duong and Hazelton (2005) and Dutta (2011) have proposed other methods of choosing the bandwidth vector h based on the given data. Let $\hat{h} = (\hat{h}_1, \ldots, \hat{h}_d)$ be a bandwidth vector obtained by any one of these bandwidth selectors. Such a $\hat{h}$ is a random vector, i.e. a $\mathbb{R}^d$ valued function of $X_1, \ldots, X_n$. A natural problem is to investigate the asymptotic properties of the resulting estimator $\hat{f}_{n,\hat{h}}$. Obtaining properties of such an estimator can be much more challenging than the estimator based on non random h. For instance, it is very difficult to compute the expectation of $E[\hat{f}_{n,\hat{h}}]$ and hence the bias.

For univariate data, Dutta and Goswami (2013) have studied the convergence of the KDE using a random bandwidth. Far less seems to be known for multivariate KDE using a random bandwidth vector. Under the assumption that $h_1 = h_2 = \cdots = h_d$, Tenreiro (2001) established the convergence

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rates of the L_2 error of these estimators for a wide range of random bandwidths (we are thankful to the reviewer for this reference). Dutta and Saha (2013) studied the L_1 consistency of the multivariate KDE for a broad class of random bandwidth vectors. Under similar assumptions on the bandwidth vector $\hat{h}$, in this paper we study the pointwise and sup-norm convergence of such estimators with increase in sample size. In particular we obtain the rates at which $P(|\hat{f}_{n,\hat{h}}(x) - f(x)| > \epsilon)$ and $P(\|\hat{f}_{n,\hat{h}} - f\| > \epsilon)$ converges to zero, where $\epsilon > 0$. These rates extend the results in Dutta and Goswami (2013) for multivariate KDE and hold for a wide variety of bandwidth selectors discussed in Sain et al. (1994), Duong and Hazelton (2005), Dutta and Saha (2013).

1.1 Notation and Assumptions.

Let us define some notation which are used in the sequel. For any function g of d variables, let $\partial_j g$ denote the partial derivative of g with respect to the jth variable, $j = 1, \ldots, d$. For $x \in \mathbb{R}^d$, $\|x\|$ denotes its Eucledian norm and for any real valued function f, $\|f\|$ denotes its sup-norm. For any event A, $I(A)$ denotes a binary valued random variable such that $I(A)(w) = 1$ for $w \in A$ and $I(A)(w) = 0$ otherwise.

Let

$$I_n = \prod_{j=1}^{d} [a_j n^{-1/(4+d)}, b_j n^{-1/(4+d)}], \tag{1.2}$$

where $a_j < b_j$, $j = 1, \ldots, d$, are arbitrary positive constants. Let us state the assumptions on the kernel K and density f.

Assumption 1: The kernel K is a differentiable density in $\mathbb{R}^d$ satisfying the following conditions

(i)

$$\|K\|, \ \|\partial_j K\|, \ \sup_{x \in \mathbb{R}^d} \left| x_j \partial_j K(x) \right| < \infty \text{ for } j = 1, \ldots, d,$$

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(ii)

$$\sup_{x\in\mathbb{R}^d} \|x\|^d K(x) < \infty \text{ where } x = (x_1, \ldots, x_d).$$

Assumption 2: $\|f\| < \infty$ and $\|\partial_i f\| < \infty$ for $i = 1, \ldots, d$.

Assumption 3: There exists $\gamma > 0$ such that $\int_{\mathbb{R}^d} \|x\|^\gamma f(x)dx < \infty$.

Assumption 4: f_j, jth marginal density, is strictly positive on some open neighbourhood of Q_{1j} and Q_{3j}, $j = 1, \ldots, d$.

2 Main results

2.1 I.I.D. case

Theorem 1. *Let $\{X_n\}_{n\geq 1}$ be i.i.d. $\mathbb{R}^d$ valued random variables with density f, satisfying Assumption 2. Let K be a kernel satisfying Assumption 1(i) and let $\hat{h} \in I_n$. Then for every $\epsilon > 0$ there exists $C > 0$ such that, as $n \to \infty$*

$$P(|\hat{f}_{n,\hat{h}}(x) - f(x)| > \epsilon) = O\left(n^{d^2/(d+4)} \exp(-Cn^{4/(4+d)})\right).$$

Theorem 2. *Let $\{X_n\}_{n\geq 1}$ be i.i.d. $\mathbb{R}^d$ valued random variables with density f. Let $\hat{h} \in I_n$. Then for every $\epsilon > 0$ there exists $C' > 0$ such that, under Assumptions $1-3$, as $n \to \infty$*

$$P(\|\hat{f}_{n,\hat{h}} - f\| > \epsilon) = O\left(n^{[d+d^2(2+1/\gamma)]/(d+4)} \exp(-C'n^{4/(4+d)})\right),$$

where $C' > 0$ and γ is a positive constant as defined in Assumption 3.

The proof of Theorems 1 and 2 are direct consequences of Lemma 4 which is stated and proved in Section 3.1. Using Borel-Cantelli Lemma and Theorem 2 we get the following Corollary.

Corollary 1. *Let $\{X_n\}_{n\geq 1}$ be i.i.d. $\mathbb{R}^d$ valued random variables with density f. Under Assumptions $1-3$, $\|\hat{f}_{n,\hat{h}} - f\| = o(1)$ completely as $n \to \infty$.*

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The coefficients $a_j,\ b_j,\ j = 1, \ldots, d$, in I_n can also be specified as functions of $X_1, X_2, \ldots, X_n$. For example, Dutta and Saha (2013) suggested to search for $\hat{h}$ in the interval

$$\prod_{j=1}^{d}[c_{1j}\eta_j^* n^{-1/(4+d)},\ c_{2j}\eta_j^* n^{-1/(4+d)}], \tag{2.1}$$

where $0 < c_{1j} < c_{2j}$ and $\eta_j^* = Q_{3j}^* - Q_{1j}^*,\ j = 1, \ldots, d$, and $Q_{1j}^*,\ Q_{3j}^*$ be the 1st and the 3rd sample quartiles based on $X_{1j}, \ldots, X_{nj}$. X_{ij} denotes the jth component of the vector $X_i,\ j = 1, \cdots, d$. The role and importance of η_j^* are discussed in detail in Dutta and Goswami (2013). See for instance the introduction and final remark number 5 in Dutta and Goswami (2013). From Dutta and Goswami (2013) we see that if the coefficients $a_j,\ b_j,\ j = 1, \ldots, d$, are random, their convergence rates effect the rate of point wise and uniform convergence of the density estimator.

In the next Theorem we extend the results in Theorems 1 and 2 to the case where $a_j,\ b_j,\ j = 1, \ldots, d$ are functions of $X_1, \ldots, X_n$ satisfying certain conditions.

Theorem 3. *Let $\{X_n\}_{n\geq 1}$ be i.i.d. $\mathbb{R}^d$ valued random variables with density f. Let*

$$H_n = \prod_{j=1}^{d}[a_{nj}n^{-1/(4+d)}, b_{nj}n^{-1/(4+d)}],$$

where $0 < a_{nj} < b_{nj},\ j = 1, \ldots, d$, are functions of $X_{1j}, \ldots, X_{nj}$ such that $P(|a_{nj} - a_j| > \epsilon) = O(r_{n,\epsilon})$ and $P(|b_{nj} - b_j| > \epsilon) = O(r_{n,\epsilon})$, where $a_j,\ b_j,\ j = 1, \ldots, d$, are positive constants and $r_{n,\epsilon} = o(1)$ as $n \to \infty$. Let $\hat{h} \in H_n$. Then under Assumptions 1 – 3, as $n \to \infty$

$$(i) P(|\hat{f}_{n,\hat{h}}(x) - f(x)| > \epsilon) = O\left(r_{n,\epsilon} + n^{d^2/(d+4)} \exp(-Cn^{4/(4+d)})\right),$$
$$(ii) P(\|\hat{f}_{n,\hat{h}} - f\| > \epsilon) = O\left(r_{n,\epsilon} + n^{[d+d^2(2+1/\gamma)]/(d+4)} \exp(-C' n^{4/(4+d)})\right),$$

where $C,\ C' > 0$.

In particular if we use $a_j = c_{1j}\eta_j^*,\ b_j = c_{2j}\eta_j^*$, as suggested by Dutta and Saha (2013), we have the following result (at the expense of one more assumption, viz. Assumption 4).

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Corollary 2. *Under the Assumptions* $1-4$*, and assuming* $\hat{h} \in H_n$ *we see that as as* $n \to \infty$

$$(i) P(|\hat{f}_{n,\hat{h}}(x) - f(x)| > \epsilon) = O\left(n^{d^2/(d+4)} \exp(-Cn^{4/(4+d)})\right),$$
$$(ii) P(\|\hat{f}_{n,\hat{h}} - f\| > \epsilon) = O\left(n^{[d+d^2(2+1/\gamma)]/(d+4)} \exp(-C'n^{4/(4+d)})\right).$$

The above corollary follows from the following inequality in page 74 of Serfling (1980)

$$P(|Q_{1j} - Q^*_{1j}| > \eta_j/4),\ P(|Q_{3j} - Q^*_{3j}| > \eta_j/4) \leq \exp(-2n\delta^2),\ j = 1, \ldots, d,$$

where $\delta = \min\{\delta_1, \ldots, \delta_d\}$, $\delta_i^2 = \min\{\delta_{1i}^2, \delta_{2i}^2\}$, and $\delta_{1i} = \min\{F_i(Q^*_{1i}+\eta_i/4)-1/4,\ 1/4-F_i(Q^*_{1i}-\eta_i/4)\}$ and $\delta_{2i} = \min\{F_i(Q^*_{3i} + \eta_i/4) - 3/4,\ 3/4 - F_i(Q^*_{3i} - \eta_i/4)\}$, where F_i is the i-th marginal cumulative distribution function. Under Assumption 4, δ is a positive constant. Clearly δ is free of ϵ. This ensures that $r_{n,\epsilon}$ goes to zero at at exponential rate (which does not depend on ϵ).

Remark 1. *(i) The Theorems 1 and 2 are obtained under the assumption* $\hat{h} \in I_n$*. Therefore it is natural to question whether such an assumption is very artificial and restrictive? A number of well known bandwidth selectors such as the unbiased, the biased, and the smoothed cross validation and the different versions of the smooth bootstrap based bandwidth selectors in Sain et al. (1994) and Dutta (2011) involve optimization of some criterion (say* F_n*) which is a continuous function of* $h_1, \ldots, h_d$*. Even the multivariate plug-in bandwidth selection rule proposed by Wand and Jones (1994) involve optimization of the asymptotic mean integrated squared error(AMISE), where the unknowns are replaced by kernel based estimates, with respect to the bandwidths* $h_1, \ldots, h_d$*.*

Let $\hat{h}$ *denote the bandwidth vector that optimizes the bandwidth selection criterion* $F_n(h_1, \cdots, h_d)$*. If we optimize the bandwidth selection criterion* $F_n(h_1, \ldots, h_d)$ *on* $(0, \infty)^d$*, the optimum* $\hat{h}$ *may not be attained. Therefore it is natural to optimize the bandwidth selection criterion on some compact set, for example* I_n *or* H_n*. In that case, the resulting random bandwidth vector* $\hat{h}$ *will automatically satisfy the condition* $\hat{h} \in I_n$ *or* $\hat{h} \in H_n$*. Therefore one can ensure that a wide variety of the above mentioned random bandwidth vectors satisfy the assumption* $\hat{h} \in I_n$ *or* H_n *simply by optimizing the*

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bandwidth selection criterion on I_n or H_n.

Moreover the assumption that $\hat{h} \in I_n$ or H_n is not very restrictive. For $d = 1$, I_n covers a wide range of bandwidths (see Dutta and Goswami (2013)). The simulation and real data analysis in Dutta and Saha (2013) also suggest that even for $d \geq 2$, the cross validation bandwidth restricted to H_n compares well with the usual unconstrained cross validation bandwidth vector.

(ii) The esteemed reviewer pointed out that the data-based cross validation selectors are known to be of the order $n^{-1/(4+d)}$. But their coefficients are unknown. Therefore is it possible to ensure that the unconstrained cross validation bandwidth satisfies the condition $\hat{h} \in I_n$ at least for large sample size?

Under the constraint $h_1 = h_2 = \cdots = h_d = h$, Sain et al. (1994) showed that as $n \to \infty$, $n^{(d+2)/(2d+8)}(\hat{h} - h_0)$ converges in law to Z, which follows normal distribution with mean zero and finite variance (see their proposition in Page 810). Where $\hat{h}$ is the cross validation bandwidth obtained under the assumption $h_1 = h_2 = \cdots = h_d = h$. h_0 is the optimal bandwidth minimizing the asymptotic mean integrated squared error, and h_0 equals some constant times $n^{-1/(d+4)}$. This result implies that for any open interval $(a,\ b),\ b > a > 0$,

$$\lim_{n\to\infty} P\left(a < n^{(d+2)/(2d+8)}(\hat{h} - h_0) < b\right) = P\left(a < Z < b\right).$$

Since h_0 is a multiple of $n^{-1/(d+4)}$, we can assume $\frac{c}{n^{1/(d+4)}} < h_0 < \frac{d}{n^{1/(d+4)}}$ for some $0 < c < d$. Then

$$\lim_{n\to\infty} P\left(\frac{c}{n^{1/(d+4)}} + \frac{a}{n^{d/(2d+8)}} < \hat{h} < \frac{d}{n^{1/(d+4)}} + \frac{b}{n^{d/(2d+8)}}\right) \geq P\left(a < Z < b\right).$$

The above result ensures that for any $0 < \eta < 1$, there exists $0 < c' < d'$ and an positive integer N (depending on η) such that

$$P\left(\hat{h} \in \left[\frac{c'}{n^{1/(d+4)}},\ \frac{d'}{n^{1/(d+4)}}\right]\right) \geq \eta, \forall\ n > N.$$

Consequently for large n there is high probability that the cross validation bandwidth $\hat{h} \in \left[\frac{c'}{n^{1/(d+4)}},\ \frac{d'}{n^{1/(d+4)}}\right]$.

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(iii) The range of bandwidths I_n can be further widened to $\prod_{j=1}^{d}\left[\frac{a_j}{n^{1/(4+d)+\delta}},\ \frac{b_j}{n^{1/(4+d)-\delta}}\right]$, where $0<\delta<1/(4+d)$, to accommodate more values of h especially the multiples of $n^{-1/(d+4)}$. In that case, under Assumptions $1-4$, we get that for $\hat{h}\in I_n$

$$P(|\hat{f}_{n,\hat{h}}(x)-f(x)|>\epsilon)=O\left(n^{(2d^2+2d)/(d+4)}\exp\left(-Cn^{4/(4+d)-d\delta}\epsilon^2\right)\right)$$
$$\text{and}\quad P(\|\hat{f}_{n,\hat{h}}-f\|>\epsilon)=O\left(n^{[4d+d^2(4+2/\gamma)]/(d+4)}\exp\left(-C'n^{4/(4+d)-d\delta}\epsilon^2\right)\right),$$

where $C,\ C'$ is positive constant. For $d=1$, the above rates are same as those obtained by Dutta and Goswami (2013) in their final remark 3.

The reviewer rightly pointed out that a random bandwidth obtained by optimizing the selection criterion on I_n or H_n can be different from a bandwidth obtained by unconstrained optimization. If the range of bandwidths is widened to $(0,\ \infty)^d$, the extension of our theoretical results appear to be quite challenging. We leave this problem for further research.

2.2 strongly mixing case

A stationary process $\{X_t\}_{t\in\mathbb{Z}}$ is a strong mixing process, with mixing coefficient α, if

$$\alpha(n)=\sup_t\sup\{|P(A\cap B)-P(A)P(B)|:A\in M_{t+n}^{\infty},B\in M_{-\infty}^{t}\}\downarrow 0,\ \text{as } n\to\infty,$$

where $M_{-\infty}^{t}$ and M_{t+n}^{∞} denote σ-fields generated by $\{X_l,\ l\le t\}$ and by $\{X_l, l\ge t+n\}$ respectively. Under very general dependence assumptions (that includes strong mixing condition), Lardjane (2007) has shown that the MSE of a KDE goes to zero at the rate similar (up to a logarithm) to the rate of convergence of the MSE under i.i.d.assumptions for $h_1=h_2=\cdots=h_d$ equal to a multiple of $(\log n/n)^{1/(d+4)}$ (see page 213, Lardjane (2007)). Therefore under strongly mixing dependence assumption, we search for the appropriate value of the random bandwidth vector $\hat{h}$ in the following

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d-dimensional rectangle.

$$I'_n = \prod_{j=1}^{d} \left[a_j \{\log n/n\}^{1/(4+d)},\ b_j \{\log n/n\}^{1/(4+d)} \right],$$

where $0 < a_j < b_j,\ j = 1, \ldots, d$, are arbitrary constants.

Theorem 4. *Let $\{X_n\}_{n\geq 1}$ be a d-dimensional strongly mixing process with marginal density f and Assumptions 1(i) and 2 hold. Also let $\hat{h} \in I'_n$ and the mixing coefficient $\alpha(n)$ satisfies $\alpha(n) \leq D\rho^n$, where $0 < \rho < 1$ and $D > 0$. Then for every $\epsilon > 0$ there exists $C > 0$, such that as $n \to \infty$*

$$P\left(|\hat{f}_{n,\hat{h}}(x) - f(x)| > \epsilon\right) = O\left((n/\log n)^{d^2/(d+4)} \exp\left(-Cn^{(4-d)/(4+d)}(\log n)^{2d/(4+d)}\right)\right).$$

The right side of the above equation goes to zero for $d \leq 3$.

Theorem 5. *Let $\{X_n\}_{n\geq 1}$ be a d-dimensional strongly mixing process with marginal density f and Assumptions $1-3$ hold. Also let $\hat{h} \in I'_n$ and the mixing coefficient $\alpha(n)$ satisfies $\alpha(n) \leq D\rho^n$ where $0 < \rho < 1$ and $D > 0$. Then for every $\epsilon > 0$ there exists $C' > 0$, such that as $n \to \infty$*

$$P(\|\hat{f}_{n,\hat{h}} - f\| > \epsilon) = O\left(\left(\frac{n}{\log n}\right)^{[d+d^2(2+1/\gamma)]/(d+4)} \exp\left\{-C'n^{(4-d)/(4+d)}(\log n)^{2d/(4+d)}\right\}\right),$$

where γ is a positive constant as defined in Assumption 2. The right side of the above equation goes to zero for $d \leq 3$.

The proof of Theorem 4 and Theorem 5 are direct consequences of Lemma 6, stated and proved in Section 3.1.

Remark 2. *(i) Like in i.i.d. case, here also the range of bandwidths I'_n can be further widened to*

$$\prod_{j=1}^{d} \left[a_j \{\log n/n\}^{1/(4+d)+\delta}, b_j \{\log n/n\}^{1/(4+d)-\delta} \right],$$

where $0 < \delta < 1/(4+d)$, to accommodate more values of h. In that case, under Assumptions $1-3$,

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we get that for $\hat{h} \in I'_n$

$$P(|\hat{f}_{n,\hat{h}}(x) - f(x)| > \epsilon) = O\left(\left(\frac{n}{\log n}\right)^{\frac{2d^2+2d}{d+4}} \exp\left(-Cn^{\frac{4-d}{4+d}-2d\delta}(\log n)^{\frac{2d}{4+d}+2d\delta}\right)\right)$$

and $$P(\|\hat{f}_{n,\hat{h}} - f\| > \epsilon) = O\left(\left(\frac{n}{\log n}\right)^{\frac{4d+d^2(4+2/\gamma)}{d+4}} \exp\left(-Cn^{\frac{4-d}{4+d}-2d\delta}(\log n)^{\frac{2d}{4+d}+2d\delta}\right)\right),$$

where $C,\ C'$ *is positive constant. Observe that if* $\delta < \frac{4-d}{(4+d)2d}$, *then right hand side of the above equations go to zero.*

(ii) Like in i.i.d. case, one may use data based coefficients $a_{nj},\ b_{nj},\ j = 1, \ldots, d$, *in* I'_n. *In that case the convergence rates in Theorems 4 and 5 are replaced by*

$$O\left(r_{n\epsilon} + (n/\log n)^{d^2/(d+4)} \exp\left(-Cn^{(4-d)/(4+d)}(\log n)^{2d/(4+d)}\right)\right)$$

and

$$O\left(r_{n\epsilon} + \left(\frac{n}{\log n}\right)^{[d+d^2(2+1/\gamma)]/(d+4)} \exp\left\{-C'n^{(4-d)/(4+d)}(\log n)^{2d/(4+d)}\right\}\right)$$

respectively, where $P(|a_{nj} - a_j| > \epsilon) = O(r_{n,\epsilon})$ *and* $P(|b_{nj} - b_j| > \epsilon) = O(r_{n,\epsilon})$ *and* $r_{n\epsilon} = o(1)$ *as* $n \to \infty$, *under the stated dependence assumption.* $0 < \epsilon,\ a_j,\ b_j,\ j = 1, \ldots, d$ *are arbitrary.*

In particular, one may use $a_{nj} = c_{1j}\eta_j^*,\ b_{nj} = c_{2j}\eta_j^*$, *where* $0 < c_{1j} < c_{2j}$ *and* $\eta_j^* = Q_{3j}^* - Q_{1j}^*,\ j = 1, \ldots, d$, *and* $Q_{1j}^*,\ Q_{3j}^*$ *be the 1st and the 3rd sample quartiles based on* $X_{1j}, \ldots, X_{nj}$. *In that case the rate of convergence of* $P\left(|a_{nj} - a_j| > \epsilon\right),\ P\left(|b_{nj} - b_j| > \epsilon\right)$ *are determined by the rate of convergence in probability of sample first and third quartiles under the stated dependence assumptions. Therefore, under Assumptions* 1 – 4 *and strong mixing dependence assumption stated in Theorem 4, following similar argument given in the proof of Theorem 3 in Section 3.2 and using* (3.15) *from*

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Lemma 7, it is easy to see that

$$P\left(|\hat{f}_{n,\hat{h}}(x) - f(x)| > \epsilon\right) = O\left((n/\log n)^{d^2/(d+4)} \exp\left(-Cn^{(4-d)/(4+d)}(\log n)^{2d/(4+d)}\right)\right),$$
$$P(\|\hat{f}_{n,\hat{h}} - f\| > \epsilon) = O\left(\left(\frac{n}{\log n}\right)^{[d+d^2(2+1/\gamma)]/(d+4)} \exp\left\{-C'n^{(4-d)/(4+d)}(\log n)^{2d/(4+d)}\right\}\right).$$

3 PROOFS

3.1 Lemmas and their proofs:

Here we state and prove a few lemmas. Lemma 1 and Lemma 2 are used to prove all the theorems stated in last section. Lemma 3 and Lemma 4 are used to prove Theorems 1, 2 and 3. Lemma 5 and Lemma 6 are used in the proof of Theorems 4 and 5. In Lemma 7, we derive the rate of convergence sample quantiles to population quantiles under strongly mixing dependence assumption. This result is used in Remark 2 (ii).

Lemma 1. *Let K be a density in $\mathbb{R}^d$ such that $\|K\|, \|z_j \partial_j K(z)\| < \infty$ for $j = 1, \ldots, d$ and $z = (z_1, \ldots, z_d) \in \mathbb{R}^d$. Then given $\epsilon > 0$, there exists $C > 0$ such that for any two bandwidth vectors $h = (h_1, \ldots, h_d)$ and $h' = (h'_1, \ldots, h'_d)$ in $I_n = \prod_{j=1}^{d}[a_j n^{-1/(4+d)},\ b_j n^{-1/(4+d)}]$, satisfying $|h_i - h'_i| < \delta_n = \frac{\epsilon}{4dCn^{(d+1)/(d+4)}},\ i = 1, \ldots, d$, we have*

$$\left|\hat{f}_{n,h}(x) - \hat{f}_{n,h'}(x)\right| < \epsilon/2.$$

Proof. Under given condition on K, the partial derivatives $\frac{\partial g_z(h)}{\partial h_i},\ i = 1, \ldots, d$, exist. Therefore, for fixed z and any two bandwidth vectors h and h' we have

$$g_z(h) - g_z(h') = \sum_{i=1}^{d}(h_i - h'_i) \int_0^1 \left[\frac{\partial g_z(h)}{\partial h_i}\right]_{h=h'+t(h-h')} dt.$$

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For any bandwidth vector $h = (h_1, \ldots, h_d) \in I_n$, it is easy to verify that

$$\left|\frac{\partial g_z(h)}{\partial h_i}\right| \leq \frac{1}{h_i^2 \prod_{l=1,\ l\neq i}^{d} h_l} \left\{ \|K\| + \sup_{-\infty<z<\infty} |z_i \partial_i K(z)| \right\} \leq C n^{(d+1)/(d+4)},\ i = 1, \ldots, d.$$

Therefore, under given conditions on K, for any two bandwidth vectors $h,\ h' \in I_n$ we see that

$$|g_z(h) - g_z(h')| \leq C n^{(d+1)/(d+4)} \sum_{i=1}^{d} \left|h_i - h_i'\right|. \tag{3.1}$$

The right side of the above inequality is free of z. A direct consequence of (1.1) and (3.1) is that for any two bandwidth vectors $h,\ h' \in I_n$,

$$\left|\hat{f}_{n,h}(x) - \hat{f}_{n,h'}(x)\right| \leq \frac{1}{n} \sum_{i=1}^{n} \left|g_{x-X_i}(h) - g_{x-X_i}(h')\right| \leq C n^{(d+1)/(d+4)} \sum_{i=1}^{d} \left|h_i - h_i'\right|. \tag{3.2}$$

The above inequality (3.2) implies that for any two random bandwidth vectors $h,\ h' \in I_n$, satisfying $|h_i - h_i'| < \delta_n,\ i = 1, \ldots, d$,

$$\left|\hat{f}_{n,h}(x) - \hat{f}_{n,h'}(x)\right| \leq d C n^{(d+1)/(d+4)} \delta_n.$$

For given $\epsilon > 0$, using $\delta_n = \frac{\epsilon}{4dCn^{(d+1)/(d+4)}}$ in the right side of the above inequality we get

$$\left|\hat{f}_{n,h}(x) - \hat{f}_{n,h'}(x)\right| < \epsilon/2.$$

This completes the proof the Lemma. □

Lemma 2. *Let K be a density in $\mathbb{R}^d$ such that $\|\partial_j K\| < \infty$ for $j = 1, 2, \ldots, d$. Then given $\epsilon > 0$ and a bandwidth vector $h = (h_1, \ldots, h_d)$, there exists $b_n > 0$ such that for any two vectors $x = (x_1, \ldots, x_d)$ and $x' = (x_1', \ldots, x_d')$ satisfying $|x_j - x_j'| < b_n,\ j = 1, \ldots, d$,*

$$\left|\hat{f}_{n,h}(x) - \hat{f}_{n,h}(x')\right| < \epsilon/2.$$

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Proof. It is easy to see that

$$\begin{aligned}\left|\hat{f}_{n,h}(x) - \hat{f}_{n,h}(x')\right| &\leq \frac{1}{n\prod_{j=1}^{d} h_j}\sum_{i=1}^{n}\left|K\left(\frac{x - X_i}{h}\right) - K\left(\frac{x' - X_i}{h}\right)\right| \\ &\leq \frac{1}{\prod_{j=1}^{d} h_j}\sum_{j=1}^{d}\frac{|x_j - x'_j|}{h_j}\|\partial_j K\| < \frac{db_n}{h_*^{d+1}}\sum_{j=1}^{d}\|\partial_j K\|,\end{aligned}$$

where $h_* = \min\{h_1, \ldots, h_d\}$. Using $b_n = \frac{\epsilon h_*^{d+1}}{4d\sum_{j=1}^{d}\|\partial_j K\|}$ in the right side of the above inequality we get

$$\left|\hat{f}_{n,h}(x) - \hat{f}_{n,h}(x')\right| < \epsilon/2.$$

This completes the proof the Lemma. □

Lemma 3. *Let $X_1, \ldots, X_n$ be i.i.d. random variables with density f, satisfying Assumptions 2 and 3. Further let $h = (h_1, \ldots, h_d)$ be a bandwidth vector such that $h_i = o(1),\ i = 1, \ldots, d$ and $nh_*^d \to \infty$ as $n \to \infty$, where $h_* = \min\{h_1, \ldots, h_d\}$. Then under Assumption 1, as $n \to \infty$*

$$P\left(\|\hat{f}_{n,h} - f\| > \epsilon\right) = O\left(\frac{1}{(h_*)^{d+d^2(1+1/\gamma)}}\exp(-Cnh_*^d)\right),\ C > 0,$$

where γ is a positive constant as stated in Assumption 3.

Proof. Let us introduce some notation

$$K_1 = \|K\|,\ K_2 = \sup_{x\in\mathbb{R}^d}\|x\|^d K(x),\ K_4 = 2^\gamma\int\|x\|^\gamma f(x)dx.$$

Given $\epsilon > 0$, let $R_n = [-a_n,\ a_n]^d$ where $a_n = \left(16K_1K_4/\epsilon h_*^d\right)^{1/\gamma}$ and $b_n = \frac{\epsilon {h_*}^{d+1}}{4d\sum_{j=1}^{d}\|\partial_j K\|}$. It is easy to see that

$$P\left(\|\hat{f}_{n,h} - f\| > \epsilon\right) \leq P\left(\sup_{x\in R_n}|\hat{f}_{n,h}(x) - f(x)| > \epsilon\right) + P\left(\sup_{x\in R_n^C}|\hat{f}_{n,h}(x) - f(x)| > \epsilon\right), \tag{3.3}$$

where R_n contains a d-dimensional sphere S_n with center $0 = (0, \ldots, 0)$ and radius a_n. If x is not

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in R_n, it is not in S_n. Therefore

$$P\left(\sup_{x\in R_n^C}|\hat{f}_{n,h}(x)-f(x)|>\epsilon\right) \leq P\left(\sup_{x\in S_n^C}|\hat{f}_{n,h}(x)-f(x)|>\epsilon\right)$$
$$= P\left(\sup_{\|x\|>a_n}|\hat{f}_{n,h}(x)-f(x)|>\epsilon\right) \tag{3.4}$$

Under Assumptions 1 and 2, and the condition $h_i = o(1),\ i = 1,\ldots,d$ as $n\to\infty$, we get

$$\|E(\hat{f}_{n,h})-f\| \leq \sum_{i=1}^{d}|h_i|\|\partial_i f\|\int|z_i|K(z_1,\ldots,z_d)dz_1\ldots dz_d = o(1). \tag{3.5}$$

Now $$P\left(\sup_{\{x:\|x\|>a_n\}}|\hat{f}_{n,h}(x)-f(x)|>\epsilon\right)$$
$$\leq P\left(\sup_{\{x:\|x\|>a_n\}}|\hat{f}_{n,h}(x)|\geq\epsilon/2\right)+P\left(\sup_{\{x:\|x\|>a_n\}}f(x)\geq\epsilon/4\right)+P\left(\|E(\hat{f}_{n,h})-f\|\geq\epsilon/4\right)$$

Under Assumption 2, $f(x)\to 0$ as $\|x\|\to\infty$ and under the stated conditions on h, $a_n\to\infty$ as $n\to\infty$. Therefore for any $\epsilon>0$, $P\left(\sup_{\{x:\|x\|>a_n\}}f(x)\geq\epsilon/4\right)$ is equal to zero for sufficiently large value of n.

Moreover from (3.5) we see that as $n\to\infty$, $P\left(\|E(\hat{f}_{n,h})-f\|\geq\epsilon/4\right)$ eventually equals zero. Therefore we see that for every $\epsilon>0$, there exists $N_\epsilon>1$ such that for every $n>N_\epsilon$

$$P\left(\sup_{\|x\|>a_n}|\hat{f}_{n,h}(x)-f(x)|>\epsilon\right) \leq P\left(\sup_{\|x\|>a_n}|\hat{f}_{n,h}(x)|\geq\epsilon/2\right)$$
$$\leq P\left(\sup_{\|x\|>a_n}\frac{1}{n\prod h_j}\sum_{\{i:\|X_i-x\|>a_n/2\}}K(\frac{x-X_i}{h})\geq\epsilon/4\right)$$
$$+P\left(\sup_{\|x\|>a_n}\frac{1}{n\prod h_j}\sum_{\{i:\|X_i-x\|\leq a_n/2\}}K(\frac{x-X_i}{h})\geq\epsilon/4\right).$$

Now $P\left(\sup_{\|x\|>a_n}\frac{1}{n\prod h_j}\sum_{\{i:\|X_i-x\|>a_n/2\}}K(\frac{x-X_i}{h})\geq\epsilon/4\right)\leq P\left(\frac{(\max h_j)^d}{\prod h_j}(a_n/2)^{-d}K_2\geq\epsilon/4\right)\to 0$

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as $n \to \infty$. This follows from the fact that $a_n \to \infty$ as $n \to \infty$. Also

$$P\left(\sup_{\|x\|>a_n} \frac{1}{n\prod h_j} \sum_{\{i:\|X_i - x\| \le a_n/2\}} K(\frac{x - X_i}{h}) \ge \epsilon/4\right) \;\le\; P\left(\frac{1}{n} \sum_{\{i:\|X_i\| \ge a_n/2\}} K_1 \ge \frac{\epsilon}{4} h_*^d\right)$$

Moreover, using Markov inequality and recalling the definition of a_n, we see that

$$P\left(\|X_1\| > a_n/2\right) \le E\|X_1\|^\gamma/(a_n/2)^\gamma = \frac{K_4}{a_n^\gamma} = \epsilon h_*^d/16K_1.$$

Therefore every $\epsilon > 0$, there exists $N'_\epsilon > N_\epsilon$ such that for every $n > N'_\epsilon$

$$\begin{aligned} P\left(\sup_{\|x\|>a_n} |\hat{f}_{n,h}(x) - f(x)| > \epsilon\right) &\le P\left(\frac{1}{n} \sum_{\{i:\|X_i\| \ge a_n/2\}} K_1 \ge \frac{\epsilon}{4} h_*^d\right) \\ &\le P\left(\frac{1}{n} \sum_{i=1}^n I_{\|X_i\|>a_n/2} \ge \frac{\epsilon h_*^d}{4K_1}\right) \\ &\le P\left(\frac{1}{n} \sum_{i=1}^n \left[I_{\|X_i\|>a_n/2} - P\left(\|X_1\| > a_n/2\right)\right] \ge \frac{\epsilon h_*^d}{K_1}(\frac{1}{4} - \frac{1}{16})\right) \end{aligned}$$

Using inequality (48) in page 187 in Rao (1983), in the right side of the above inequality we get the following equation

$$P\left(\sup_{\|x\|>a_n} |\hat{f}_{n,h}(x) - f(x)| > \epsilon\right) = O\left(\exp(-C_1 n (h_*)^d)\right), \text{ as } n \to \infty. \tag{3.6}$$

From (3.3), (3.4) and the above equation we get as $n \to \infty$,

$$P\left(\|\hat{f}_{n,h} - f\| > \epsilon\right) \le P\left(\sup_{x \in R_n} |\hat{f}_{n,h}(x) - f(x)| > \epsilon\right) + O\left(\exp(-Cn(h_*)^d)\right),\ C > 0. \tag{3.7}$$

We partition R_n into $k'(n) = (1 + [2a_n/b_n])^d$ (where $[y]$ is the largest integer not exceeding y) subintervals $\{J_{ni} : i = 1, \ldots, k'(n)\}$ each of volume b_n^d. $\{J_{ni} : i = 1, \ldots, k'(n)\}$ are d-fold product of the non-overlapping intervals of a partition of $[-a_n,\ a_n]$ into sub-intervals of length b_n. We see that

$$P\left(\sup_{x \in R_n} |\hat{f}_{n,h}(x) - f(x)| > \epsilon\right) \le P\left(\max_{1 \le i \le k'(n)} \left\{|\hat{f}_{n,h}(y_i) - f(y_i)| + \sup_{x \in J_{ni}} |\hat{f}_{n,h}(y_i) - \hat{f}_{n,h}(x)|\right\} > \epsilon\right),$$

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where $y_i \in J_{ni},\ i = 1, \ldots, k'(n)$. For $x \in J_{ni},\ i = 1, \ldots, k'(n)$, x and y_i satisfy the conditions of Lemma 2. Therefore by Lemma 2

$$|f_{n,h}(y_i) - \hat{f}_{n,h}(x)| < \epsilon/2, \forall x \in J_{ni}, i = 1, \ldots, k'(n).$$

Therefore

$$\begin{aligned} P\left(\sup_{x \in R_n} |\hat{f}_{n,h}(x) - f(x)| > \epsilon\right) &\leq P\left(\max_{1 \leq i \leq k'(n)} \left\{|\hat{f}_{n,h}(y_i) - f(y_i)|\right\} > \epsilon/2\right) \\ &\leq \sum_{i=1}^{k'(n)} P\left(|\hat{f}_{n,h}(y_i) - f(y_i)| > \epsilon/2\right). \end{aligned} \tag{3.8}$$

Using (3.5), we see that for $\epsilon > 0$ there exists N_1 such that for

$$\sum_{i=1}^{k'(n)} P\left(|\hat{f}_{n,h}(y_i) - f(y_i)| > \epsilon/2\right) \leq \sum_{i=1}^{k'(n)} P\left(|\hat{f}_{n,h}(y_i) - E\,\hat{f}_{n,h}(y_i)| > \epsilon/4\right).$$

Now using inequality (27) in the proof of Theorem 3.1.5. in Rao (1983), we get that as $n \to \infty$

$$P\left(|\hat{f}_{n,h}(x) - E[\hat{f}_{n,h}(x)]| > \epsilon/4\right) = O(\exp(-Cnh_*^d\epsilon^2). \tag{3.9}$$

Therefore as $n \to \infty$,

$$P\left(\sup_{x \in R_n} |\hat{f}_{n,h}(x) - f(x)| > \epsilon\right) = O\left(k'(n) \exp(-C_2 nh_*^d)\right),\ C_2 > 0.$$

Substituting the above equation in the right side of (3.7) we get as $n \to \infty$

$$P\left(\|\hat{f}_{n,h} - f\| > \epsilon\right) = O\left(k'(n) \exp(-Cnh_*^d)\right),\ C > 0.$$

Recalling the notions $a_n,\ b_n$ we see that $1+[2a_n/b_n]$ is a multiple of $\frac{1}{(h_*)^{d+1+d/\gamma}}$. $k'(n) = (1+[2a_n/b_n])^d$ is a multiple of $\frac{1}{(h_*)^{d^2(1+1/\gamma)+d}}$. This completes the proof. □

Lemma 4. *Let $X_1, \ldots, X_n$ be i.i.d. random variables with density f, satisfying Assumptions 2 and*

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3. Under Assumption 1, we see that as $n \to \infty$

$$(i) P(\sup_{h \in I_n} |\hat{f}_{n,h}(x) - f(x)| > \epsilon) = O\left(n^{d^2/(d+4)} \exp(-Cn^{4/(4+d)})\right),$$
$$(ii) P(\sup_{h \in I_n} \|\hat{f}_{n,h} - f\| > \epsilon) = O\left(n^{[d+d^2(2+1/\gamma)]/(d+4)} \exp(-C' n^{4/(4+d)})\right),$$

where $I_n = \prod_{j=1}^{d} [a_j n^{-1/(4+d)},\ b_j n^{-1/(4+d)}]$, $C, C' > 0$ *and* γ *is a positive constant as defined in Assumption 2.*

Proof. Given $\epsilon > 0$, we partition $I_n = \prod_{j=1}^{d} [a_j n^{-1/(4+d)},\ b_j n^{-1/(4+d)}]$ into $k(n)$ non-overlapping d-dimensional rectangles $\{I_{ni},\ i = 1, \ldots, k(n)\}$. To achieve this, each $[a_j n^{-1/(4+d)},\ b_j n^{-1/(4+d)}]$, $j = 1, \ldots, d$, is partitioned into non-overlapping sub-intervals of length proportional to $\delta_n = \frac{\epsilon}{4dCn^{(d+1)/(d+4)}}$. $\{I_{ni},\ i = 1, \ldots, k(n)\}$ are the d-fold product of these sub-intervals of length δ_n. Consequently $k(n)$ is proportional to $\frac{1}{n^{d/(4+d)}\delta_n^d} = n^{d^2/(d+4)}$. Now using Lemma 1, we get that

$$\begin{aligned} P(\sup_{h \in I_n} |\hat{f}_{n,h}(x) - f(x)| > \epsilon) &\leq P(\max_{1 \leq i \leq k(n)} \{|\hat{f}_{n,h^i}(x) - f(x)| + \sup_{h \in I_{ni}} |\hat{f}_{n,h}(x) - f_{n,h^i}(x)|\} > \epsilon) \\ &\leq P(\max_{1 \leq i \leq k(n)} |\hat{f}_{n,h^i}(x) - f(x)| > \epsilon/2) \\ &\leq \sum_{i=1}^{k(n)} P\left(|\hat{f}_{n,h^i}(x) - f(x)| > \epsilon/2\right) \end{aligned} \tag{3.10}$$

and similarly

$$P(\sup_{h \in I_n} \|\hat{f}_{n,h} - f\| > \epsilon) \leq \sum_{i=1}^{k(n)} P\left(\|\hat{f}_{n,h^i} - f\| > \epsilon/2\right) \tag{3.11}$$

where $h^i \in I_{ni}, i = 1, \ldots,\ k(n)$ and each h^i is a non random bandwidth vector.

Using (3.5) we see that for $\epsilon > 0$ there exists N_1 such that for $1 \leq i \leq k(n)$,

$$\|E(\hat{f}_{n,h^i}) - f\| < \epsilon/4,\ \forall\ n > N_1.$$

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Hence for $n > N_1$, we get from (3.10) that

$$P(\sup_{h\in I_n} |\hat{f}_{n,h}(x) - f(x)| > \epsilon) \leq \sum_{i=1}^{k(n)} P\left(|\hat{f}_{n,h^i}(x) - E[\hat{f}_{n,h^i}(x)]| > \epsilon/4\right) \tag{3.12}$$

Now using inequality (27) in the proof of Theorem 3.1.5. in Rao (1983), we get that as $n \to \infty$

$$P\left(|\hat{f}_{n,h^i}(x) - E[\hat{f}_{n,h^i}(x)]| > \epsilon/4\right) = O(\exp(-Cn(h_*^i)^d\epsilon^2) = O\left(\exp(-Cn^{4/(d+4)}\epsilon^2)\right),$$

where $h_*^i = \min\{h_1^i, \ldots, h_d^i\},\ i = 1, \ldots, k(n)$. Substituting the above result in the right side of (3.12) and recalling that $k(n)$ is a multiple of $n^{d^2/(d+4)}$, we get the result (i) of Lemma 4.

To prove (ii), we use Lemma 3 and see that as $n \to \infty$

$$P\left(\|\hat{f}_{n,h^i} - f\| > \epsilon\right) = O\left(\frac{1}{(h_*^i)^{d+d^2(1+1/\gamma)}} \exp(-Cn(h_*^i)^d)\right) \text{ where } C > 0,$$

γ is a positive constant as stated in Assumption 3. Substituting the above equation in the right side of (3.11) and using the fact that $k(n)$ is a multiple of $n^{d^2/(d+4)}$, we get the following equation

$$P(\sup_{h\in I_n} \|\hat{f}_{n,h} - f\| > \epsilon) = O\left(n^{[d+d^2(2+1/\gamma)]/(d+4)} \exp(-C'n^{4/(4+d)})\right).$$

This completes the proof of (ii). □

Lemma 5. *Let $\{X_n\}_{n\geq 1}$ be a d-dimensional strongly mixing process with marginal density f, satisfying Assumptions 2 and 3. Further let the mixing coefficient $\alpha(n)$ satisfies $\alpha(n) \leq D\rho^n$ where $0 < \rho < 1$ and $D > 0$ and $h = (h_1, \ldots, h_d)$ be a bandwidth vector such that $h_i = o(1),\ i = 1, \ldots, d$ and $nh_*^d \to \infty$, $h_*^d \log n \log\log n \to 0$ as $n \to \infty$, where $h_* = \min\{h_1, \ldots, h_d\}$. Then under Assumption 1, as $n \to \infty$*

$$P\left(\|\hat{f}_{n,h} - f\| > \epsilon\right) = O\left(\frac{1}{(h_*)^{d+d^2(1+1/\gamma)}} \exp\left(-Cnh_*^{2d}\right)\right),\ C > 0,$$

where γ is a positive constant as stated in Assumption 3.

Proof. Proof of this lemma is exactly similar to the proof of Lemma 3. Observe that most of the

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arguments used in the proof of Lemma 3 do not depend on independence of $\{X_n\}$. Only in (3.6) and (3.9), we used independence. In particular, we used Bernstein type inequality for i.i.d. entries to find the convergence rates of $P\left(\frac{1}{n}\sum_{i=1}^{n} I_{\|X_i\|>a_n/2} \geq \frac{\epsilon h_*^d}{4K_1}\right)$ and $P\left(\left|\hat{f}_{n,h}(y_l) - E\left[\hat{f}_{n,h}(y_l)\right]\right| > \epsilon/4\right)$, respectively.

Here we use Bernstein type inequality from Merleveede et al. (2009) for strongly mixing process to prove (3.6) and (3.9). Observe that

$$P\left(\left|\hat{f}_{n,h}(y_l) - E\left[\hat{f}_{n,h}(y_l)\right]\right| > \epsilon/4\right) \leq P(|\sum_{j=1}^{n} Y_{nj}| > nh_*^d\epsilon/4), \tag{3.13}$$

where $Y_{nj} = K\left(\frac{x-X_j}{h}\right) - E[K\left(\frac{x-X_j}{h}\right)]$. If $\{X_n\}$ is a stationary strongly mixing process with mixing coefficient $\alpha(n)$, then each row of the triangular array $\{Y_{nj}, j = 1, \ldots, n\}$ represents a strongly mixing stationary sequence of mean zero bounded random variables with a sequence of mixing coefficients $\{\alpha_j(n)\}$ bounded above by $\{\alpha(n)\}$ (see Dutta and Goswami (2013)). Under the stated condition on $\alpha(n)$, the sequence $\{\alpha_j(n)\}$, $j = 1, \ldots, n$ also satisfies the condition $\alpha_j(n) \leq exp(-2cn)$ for some $c > 0$. Under this, using Theorem 1 of Merleveede et al. (2009) we get that for $\epsilon > 0$ and $n \geq 4$,

$$P(|\sum_{j=1}^{n} Y_{nj}| > nh_*^d\epsilon/4) \leq \exp\left(-\frac{Cnh_*^{2d}\epsilon^2}{4K_1^2 + 2K_1\epsilon h_*^d \log n \log\log n}\right)$$

where C, $K_1 > 0$. Under the stated assumption on the bandwidth vector h, the right side of the above inequality equals $O\left(\exp(-C'nh_*^{2d})\right)$ as $n \to \infty$. Hence under the stated dependence assumptions and the other conditions, we have as $n \to \infty$

$$P\left(\left|\hat{f}_{n,h}(y_j) - E\left[\hat{f}_{n,h}(y_j)\right]\right| > \epsilon/4\right) = O\left(\exp(-C'nh_*^{2d})\right), \tag{3.14}$$

for $j = 1, \ldots, k'(n)$ and this is analogue of (3.9) of Lemma 3.

Similarly, using Theorem 1 of Merleveede et al. (2009), we see that

$$P\left(\frac{1}{n}\sum_{i=1}^{n} I_{\|X_i\|>a_n/2} \geq \frac{\epsilon h_*^d}{4K_1}\right) \leq \exp\left(-\frac{Cnh_*^{2d}\epsilon^2}{4K_1^2 + 2K_1\epsilon h_*^d \log n \log\log n}\right) = O\left(\exp(-C_1nh_*^{2d})\right)$$

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and we get (3.6) for strongly mixing case. This completes the proof of this lemma. □

Lemma 6. *Let $\{X_n\}_{n\geq 1}$ be a d-dimensional strongly mixing process with marginal density f, satisfying Assumptions 2 and 3. Also let the mixing coefficient $\alpha(n)$ satisfies $\alpha(n) \leq D\rho^n$ where $0 < \rho < 1$ and $D > 0$. Under Assumption 1, we see that as $n \to \infty$*

$$(i) P(\sup_{h\in I_n} |\hat{f}_{n,h}(x) - f(x)| > \epsilon) = O\left(\left(\frac{n}{\log n}\right)^{d^2/(d+4)} \exp\left(-Cn^{(4-d)/(4+d)}(\log n)^{2d/(4+d)}\right)\right),$$

$$(ii) P(\sup_{h\in I_n} \|\hat{f}_{n,h} - f\| > \epsilon) = O\left(\left(\frac{n}{\log n}\right)^{[d+d^2(2+1/\gamma)]/(d+4)} \exp\left\{-C'n^{(4-d)/(4+d)}(\log n)^{2d/(4+d)}\right\}\right),$$

where $I_n = \prod_{j=1}^{d}[a_j(\log n/n)^{1/(4+d)},\ b_j(\log n/n)^{1/(4+d)}]$, $C, C' > 0$ and γ is a positive constant as defined in Assumption 3.

Proof. We follow the similar arguments given in the proof of Lemma 4. From the proof of the Lemma 4, we see that the arguments up to the inequality (3.12) do not depend on independence of $X_1, X_2, \ldots, X_n$. Therefore under strong mixing dependence assumption, we repeat these arguments, replacing $n^{1/(4+d)}$ by $(\frac{\log n}{n})^{1/(4+d)}$. Consequently the inequalities (3.11) and (3.12) hold, with $k(n)$ proportional to $(\frac{n}{\log n})^{d^2/(d+4)}$, under the stated dependence assumption. Now we complete the proof of (i) by using (3.14) in the right side of (3.12) and of (ii) by using Lemma 5 in the right side of (3.11). Most of the arguments used in the proof of Lemma 3 do not depend on any dependence assumption. For instance, under the stated conditions, the inequalities (3.5) and (3.8) do not require the independence assumption to be true. These results can be repeated as it is in the presence of mixing type dependence.

Only the convergence rate of $P\left(\left|\hat{f}_{n,h}(y_l) - E\left[\hat{f}_{n,h}(y_l)\right]\right| > \epsilon/4\right),\ l = 1, \ldots, k'(n)$, is obtained under i.i.d. assumption in Lemma 3. In Lemma 5, we obtain the same convergence rate under strongly mixing dependence assumption, where $\alpha(n) \leq D\rho^n,\ 0 < D,\ \rho < 1$. From (3.13) we see that

$$P\left(\left|\hat{f}_{n,h}(y_l) - E\left[\hat{f}_{n,h}(y_l)\right]\right| > \epsilon/4\right) \leq P(|\sum_{j=1}^{n} Y_{nj}| > nh_*^d\epsilon/4),$$

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where $Y_{nj} = K\left(\frac{x-X_j}{h}\right) - E[K\left(\frac{x-X_j}{h}\right)],\ j = 1, \ldots, n.$ If $\{X_n\}$ is a stationary strongly mixing process with mixing coefficient α, each row of the triangular array $\{Y_{nj}, j = 1, ..., n, n \in N\}$ represents a strongly mixing stationary sequence of mean zero bounded random variables with a sequence of mixing coefficients bounded above by $\{\alpha(n)\}$ (see Dutta and Goswami (2013)). Under the stated condition on α, the sequence of mixing coefficients of $\{Y_{nj}, j = 1, ..., n, n \in N\}$ also satisfies the condition $\alpha(n) \leq exp(-2cn),\ c > 0.$ Under this using the Theorem 1 in page 3 of Merleveede et al. (2009) we get the following inequality. For $n \geq 4$ and $\epsilon > 0$,

$$P(|\sum_{j=1}^{n} Y_{nj}| > nh_*^d\epsilon/4) \leq \exp\left(-\frac{Cnh_*^{2d}\epsilon^2}{4K_1^2 + 2K_1\epsilon h_*^d \log n \log\log n}\right),$$

where $C,\ K_1 > 0.$ Under the stated assumption on the bandwidth vector h, the right side of the above inequality equals $O\left(\exp(-C'nh_*^{2d})\right)$ as $n \to \infty.$ Hence under the stated dependence assumptions and the other conditions we have as $n \to \infty$

$$P\left(\left|\hat{f}_{n,h}(y_l) - E\left[\hat{f}_{n,h}(y_l)\right]\right| > \epsilon/4\right) = O\left(\exp(-C'nh_*^{2d})\right),\ l = 1, \ldots, k'(n),$$

Now using t the condition that $h_i = o(1),\ i = 1, \ldots, d,$ and the inequalities (3.5) and (3.8) we get the stated rate of convergence $P\left(||\hat{f}_{n,h} - f|| > \epsilon\right)$. This completes the proof. □

In the following lemma we prove the rate of convergence of sample quantiles to population quantiles for strongly mixing case.

Lemma 7. *Let $\{X_n\}_{n\geq 1}$ be a d-dimensional strongly mixing process with marginal density f satisfying Assumption 4. Also let the mixing coefficient $\alpha(n)$ satisfies $\alpha(n) \leq D\rho^n$ where $0 < \rho < 1$ and $D > 0$. Then for $i = 1, 3$ and $j = 1, 2, \ldots, d$,*

$$P\left(|Q_{ij} - Q_{ij}^*| > \epsilon\right) = O\left(\sqrt{n}\exp(-\sqrt{n}s)\right) \tag{3.15}$$

where s is a positive constant.

Proof. If $\{X_n = (X_{n1}, X_{n2}, \ldots, X_{nd})\}_{n\geq 1}$ is d-dimensional strongly mixing process with mixing coef-

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ficient $\alpha(n)$ then for $1 \leq j \leq d$, $\{X_{nj}\}_{n\geq 1}$ is one dimensional strongly mixing process with mixing coefficient $\alpha_j(n)$ and $\alpha_j(n) \leq \alpha(n)$. With this observation, this lemma follows from Lemma 4 of Dutta and Goswami (2013). □

3.2 Proof of Theorems

Theorems 1 and 2 follow from Lemma 4 and Theorems 4 and 5 follow from Lemma 6. It remains to prove only Theorem 3.

Proof of Theorem 3. Let $I_n = \prod_{j=1}^{d} \left[\frac{a_j}{2} n^{-1/(4+d)},\ \frac{3b_j}{2} n^{-1/(4+d)}\right]$. We note that

$$\begin{aligned} P(|\hat{f}_{n,\hat{h}}(x) - f(x)| > \epsilon) &\leq P\left(\hat{h} \notin I_n\right) + P\left(|\hat{f}_{n,\hat{h}}(x) - f(x)| > \epsilon,\ \hat{h} \in I_n\right) \\ &\leq P\left(\hat{h} \notin I_n\right) + P\left(\sup_{h\in I_n} |\hat{f}_{n,h}(x) - f(x)| > \epsilon\right). \end{aligned} \tag{3.16}$$

Similarly

$$P(\|\hat{f}_{n,\hat{h}} - f\| > \epsilon) \leq P\left(\hat{h} \notin I_n\right) + P\left(\sup_{h\in I_n} \|\hat{f}_{n,h} - f\| > \epsilon\right) \tag{3.17}$$

We assume that $\hat{h} \in H_n = \prod_{j=1}^{d} [a_{nj} n^{-1/(4+d)},\ b_{nj} n^{-1/(4+d)}]$. Therefore it is easy to verify that

$$\begin{aligned} P\left(\hat{h} \notin I_n\right) &\leq \sum_{j=1}^{d} [P(|a_{nj} - a_j| > a_j/2) + P(|b_{nj} - b_j| > b_j/2)] \\ \Rightarrow P\left(\hat{h} \notin I_n\right) &= O\left(r_{n,\epsilon}\right) \end{aligned} \tag{3.18}$$

The convergence rates of $P\left(\sup_{h\in I_n} |\hat{f}_{n,h}(x) - f(x)| > \epsilon\right)$ and $P\left(\sup_{h\in I_n} \|\hat{f}_{n,h} - f\| > \epsilon\right)$ are obtained in Lemma 4 (*i*) and (*ii*). Hence Theorem 3 is a consequence of (3.18) and Lemma 4. □

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$\alpha-$, $\beta-$phenomena in the post-symmetry break for the flow past a circular cylinder

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α-, β-phenomena in the post-symmetry break for the flow past a circular cylinder

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In the existing literature, the so-called α- and β-phenomena have been reported only for the early stages for the flow past an impulsively started circular cylinder. The current study endeavours to explore the possible existence of these phenomena even in the later stages of the flow. The flow is computed using a recently developed compact finite difference method for the biharmonic form of the two-dimensional Navier-Stokes equations for a wide range of Reynolds numbers (Re). We establish that these secondary phenomena not only appear once the wake becomes asymmetric but also periodically during the post-vortex shedding period for $Re = 1000$. Further, the recently reported sub-α- and sub-β-phenomena for $Re = 5000$ at the tertiary level during the early stages of the flow could be identified even during the later stages of the flow as well. The formation of these tertiary structures has been explained through a detailed theoretical characterization of the topological aspects of the boundary layer separation. Both qualitative and quantitative results are provided to substantiate our claim. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4979065]

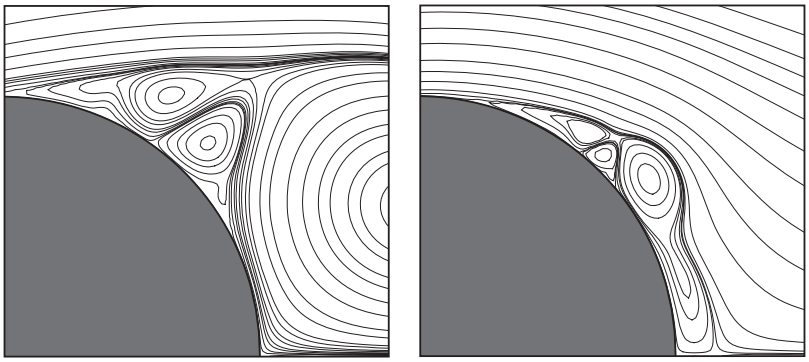
I. INTRODUCTION

The flow past a circular cylinder problem has been the centerpiece of attraction amongst fluid dynamists over the past century in all the three approaches, namely, experimental, theoretical, and numerical. The prime reason for the huge interest in this flow is the exhibition of a highly complicated and fascinating interplay of primary, secondary, and tertiary vortices along with the phenomenon of vortex shedding in the simplest of geometric settings; the flow characteristics could vary widely over different ranges of Reynolds numbers (Re), thus making the study of this problem even more exciting. Two of such intriguing secondary vortex phenomena are the α- and β-phenomena, which are typical of the flow past an impulsively started circular cylinder in the early stages of the flow.

The α- and β-phenomena have been studied a bit scarcely over the last four decades. Although α-phenomenon can be traced back to the computation of Thoman and Szewczyk[2] in the year 1969, the term was coined by Bouard and Coutanceau in 1980,[1] almost a decade later. In the literature, this secondary phenomenon has been reported at the initial stage of flow with the primary vortex in the wake of the circular cylinder remaining stable. Here streamlines close to the cylinder initially deviate from the surface causing a bulge pattern eventually giving rise to a secondary eddy. This eddy grows in size to such an extent that it touches the boundary of the primary eddy, thereby splitting the primary one into two parts and isolating the region of the wake next to the separation point where another secondary eddy becomes visible. α-phenomenon during the early stages of the flow has been chronicled with great precision in several experimental and numerical works of Bouard and Coutanceau,[1] Loc,[3] and Coutanceau and Defaye.[4] A clear evidence of the α-phenomenon can be seen in Figure 1(a) for $Re = 3000$ at time $t = 2.50$. For higher Re values, another secondary phenomena called β-phenomena precede α-phenomenon. The β-phenomenon, documented both numerically and experimentally in the works of Loc,[3] Loc and Bouard,[5] Chang and Chern,[6] Chou and Huang,[7] Sanyasiraju and Manjula,[8] Kalita and Ray,[9] and Kalita and Sen,[10] emanates with the formation of a very thin recirculating wake. Here the core of the recirculating zone rotates in one piece with a speed which is much faster than the other part of the separated zone forming a vortex which gains strength and size with time. After a while, this vortex separates the initial wake into two parts where the one situated near the point of separation is occupied by two secondary eddies. This phenomenon is clearly exemplified by the flow patterns for $Re = 9500$ at time $t = 1.00$ and is being depicted in Figure 1(b). A complete review of these two secondary phenomena can be found in the work of Kalita and Sen.[11] It is worthwhile to note that the presence of these two phenomena can be found in the whole range $4000 \leq Re \leq 8000$ as evidenced by the work of Kalita and Sen.[11] In Figure 2, we depict the phenomena for two other Reynolds numbers in this range, viz., 4500 and 5500. However, $Re = 5000$ has been a popular choice among the computational fluid dynamics community[8–11] as it exhibits both α- and β-phenomena at certain stages of the early flow evolution for which experimental results are also available.

Kalita and Sen[11] reported the existence of two new tertiary vortex phenomenon for $Re = 5000$ which they termed as sub-α- and sub-β-phenomenon. The sub-β-phenomenon which follow events analogous to β-phenomenon is defined as one where a secondary vortex, part of which is bounded by the free stream, is split by the tertiary vortex and is observed

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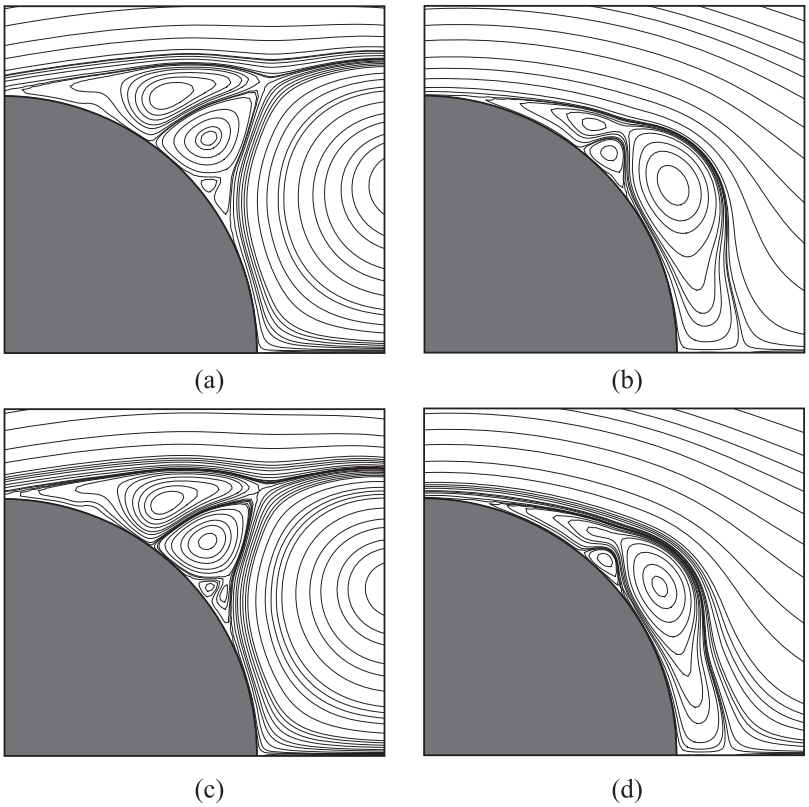
(a) (b)

FIG. 1. (a) α-phenomenon for $Re = 3000$ at $t = 2.50$ and (b) β-phenomenon for $Re = 9500$ at $t = 1.00$ form the current computation. The Reynolds number and time are chosen as in the experiments of Bouard and Coutanceau.[1]

with the formation of a tertiary vortex on the surface of the cylinder cushioned by the secondary vortex. The sub-α-phenomenon defined similar to the α-phenomenon where a pair of tertiary vortices is either equivalent in strength or size. Both these phenomena were captured successfully for $Re = 5000$, 8000, 9500, and 10 000 at early times. Along with categorizing this phenomenon in strong and weak sense, they also observed the presence of some α- and β-like phenomena.

For all the secondary and tertiary vortex phenomena described above, their occurrence had been reported only for the early part of the flow beginning from the impulsive start. Even the occurrence of the secondary phenomena, which are quite well reported in the existing literature, has never been reported in the post-symmetry-breaking period. In the present work, we carry out long term simulation and report complete evolution of the flow for $Re = 1000$ and $Re = 5000$ in the post-symmetry break regime. We also record occurrences of sub-α- and sub-β-phenomena at large times for $Re = 5000$ which has not been reported earlier. We demonstrate that these phenomena continue to make their presence even during the vortex shedding period. More interestingly, we observe the periodic occurrence of the α-phenomenon for $Re = 1000$ during the vortex shedding period.

This paper is organized in the following way. In Section II, we briefly discuss the governing equations and the numerical scheme, Section III deals with the observations on the results obtained from our long time simulations for $Re = 1000$ and $Re = 5000$ including a discussion on the underlying mechanism leading to the formation of tertiary vortices, and finally in Section IV, we summarize our achievements.



(a) (b)

(c) (d)

FIG. 2. (a) α-phenomenon for $Re = 4500$ at $t = 2.55$, (b) β-phenomenon for $Re = 4500$ at $t = 1.10$, (c) α-phenomenon for $Re = 5500$ at $t = 2.40$, and (d) β-phenomenon for $Re = 5500$ at $t = 1.00$ form the current computation.

II. THE GOVERNING EQUATION AND ITS DISCRETIZATION

Here, unsteady two-dimensional Navier-Stokes equations are solved using the biharmonic pure stream function formulation following the work of Kalita and Sen.[10,11] The formulation has also been used by the authors with great success in simulating long time flows around various bluff bodies.[12] In the following, we provide a brief outline of the formulation and the numerical scheme used for discretizing the equations governing the flow.

In the Cartesian (x, y) coordinate system, the non-dimensional unsteady pure streamfunction form of Navier-Stokes equations for incompressible flow is

$$\frac{\partial}{\partial t}(\Delta\psi) + [(\nabla^{\perp}\psi).\nabla]\Delta\psi = \frac{1}{Re}\Delta^2\psi. \tag{1}$$

The stream function ψ is related to the velocity field by $(u, v) = \nabla^{\perp}\psi = (\psi_y, -\psi_x)$. We intend to solve the pure stream-function equation in an annular region with inner radius $a = 1$ and outer radius R_∞. This is achieved by converting from the physical (x, y) plane into the computational (ξ, η) plane by using transformation $x + \mathrm{I}y = e^{\pi(\xi+\mathrm{I}\eta)}$, where $\mathrm{I} = \sqrt{-1}$. Following the work of Bouard and Coutanceau,[1] we non-dimensionalise the flow variables as $x = x^*/a$, $y = y^*/a$, $u = u^*/U_\infty$, $v = v^*/U_\infty$, $t = t^*U_\infty/D$, and $Re = DU_\infty/\nu$; u^*, v^* being the dimensional velocities along the x^*-, y^*-directions, respectively, and ν is the kinematic viscosity with $D = 2a$ being the diameter of the cylinder. The semi-discrete second order spatially accurate pure stream-function equation in computational plane can be written as

$$\frac{\partial}{\partial t}\Delta_h\psi_{i,j} = a_{i,j}\Delta_h^2\psi_{i,j} + b_{i,j}\Delta_h\psi_{\xi_{i,j}} + c_{i,j}\Delta_h\psi_{\eta_{i,j}} + d_{i,j}\Delta_h\psi_{i,j}, \tag{2}$$

where

$$a_{i,j} = \frac{4}{\pi^2 e^{2\pi\xi_i} Re}, \qquad b_{i,j} = -\frac{2}{\pi^2 e^{2\pi\xi_i}}\left(\frac{8\pi}{Re} + \psi_{\eta_{i,j}}\right),$$
$$c_{i,j} = \frac{2}{\pi^2 e^{2\pi\xi_i}}\psi_{\xi_{i,j}}, \qquad d_{i,j} = \frac{4}{\pi e^{2\pi\xi_i}}\left(\frac{4\pi}{Re} + \psi_{\eta_{i,j}}\right).$$

The compact discretization operators $\Delta_h^2\psi$ and $\Delta_h\psi$ are defined as

$$\Delta_h^2\psi_{i,j} = \delta_\xi^4\psi_{i,j} + 2\,\delta_\xi^2\delta_\eta^2\psi_{i,j} + \delta_\eta^4\psi_{i,j} + O(h^2), \tag{3}$$

$$\Delta_h\psi_{i,j} = \delta_\xi^2\psi_{i,j} + \delta_\eta^2\psi_{i,j} + O(h^2), \tag{4}$$

with δ_ξ^4 denoting the Stephenson finite difference operator[12]

$$\delta_\xi^4\psi_{i,j} \equiv \frac{12}{h^2}(\delta_\xi\psi_{\xi_{i,j}} - \delta_\xi^2\psi_{i,j}).$$

As noted by Sen *et al.*,[12] we use compatible Padé approximations for ψ_ξ and ψ_η. Finally, Crank-Nicolson time discretization is used to obtain second order temporal accuracy.

The initial and boundary conditions required to compute the flow field are obtained following the work of Kalita and Sen.[11] Although we carried out computations on seven meshes of sizes ranging from 181×301 to 301×1201, unless otherwise stated, the results presented here are with the ones with a grid of size 181×451 and 241×401 where the outer radius R_∞ of the annular computational domain was set as $R_\infty \cong 12$ and $R_\infty \cong 43$, respectively. In case of the former, R_∞ is kept small as we are interested in the flow only in the vicinity of the surface of the cylinder at the earlier stages of the flow, while the latter is specifically chosen for examining the flow in the post-vortex shedding period. Though we computed the flow with time steps varying from 10^{-3} to 10^{-5}, all the subsequent results presented here are from employing a time step $\Delta t = 5 \times 10^{-4}$.

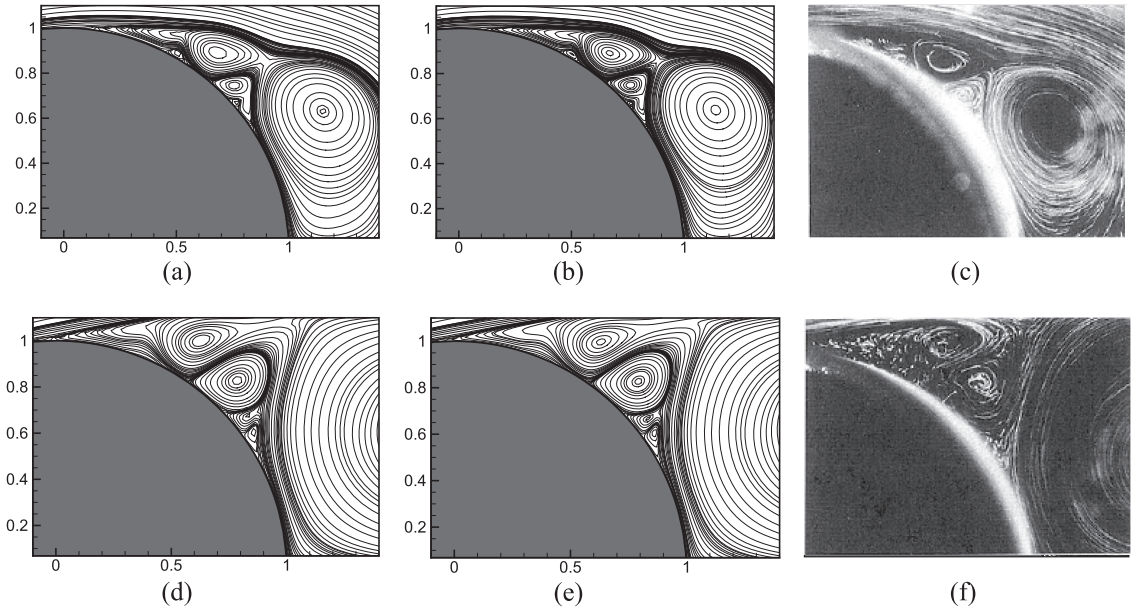


FIG. 3. Comparison of the present simulation with the experimental results of Coutanceau and Defaye[4] for $Re = 4500$: (a) $t = 1.40$ (num. 181×451). (b) $t = 1.40$ (num. 271×676). (c) $t = 1.40$. (Exp., reproduced with permission from M. Coutanceau and J.-R. Defaye, Appl. Mech. Rev. **44**, 255–305 (1991). Copyright 1991 American Society of Mechanical Engineers.) (d) $t = 2.80$ (num. 181×451). (e) $t = 2.80$ (num. 271×676). (f) $t = 2.80$. (Exp., reproduced with permission from M. Coutanceau and J.-R. Defaye, Appl. Mech. Rev. **44**, 255–305 (1991). Copyright 1991 American Society of Mechanical Engineers.)

In order to validate our computed results obtained by using the finite difference scheme described above, we compare our numerical results with some experimental results available in the literature and also perform a grid independence study. In Figure 3, we compare the streamlines obtained through our computation on two different grids of size 181×451 and 271×676 with $R_\infty \cong 12$ at $t = 1.40$ and 2.80 for $Re = 4500$ with the experimental visualization of Coutanceau and Defaye,[4] and one can see that they are very close to each other. This figure also exemplifies the grid independence of our results and better resolution of the tertiary vortices on finer grids.

We also perform an exhaustive grid independence study for both spatial and temporal variations of the grid for $Re = 5000$ when $R_\infty \cong 43$. In Figure 4(a), we plot the time history of drag coefficient C_D till the onset of shedding on four different space grid sizes 181×451, 211×451, 241×401, and 271×451. One can see that the graphs resulting from the last two grids are extremely close to each other. As such, the next study for the temporal grid independence was performed on a spatial grid of size 241×401 by employing the time steps $\Delta t = 0.001$, $\Delta t = 0.0005$, and $\Delta t = 0.000\,125$, respectively. As one can see from Figure 4(b), the graphs of the time history of C_D computed with these time steps are literally indistinguishable from each other indicating the time grid independence of our computed solutions. Further we show the effect of the grid size on the flow characteristics in a tabular form as well. In Table I, we show the separation angle θ_S, the wake length

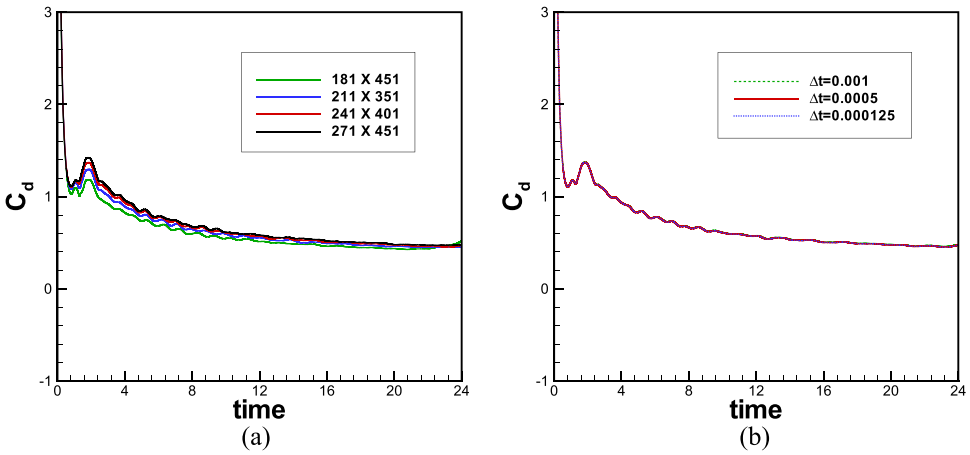


FIG. 4. Grid independence study of time history of drag coefficients for $Re = 5000$ on (a) different space grids and (b) different time grids.

TABLE I. Effect of the mesh size and the outer radius location on flow characteristics for $Re = 5000$ in the upper half of the cylinder.

Mesh		M_1	M_2	M_3	M_4
Grid size		181×451	211×351	241×401	271×451
$R_\infty \cong$		12	43	43	43
$t = 1$	θ_S	80.38	79.34	79.84	80.38
	L	0.216	0.203	0.205	0.207
	C_D	1.181	1.138	1.159	1.159
	$(\bar{x}, \bar{y})$	(0.9484, 0.6498)	(0.9578, 0.6448)	(0.9521, 0.6478)	(0.9378, 0.6373)
$t = 3$	θ_S	94.75	95.73	94.53	95.48
	L	1.672	1.677	1.664	1.668
	C_D	1.129	1.013	1.070	1.102
	$(\bar{x}, \bar{y})$	(1.7539, 0.5834)	(1.7458, 0.5847)	(1.7459, 0.5974)	(1.7548, 0.5702)
$t = 5$	θ_S	97.30	97.88	98.92	98.08
	L	2.411	2.433	2.412	2.419
	C_D	0.900	0.790	0.830	0.850
	$(\bar{x}, \bar{y})$	(1.9488, 0.7716)	(1.9768, 0.7819)	(1.9763, 0.7819)	(1.9454, 0.7702)
$t = 10$	θ_S	100.80	100.52	100.79	100.80
	L	3.664	3.703	3.645	3.611
	C_D	0.659	0.596	0.608	0.612
	$(\bar{x}, \bar{y})$	(2.4288, 0.8843)	(2.3816, 0.8933)	(2.4152, 0.8698)	(2.3825, 0.9042)
$t = 15$	θ_S	101.59	102.88	102.60	102.42
	L	4.255	4.418	4.506	4.448
	C_D	0.590	0.504	0.529	0.538
	$(\bar{x}, \bar{y})$	(2.5558, 0.9713)	(2.6528, 0.9949)	(2.6363, 0.9967)	(2.5403, 1.0049)

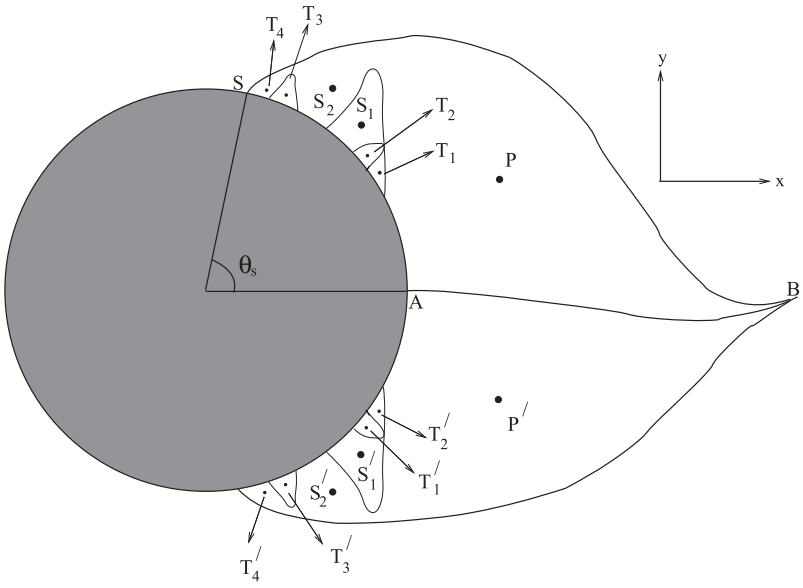


FIG. 5. Schematic diagram of the vortex structures: P, S_i $(1 \le i \le 2)$, T_j $(1 \le j \le 4)$ denote the centers of the primary, secondary, and tertiary vortices, respectively, in the upper half of the cylinder, and the primed letters denote the same in the lower half.

L, and the location of the centre of the upper primary vortex $(\overline{x}, \overline{y})$ at a different time juncture till the onset of asymmetry computed on four grids in the upper half of the cylinder. It is heartening to note that the maximum differences for θ_S, L, and $(\overline{x}, \overline{y})$ values in the last two grids are below 5% for all the flow parameters.

III. RESULTS AND DISCUSSION

A detailed description of the flow past an impulsively started circular cylinder in the pre-symmetry breaking regime can be found in the authors' previous work.[11] The present work is concerned with the post-symmetry breaking period leading to vortex shedding. Maintaining the same convention used by the authors in earlier work[11] for the primary, secondary, and tertiary vortices, we reproduce the schematic diagram of the vortex structures in Figure 5, now at the juncture of onset of asymmetry. Here the rounded structure represents the circular cylinder, S is the point of separation, θ_s is the angle of separation, A is the rear most point on the cylinder, and B is the point on the primary wake farthest from the cylinder. The letters P, P' denote the centers of the primary, S_1,S_2, S_1', S_2', the centers

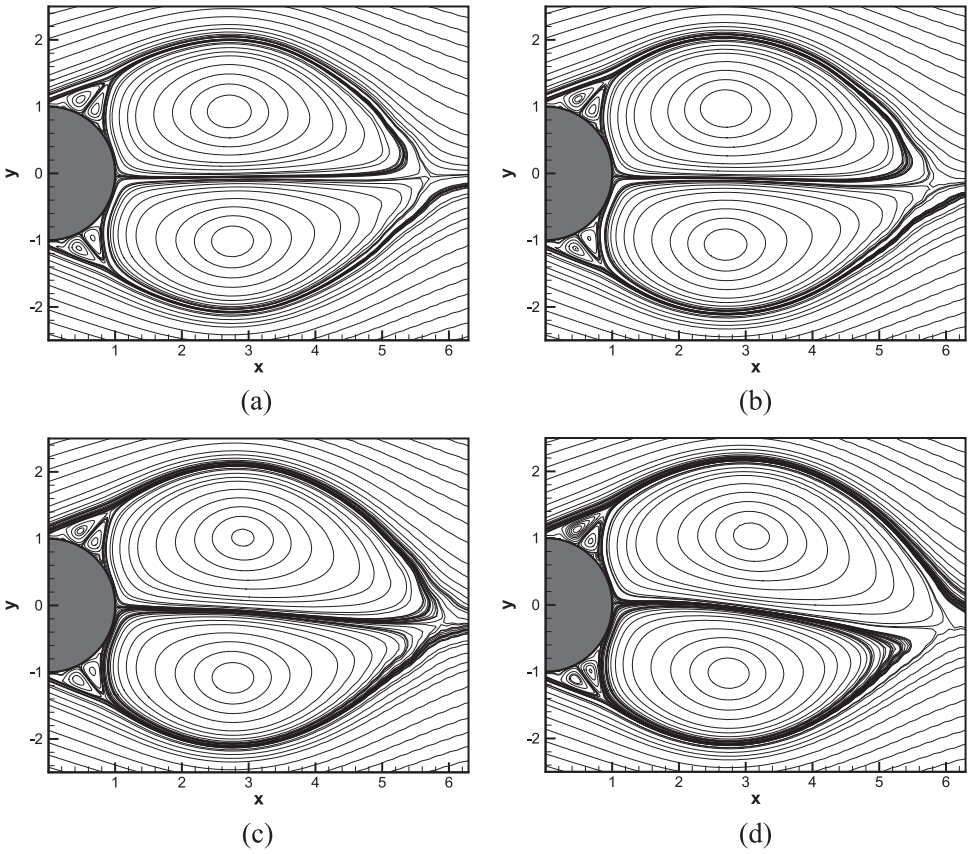


FIG. 6. Onset of asymmetry for the flow past circular cylinder problem for Re = 5000, streamlines at (a) t = 17.00, (b) t = 18.00, (c) t = 19.00, and (d) t = 20.00.

of the secondary, and T_1-T_4, T_1'-T_4', the centers of the tertiary vortices behind the cylinder. The primed letters denote vortices in the lower half of the cylinder while non-primed ones refer to vortices in the upper half. Note that, almost instantly after the impulsive start, the flow separation takes place on the surface of the cylinder and a pair of symmetric vortices grow behind the cylinder. These two primary vortices grow in size and strength with their centers P, P' shifting towards the downstream of the flow. Meanwhile, the interplay amongst the primary, secondary, and tertiary vortices continues on the surface of the cylinder giving rise to the intriguing phenomena, namely, α-, β-, sub-α-, sub-β-, and β-like phenomena. After a while, the symmetry of the flow is broken presaging the onset of vortex shedding.

A. The sub-α-phenomenon preceded by the sub-β-phenomenon

For $Re = 5000$, the upper and lower primary vortices lose their symmetry about the x-axis at the tail of the wake around time $t = 18.00$ and the break in the symmetry becomes completely obvious at $t = 20.00$. We depict the onset of asymmetry of the wake in Figure 6. Prior to this, the secondary and tertiary vortex phenomena take place at the early stages of the flow. It is well known that for $Re = 5000$, the α-phenomenon is preceded by the β-phenomenon. Likewise, it was found by Kalita and Sen[11] that at times the sub-α-phenomenon is preceded by the sub-β-phenomenon in early stages of the flow when the flow is still symmetric about the x-axis. We now show that such phenomena occur even after the symmetry in the flow is broken. However, the duration of such an occurrence is very short. The first of this occurs when the tertiary vortex $\mathbf{T}_3$ reappears at around $t = 22.60$ on the upper half surface and becomes clearly visible at $t = 22.85$ as can be seen from Figures 7(a)–7(b). It grows in size and strength and splits the secondary vortex $\mathbf{S}_2$ into two parts $\mathbf{T}_3$ and $\mathbf{T}_4$ (Figures 5 and 7(c)–7(e)) leading to the formation of a pair of tertiary vortices (Figure 7(f), $t = 23.40$: ψ values at the centers are −0.000 642 3 and 0.000 440 5). However, they are quickly washed away by the free-stream just on the eve of the vortex shedding process.

A similar event can be seen at the lower half of the cylinder as well which last almost three times in duration than the one at the upper half described above. Here the tertiary vortex $\mathbf{T'}_3$ surrounded by $\mathbf{S'}_2$ reappears at around $t = 20.00$ and the pair of tertiary vortices $\mathbf{T'}_3$ and $\mathbf{T'}_4$ (Figure 5) forming the sub-α-phenomenon is closest to each other in strength at $t = 25.00$ with ψ values −0.004 125 7 and 0.003 869 4 at their centers. The sequence of events can be seen in Figures 8(a)–8(d). Gradually, the tertiary vortices lose their strengths, merge with one another somewhere around $t = 26.00$, and eventually are swept away by the free stream.

The evidence of the sub-α-phenomenon preceded by the sub-β-phenomenon will also be observed in the post-vortex shedding period in Sec. III B.

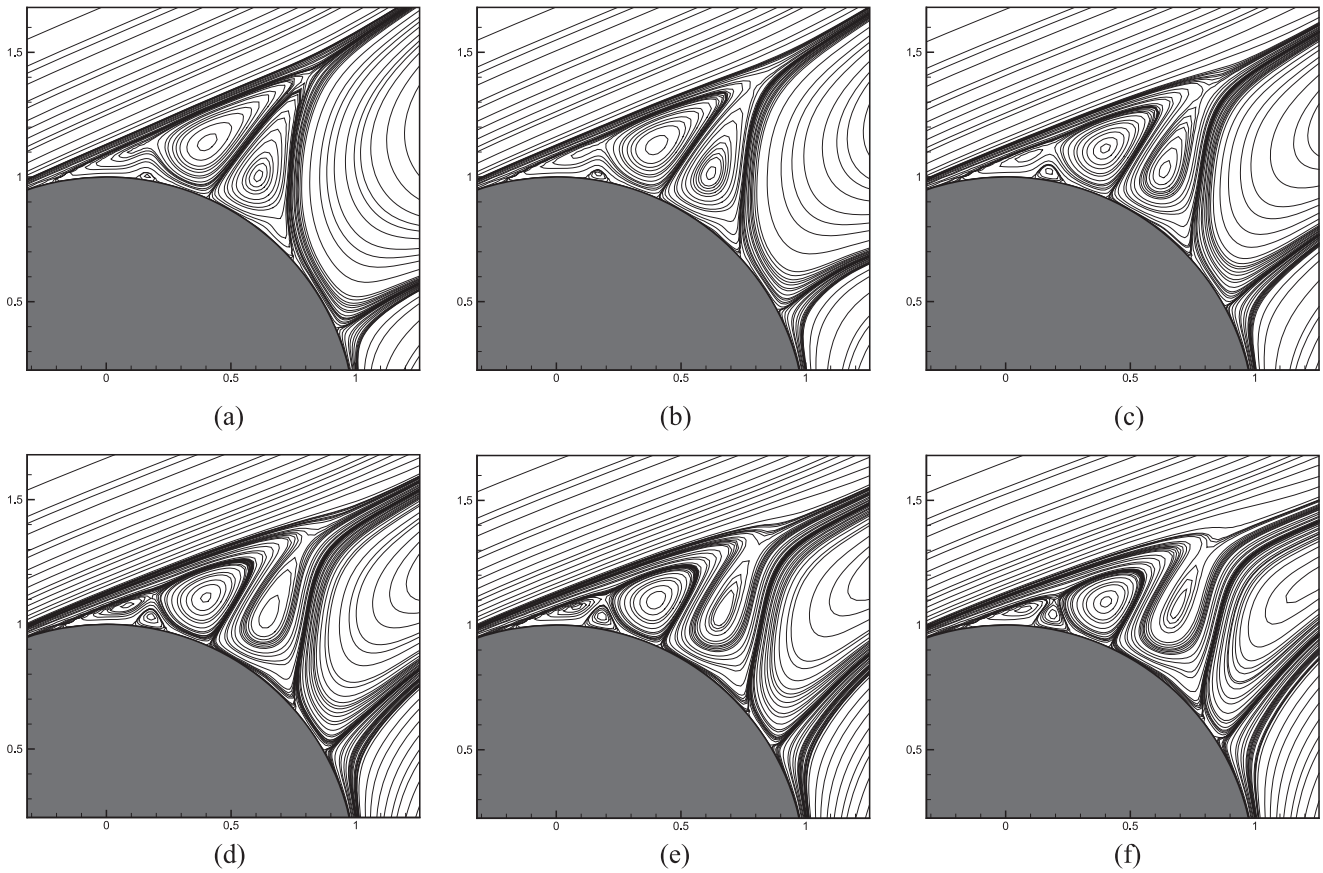


FIG. 7. The sub-β-phenomenon followed by the sub-α-phenomenon, streamlines in the upper half: (a) $t = 22.60$, (b) $t = 22.85$, (c) $t = 23.10$, (d) $t = 23.20$, (e) $t = 23.30$, and (f) $t = 23.40$ for $Re = 5000$.

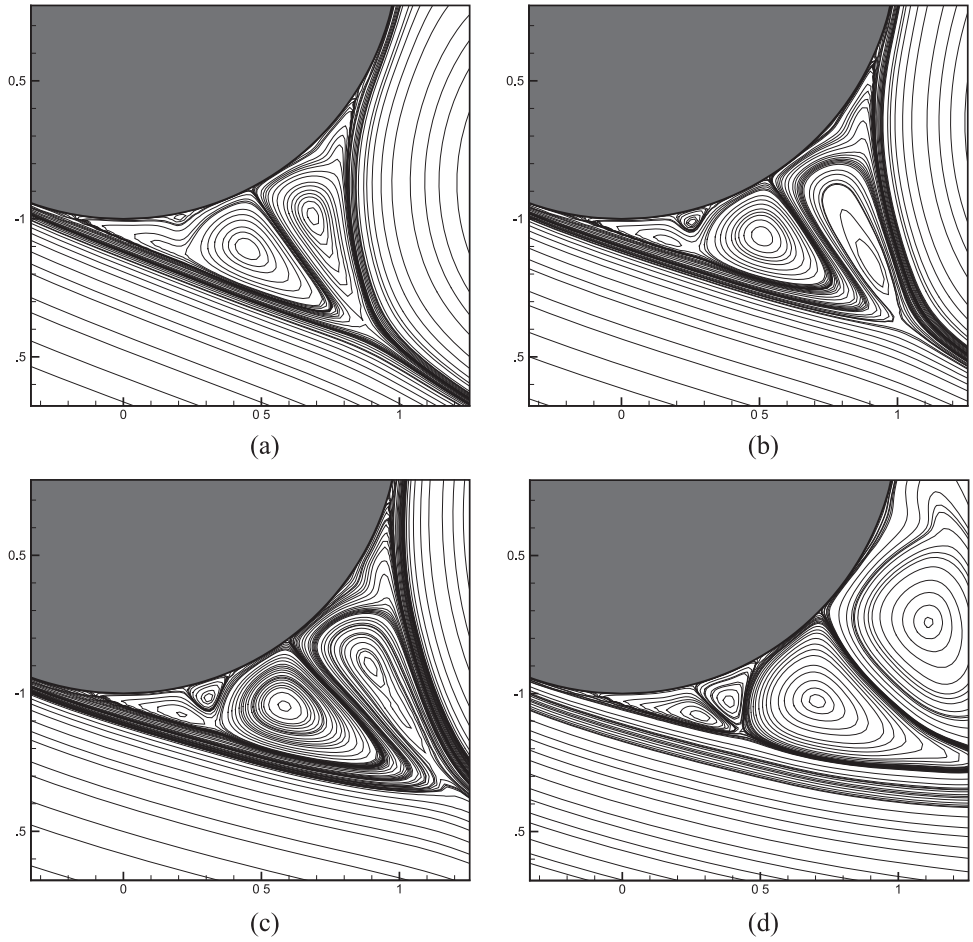


FIG. 8. The sub-β-phenomenon followed by the sub-α-phenomenon, streamlines in the lower half: (a) t = 20.25, (b) t = 22.00, (c) t = 23.00, and (d) t = 25.00 for Re = 5000.

B. Post-vortex shedding secondary and tertiary phenomena

Long time flow computations on the impulsively started circular cylinder by Thoman and Szewczyk[2] and Chou and Huang[7] indicate the occurrence of vortex shedding for Reynolds number as high as 40 000. It will be interesting to see whether the secondary and tertiary vortex phenomena described in Sec. III A still persist in the post-shedding regime as well and if so, whether the event is periodic or not. In order to check this, at first, we probe the flow for Re = 1000. As in the work of Chou and Huang,[7] we also found the flow to be periodic in the post-vortex shedding regime; in Figure 9 and Table II, we present our results for a complete periodic cycle over five successive instants of time. In this sequence, Figures 9(a) and 9(c); 9(c) and 9(e); and 9(b) and 9(d) are half a vortex shedding cycle apart and each of these combinations is a mirror image of the other. Figures 9(a) (also 9(e)) and 9(c) are indicative of the presence of a "pair of secondary vortex" constituting the α-phenomenon at the top and bottom surfaces respectively during these instants of time. This is further confirmed by Table II where we have tabulated the strength of the primary and secondary vortices over those instants. The extreme closeness of the strengths of $\mathbf{S}_1$ and $\mathbf{S}_2$ at times $t = T$ and $t = T + T_0$ and the strengths of $\mathbf{S'}_1$ and $\mathbf{S'}_2$ at time $t = T + \frac{T_0}{2}$ is clear evidence of α-phenomenon during those instants.

Further, we see that the magnitudes and the signs of the values of $(\mathbf{P}, \mathbf{P}, \mathbf{P})$ and $(\mathbf{S}'_2, \mathbf{S}_2, \mathbf{S}'_2)$ at $(T, T + \frac{T_0}{2}, T + T_0)$, and $(\mathbf{P}, \mathbf{P})$ and $(\mathbf{S}_1, \mathbf{S}'_1)$ at $(T + \frac{T_0}{4}, T + \frac{3T_0}{4})$ assert the periodicity of the flow, and also the fact at half a period apart, the streamlines (as well as vorticity contours, not shown here) are mirror images of each other. Here T denotes a particular instant of time after periodic shedding and T_0 is the period of one vortex shedding cycle. It is interesting to note that while α-phenomenon was completely absent in the strong sense in the early stages of the flow, it can be found in the post-shedding period for this Re.

Next, we move our attention towards the post-vortex shedding period for Re = 5000. Note that the cycles of vortex shedding are not exactly periodic for this Reynolds number and as such it is not possible to get mirror images like Figure 9. However, even during the post-shedding period, the flow is replete with the presence of α-phenomenon, sub-α-phenomenon in isolation, and sub-α-phenomenon preceded by sub-β-phenomenon. In Figure 10(a), we depict the sub-α-phenomenon in isolation in the lower half of the cylinder at t = 35.74 while the events leading to sub-α-phenomenon preceded by sub-β-phenomenon in the upper half of the cylinder are presented in Figures 11(a)–11(c) (also see Figure 10(b)). In Figure 11(d), we depict the α-phenomenon which occur shortly after at time t = 42.15. The strengths of the secondary and tertiary vortices corresponding to these events

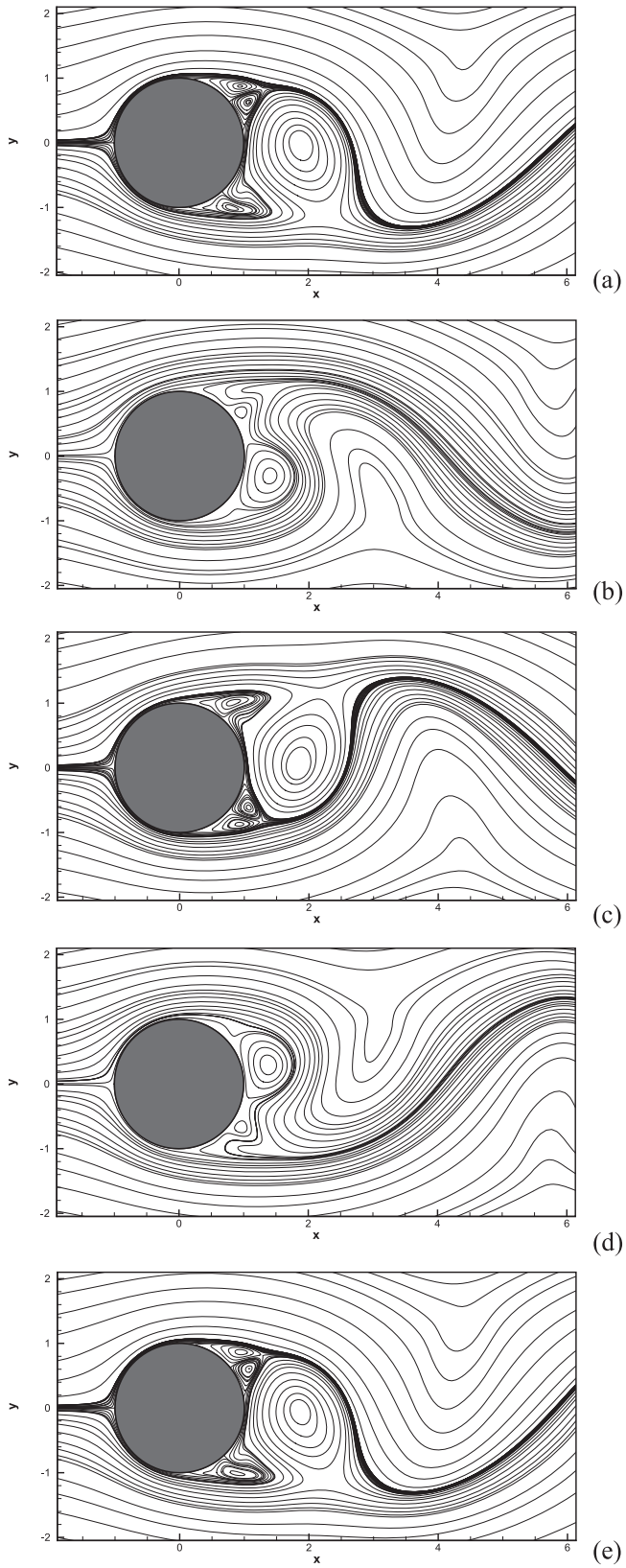


FIG. 9. The streamfunction contours depicting the wake behind the cylinder for $Re = 1000$ for five successive instants of time (a) $t = T$, (b) $t = T + \frac{T_0}{4}$, (c) $t = T + \frac{T_0}{2}$, (d) $t = T + \frac{3T_0}{4}$, (e) $t = T + T_0$.

are presented in Table III. These figures are also indicative of the pace at which the secondary and tertiary vortices are formed.

C. Unsteady flow separation in the post-symmetry break regime

Earlier studies by the authors (Kalita and Sen[11]) and Figures 6(a)–6(d) reveal that the α-phenomenon continues to persist even after the symmetry in the flow is broken. However, the occurrence of the sub-α- and sub-β-phenomenon which is typical of tertiary vortex dynamics in the post-symmetry breaking and post-shedding period is a new observation. Juxtaposed to the longer duration of the secondary vortex phenomena, one can observe intermittent appearance and disappearance and sometimes persistent stay of tertiary vortices around a particular location without leading to the sub-α- and sub-β-phenomenon. The underlying mechanism behind the formation of these structures before the symmetry break during the early stages of the flow has already been described by the authors.[11] In this section, we shall illustrate one of the many such tertiary vortex phenomena after the symmetry break by the structural bifurcation theory proposed by Ma and Wang[13] and Ghil *et al.*[14,15] and also the viscous-inviscid interaction of unsteady boundary layer separation induced by a vortex by Obabko and Cassel.[16,17]

The formation of the primary secondary and tertiary vortices, which gives way to multiple separation zones on the surface of the cylinder in the early stages of the flow, can also be seen in the post-symmetry break period. For complete understanding of the interplay of these vortices, one needs to carefully examine the vorticity distribution on the surface of the cylinder. The extent and size of these vortices can be gauged by the alternate positive and negative values of vorticity on the surface of the cylinder. In Figure 12(a), we show several recirculation zones on the surface of the cylinder by plotting the streamlines over the whole cylinder just before the onset of sub-α-phenomenon on the lower half of the cylinder preceded by the sub-β-phenomenon at $t = 23.20$. The corresponding vorticity distribution on the surface of the cylinder at the same instant of time is shown in Figure 12(b). The separation points and the recirculation zones enclosed by a pair of such points have been indicated by marking them with numbers 1-13. This figure clearly demonstrates the one to one correspondence between the recirculation zones and vorticity distribution which will be used in the subsequent discussion to delve more into the making up of these vortices.

The boundary layer separation theory proposed by Ghil *et al.*[14,15] also uses vorticity as a tool to predict the location and time of the formation of a recirculation zone. The conditions for the separation of flow on the surface of a wall are given as follows:

$$\omega(P,T) = 0, \quad \frac{\partial \omega}{\partial \tau} = 0, \quad \frac{\partial^2 \omega}{\partial \tau^2} > 0, \quad \frac{\partial \omega}{\partial t} < 0, \tag{5}$$

where ω is the vorticity, t the time, and τ is the direction tangential to the wall. It is assumed that structural bifurcation occurs at time T and the normal derivative of the velocity field $\frac{\partial \mathbf{u}}{\partial n}$ ($\mathbf{u} = (u, v)$ is the velocity field and n is the direction normal to the wall) has a degenerate singular point P on the wall; it is also assumed that on the wall, no slip boundary conditions are imposed and the shear flow is downward. When the flow is upward, the last two inequalities in Equation (5) undergo change of signs.

TABLE II. Strengths of primary and secondary vortices during the vortex shedding period for Re = 1000 corresponding to Figure 9.

	Vortices				
Time	**P**	$\mathbf{S}_1$	$\mathbf{S}_2$	$\mathbf{S'}_1$	$\mathbf{S'}_2$
$t = T$	−0.589 1259	0.028 989 4	−0.029 043 6	...	0.011 569 2
$t = T + \frac{T_0}{4}$	0.193 834 2	0.015 063 9	...	...	...
$t = T + \frac{T_0}{2}$	0.571 003 3	...	−0.015 594 1	−0.030 210 8	0.029 752 8
$t = T + \frac{3T_0}{4}$	−0.189 872 1	...	...	−0.016 458 9	...
$t = T + T_0$	−0.579 389 2	0.027 626 7	−0.027 493 2	...	0.016 660 9

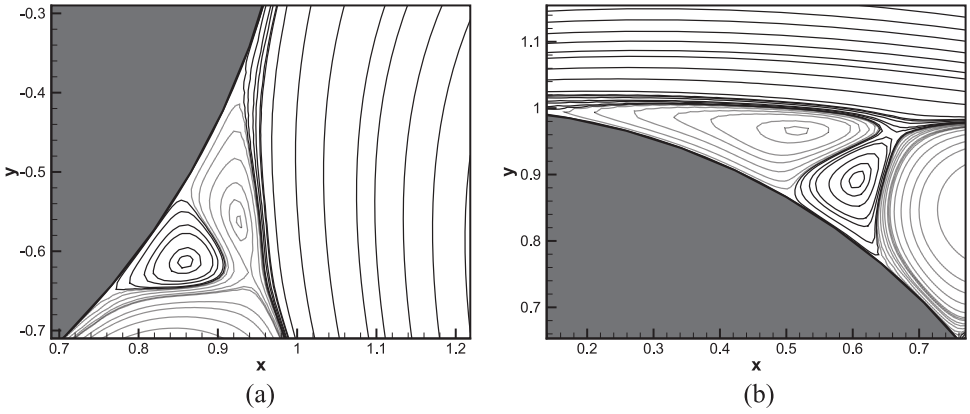


FIG. 10. Streamfunction contours representing the sub-α-phenomenon at (a) t = 35.74 and (b) t = 42.03 for Re = 5000 (solid lines for positive and dotted lines for negative values, grid size: 481×801).

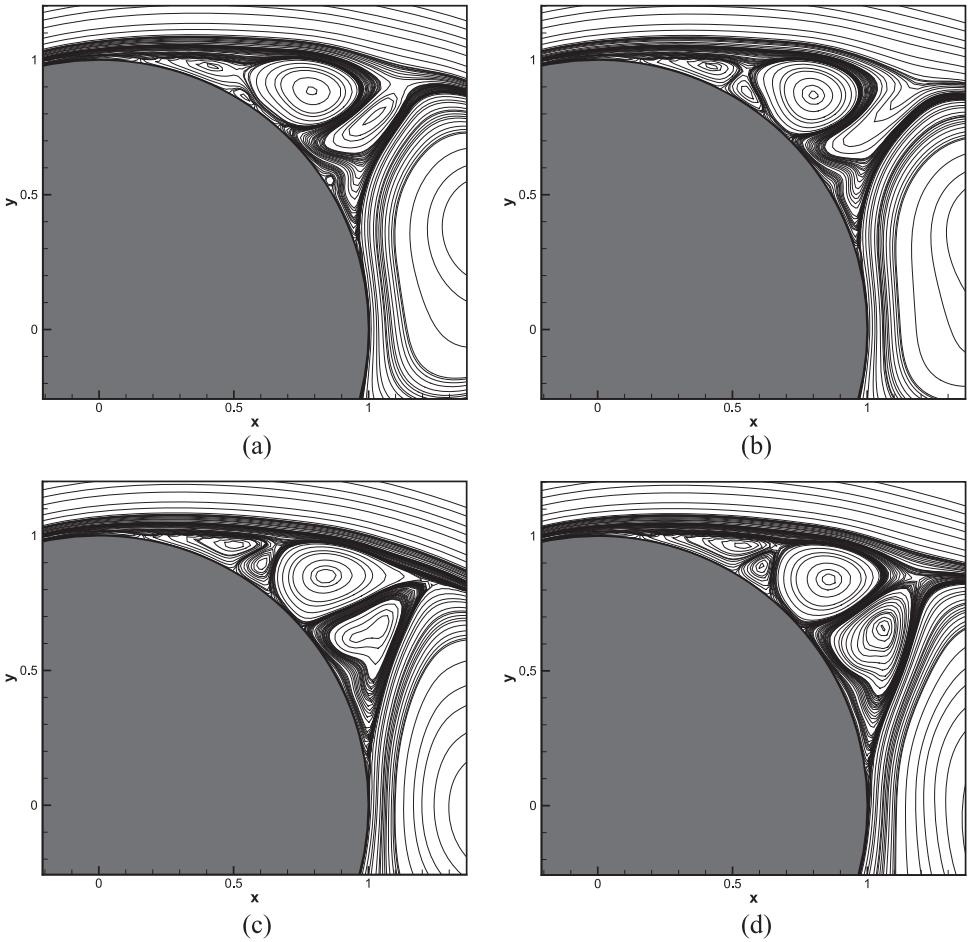


FIG. 11. The post-shedding period, the sub-β-phenomenon followed by the sub-α-phenomenon, streamlines in the upper half: (a) t = 41.615, (b) t = 41.70, and (c) t = 42.00; the α-phenomenon at (d) t = 42.15 for Re = 5000.

TABLE III. Strengths of secondary and tertiary vortices during the vortex shedding period for $Re = 5000$.

	Vortices					
Time	$\mathbf{S}_1$	$\mathbf{S}_2$	$\mathbf{T}_3$	$\mathbf{T}_4$	$\mathbf{T'}_1$	$\mathbf{T'}_2$
$t = 35.74$ (Figure 10(a))	...	...	...	...	−0.001 651 9	0.001 664 5
$t = 42.03$ (Figure 10(b))	...	...	0.002 9567	−0.002 939 2	...	...
$t = 42.15$ (Figure 11(d))	0.062 850 9	−0.061 122 8	...	...	...	...

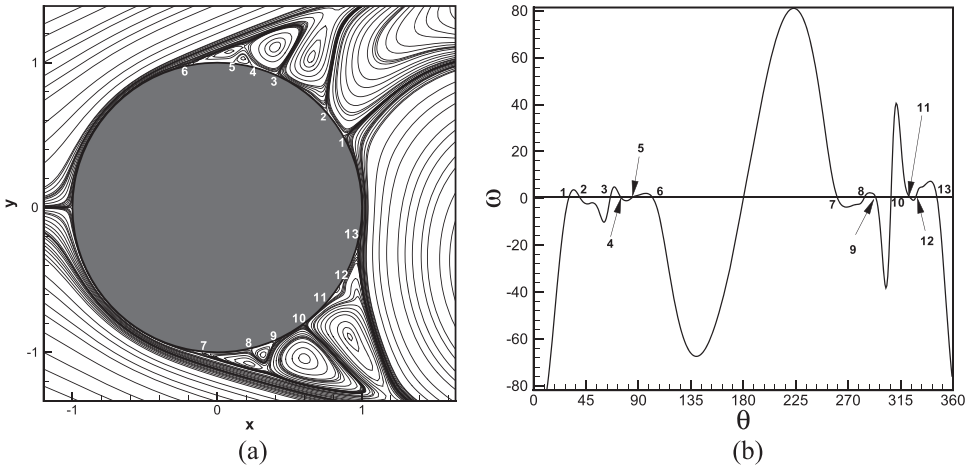


FIG. 12. (a) Recirculation zones depicted by streamlines and (b) vorticity distribution on the surface of the cylinder at $t = 23.20$ for $Re = 5000$ on grid size 361×901.

Note that during the early stages of the flow, tertiary vortices responsible for the sub-α- and sub-β-phenomenon, immediately after inception, grow in strength and size to split the secondary vortices cushioning them eventually leading to the sub-α-phenomenon. However, the tertiary vortices causing these phenomena after symmetry break lack the strength required to grow in size, and as such sits in the same location for quite some time. For example, the tertiary vortices between the pairs of points (4, 5) and (8, 9) in Figure 12 in the upper and lower halves of the cylinder, respectively, had their inception much before the symmetry break at a very early stage around $t = 2.00$. They alternate between growing and shrinking slenderly in size and shifting their centers in small neighbourhoods eventually making the final push at around $t = 22.00$ to split the secondary vortices (lying between the pair of points (3, 6) and (7, 10) in Figure 12) leading to the sub-α- and sub-β-phenomenon.

Authors in their earlier work[11] discussed structural bifurcation and unsteady flow separation in the very early stages of the flow when the flow is essentially symmetric and as such the upper half was a true representation of the whole cylinder for explaining the tertiary vortex dynamics. Here we have chosen the lower half of the cylinder to explain the formation of one of the tertiary vortices after the symmetry break around time $t = 23.20$. In Figures 13(a) and 13(b), we show the close up view of the streamfunction contours at $t = 22.82$ and $t = 23.40$, respectively, on the lower half of the cylinder. In Figure 13(a), one can clearly see the trace of a tertiary vortex between points 11 and 12 in the vicinity of $\theta = 324.8°$ (refer to Figure 12(b) as well), which is more evident in Figure 13(b). This is corroborated by Figure 14(a), which shows the surface vorticity distribution on the lower half of the cylinder at the same time junctures and suggests that the birth of this vortex actually takes place sometimes between $t = 22.82$ and $t = 23.40$.

The primary and secondary vortices are formed behind the cylinder much earlier; details of it can be found in the authors' work.[11] The primary vortices produce adverse pressure gradients leading to the formation of secondary vortices,

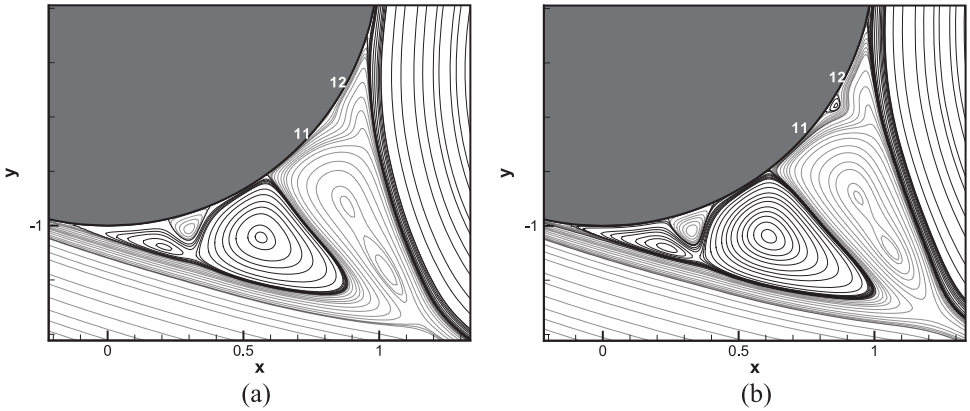


FIG. 13. Close up view of the streamfunction contours on the lower half of the cylinder at (a) $t = 22.82$ and (b) $t = 23.40$ for $Re = 5000$ on grid size 361×901.

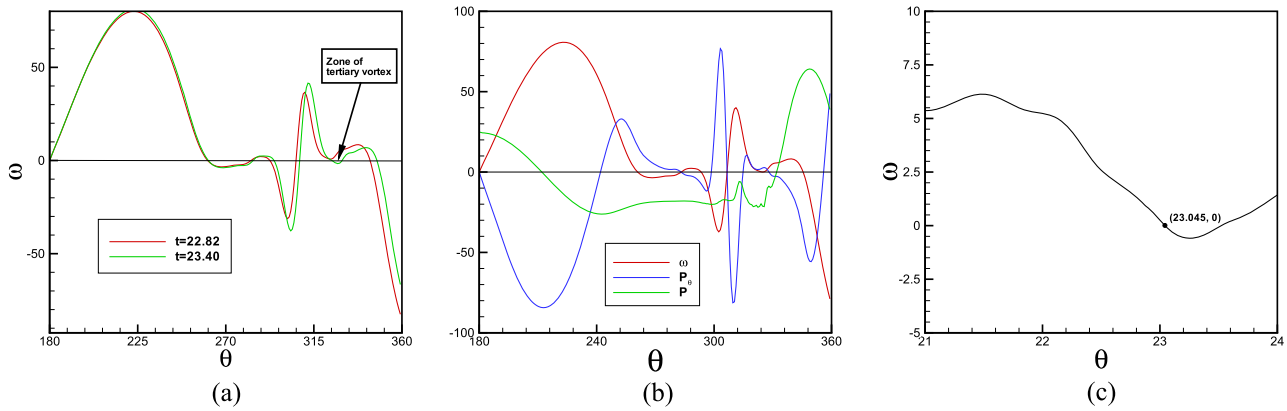


FIG. 14. (a) Vorticity distribution just before and after the formation of the tertiary vortex and (b) distribution of vorticity, pressure, and pressure gradient at t = 23.045 on the surface of the cylinder in the lower half and (c) time history of vorticity at the point $\theta = 324.8°$ for $Re = 5000$.

which in turn produce adverse pressure gradients in the opposite direction of the previous one eventually resulting in the formation of tertiary vortices. In Figure 14(b), we show the distribution of vorticity ω (which is scaled down in order to make it comparable with pressure and pressure gradient), the pressure p and $\frac{\partial p}{\partial \theta}$ along the surface of the cylinder at time t = 23.045. It is clear from Figure 12(b) that the vorticity value reaches zero at $(\theta, t) = (324.8°, 23.045)$ as a local minimum in space and decreases in time, that is, $\omega(324.8°, 23.045) = 0$, $\frac{\partial \omega}{\partial \theta}(324.8°, 23.045) = 0$, $\frac{\partial^2 \omega}{\partial \theta^2}(324.8°, 23.045) > 0$. In Figure 14(c), we show the time history of the vorticity value at the point $\theta = 324.8°$ on the surface; this figure clearly shows that $\frac{\partial \omega}{\partial t}(324.8°, 23.045) < 0$. Thus all the conditions given by Equation (5) are satisfied confirming the occurrence of a structural bifurcation at time t = 23.045 at $\theta = 324.8°$ on the surface of the cylinder.

In Figures 15(a) and 15(b), we show the combined streamfunction and vorticity contours in the neighborhood of the point $\theta = 324.8°$ in the θ-r plane at the same time junctures discussed above. Here the r scale has been magnified 10 times in order to have a better view. These figures will help us explain the formation of this vortex in the light of the theory of viscous-inviscid interaction of unsteady boundary layer separation induced by a vortex by Obabko and Cassel.[16,17] The secondary vortex lying between the points 10 and 13 on the surface of the cylinder in Figure 12 produces an adverse pressure gradient. Just before the formation of the tertiary vortex (refer to Figures 13(a) and 15(a)), the pressure gradient

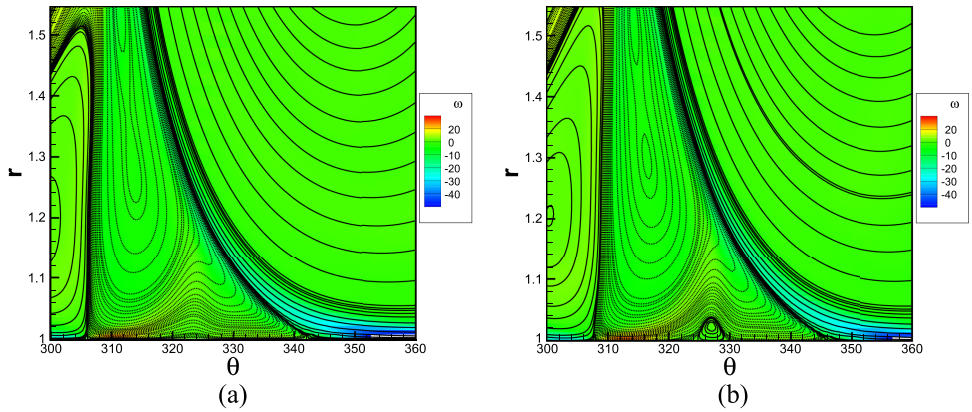


FIG. 15. Combined streamfunction and vorticity contours for Re = 5000 at (a) t = 22.82 and (b) t = 23.40. (Streamfunction with line option and vorticity contours with flood option. The solid lines for positive and dotted lines for negative streamfunction values, grid size: 361 × 901.).

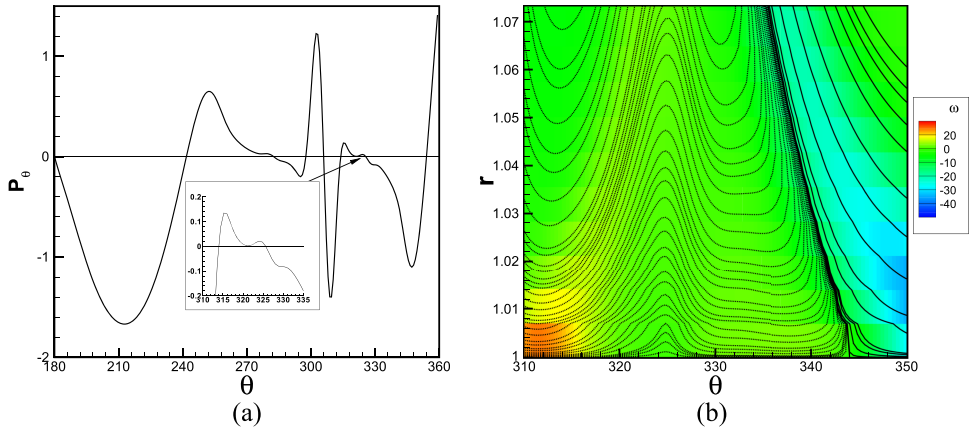


FIG. 16. (a) Distribution of surface pressure gradient at t = 22.82 on the lower half of the cylinder and (b) combined streamfunction and vorticity contours at t = 23.04 for Re = 5000. (Streamfunction with line option and vorticity contours with flood option. The solid lines for positive and dotted lines for negative streamfunction values, grid size: 361 × 901.).

begins changing in the vicinity of its minimum exactly where the structural bifurcation takes place eventually. As one can clearly see in Figure 16(a) which depicts the distribution of surface pressure gradient on the lower half of the cylinder at $t = 22.82$, there is a presence of a local minimum for p_θ at $\theta \cong 324.8°$. This local minimum having a positive value becomes locally adverse immediately afterwards attaining a negative value leading to the formation of the tertiary vortex in Figures 13(b) and 15(b).

In Figure 16(b), we again show the close up view of the streamfunction contours at time $t = 23.04$ in the θ-r plane where the scales have been further magnified to show the flow pattern more clearly within the boundary layer which is of the order $Re^{-\frac{1}{2}}$. Note that, in the later stages, after the formation of the secondary vortex represented by the zone between the points 10 and 13 in Figure 12, the boundary layer concentrates on a narrow region that forms on the upstream side of the recirculation zone. Since the fluid particle at the separation point is rapidly compressed in the streamwise direction, it elongates in the direction normal to the wall. The elongation of the fluid particle in the normal direction leads to the growth of a sharp spike (Obabko and Cassel,[16] Obabko and Cassel,[17] Cassel *et al.*[18]) leading to what is called as the spike formation just before the formation of tertiary vortex. This spike is clearly visible from our computation as can be seen at time $t = 23.04$ in Figure 16(b), thus validating the birth of the tertiary vortex at time $t = 23.045$ discussed above.

IV. CONCLUSION

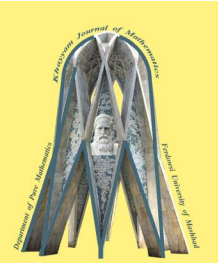
In this study, we make some interesting observations on the flow phenomena that can be observed during α- and β-phenomena which are typical of the flow past of an impulsively started circular cylinder at moderately high Reynolds numbers by numerically solving the biharmonic pure streamfunction form of the Navier-Stokes equations. Contrary to the belief that these phenomena occur at the very early stage of the flow, we have shown that they are very much in existent even during the post-shedding period. The evidence of their existence is substantiated by the strengths of the secondary vortices forming such phenomena at later stages of the flow. Events similar to the α- and β-phenomena were also observed at the tertiary level in either half of the cylinder once the symmetry of the flow was broken. We have also carried out a detailed analysis on the topological structures of the tertiary recirculation zones using the boundary layer separation theory based on the structural bifurcation concept. This has been further strengthened by explaining the birth of the tertiary structures from the angle of viscous-inviscid interaction. To the best of our knowledge, no report on the existence and quantification of these phenomena is seen in the available literature and our observations may lead to providing more insights into the secondary and tertiary vortex dynamics for flow past bluff bodies other than the circular cylinder.

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ERROR LOCATING CODES BY USING BLOCKWISE-TENSOR PRODUCT OF BLOCKWISE DETECTING/CORRECTING CODES

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Abstract. In this paper, we obtain lower and upper bounds on the number of parity check digits of a linear code that corrects e or less errors within a sub-block. An example of such a code is provided. We introduce blockwise-tensor product of matrices and using this, we propose classes of error locating codes (or EL-codes) that can detect e or less errors within a sub-block and locate several such corrupted sub-blocks.

1. Introduction

Error detecting codes are used by the receiver merely to detect the presence of errors in a block of received digits, whereas error correcting codes at the receiver are used to correct errors that may have occurred in transmission. To have a good error correcting capability of the codes, long code length is required to be considered, but this results in decreasing the information rate of the system. This problem had been somewhat sorted out by Wolf and Elspas [15]. They introduced the concept of error location by which a compromise between short and long code length can be obtained. The technique in error location concept is that the block of received digits is subdivided into mutually exclusive sub-blocks. The length of the sub-blocks can be chosen relatively smaller requiring less number of parity checks which can improve the information rate of the system. In this concept, errors occurring within sub-block(s) are detected at the receiver, and in addition the receiver is able to specify which particular sub-block(s) contains errors. Of

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course, the precise location of erroneous digit positions can not be located, but the sub-block(s) containing errors can be identified. The codes following this technique is referred to as *error locating codes* (in short *EL-codes*).

Wolf and Elspas [15] devised codes that could locate a single corrupted sub-block containing a given number of random errors. Later on, Wolf in [13] obtained a class of EL-codes capable of identifying corrupted sub-blocks by considering tensor product of parity check matrices of two codes. In EL-codes, the errors are considered to occur within sub-block(s). Such type of errors are found to be in mostly storage and memory systems [6, 9]. Thus, there is a need to study blockwise error detecting/correcting codes. For more study on error locating codes and blockwise error correcting codes, one may refer to [2, 3, 4, 5, 6, 7, 8, 10, 16, 17].

1.1. **Outline:** The paper is organized as follows: In Section 2, we recall basic definition and notations to make the paper self-contained. We present lower and upper bounds on the number of parity check digits for a linear code correcting e or less errors within a sub-block in Section 3. An example is given to support our result. Wolf in [13, 14] studied construction of a class of EL-codes that can locate several corrupted sub-blocks by considering tensor product of parity check matrices of two codes. We introduce the concept of blockwise tensor product and by using this we construct another classes of EL-codes (but of relatively smaller length given in [13, 14]) that can locate several corrupted sub-blocks in Section 4. In Section 5, we present a comparative study between the bounds obtained in Section 3-4, and some existing bounds. We verify that the development of the codes obtained in Section 3-4 improves the efficiency of the communication channel.

2. Basic Tools

In this section, we recall basic definition, notations and results to make the paper self-contained. The largest integer less than or equal to x is denoted by $\lfloor x \rfloor$. An (n, k) linear code will be considered as a subspace of the space of all n-tuples over GF(q). The distance between two vectors shall be considered in the Hamming sense. The minimum distance of a linear code is the minimum of distances between any two code vectors. The information rate of the code is $\frac{k}{n}$. By a burst of length b, we mean a vector with nonzero entries in some b consecutive positions and zeros elsewhere.

Next we consider following conditions for an (n, k) linear codes over $GF(q)$ of which the code length is divided into m mutually exclusive sub-blocks.

(1) The syndrome resulting from the occurrence of e or fewer errors within a sub-block must be distinct from the all zero syndrome.
(2) The syndrome resulting from the occurrence of e or fewer errors within a sub-block must be distinct from the syndrome resulting likewise from e or fewer errors within any *other* sub-block.

(3) The syndrome resulting from the occurrence of e or fewer errors within a single sub-block must be distinct from the syndrome resulting likewise from e or fewer errors within the same sub-block.
(4) The syndrome resulting from the occurrence of such errors within any $l(< m)$ or less sub-blocks must be non-zero and distinct from the syndrome resulting likewise from within any *same* $l(< m)$ or less sub-blocks.
(5) The syndrome resulting from the occurrence of such errors within any $l(< m)$ or less sub-blocks must be distinct from the syndrome resulting likewise from within any *other* $l(< m)$ or less sub-blocks.

The code capable of detecting e or fewer errors within a sub-block must satisfy condition (1). The code capable of detecting and locating such error within a sub-block must satisfy conditions (1) and (2). The code capable of correcting e or fewer errors within a sub-block, the code should satisfy conditions (1) − (3). Further, to locate any $l(< m)$ sub-blocks containing e or fewer errors within a sub-block, conditions (4) and (5) must be satisfied.

To conclude the section, we give the definition of tensor product of matrices which can be found in literature (for example see, [1]).

Definition 2.1. Let $A = [a_{ij}]$ be a m-by-n matrix and let $B = (b_{ij})$ be a p-by-q matrix. The tensor product of A and B is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & . & . & . & . & a_{1n}B \\ a_{21}B & a_{22}B & . & . & . & . & a_{2n}B \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{m1}B & a_{m2}B & . & . & . & . & a_{mn}B \end{bmatrix}.$$

3. Bounds on the number of parity check digits

Peterson and Weldon [11] gave lower and upper bounds for the linear codes that correct e or less errors (see Theorem 4.5 and Theorem 4.7 in [11], also see [12]). By using the technique given in [11], we present lower and upper bounds for the linear codes that correct e or less errors within a sub-block.

Theorem 3.1. [*Lower Bound*] *The number of parity check digits in an* (n, k) *linear code over* $GF(q)$ *subdivided into* m *sub-blocks of length* t *each, that corrects* e *or less errors within a sub-block is bounded from below by*

$$n - k \geq \log_q \left\{ 1 + m \sum_{i=1}^{e} \binom{t}{i} (q-1)^i \right\}. \tag{3.1}$$

Proof. We prove the theorem by counting the number of syndromes that are required to be nonzero and distinct by conditions (1), (2) and (3) (in Section 2), and then setting this number less than or equal to q^{n-k}, the number of maximum possible syndromes. By conditions (1), (2) and (3), the syndromes produced by e or less errors within a sub-block are nonzero and distinct, whether in the same sub-block or in different sub-blocks. The total number of such syndromes,

including the vector of all zeros, is given by (see Wolf [13])

$$1 + m\sum_{i=1}^{e}\binom{t}{i}(q-1)^i.$$

Therefore, we must have

$$q^{n-k} \geq 1 + m\sum_{i=1}^{e}\binom{t}{i}(q-1)^i$$

or,

$$n-k \geq \log_q\left\{1 + m\sum_{i=1}^{e}\binom{t}{i}(q-1)^i\right\}.$$

The result is proved. □

Remark 3.2. For $m = 1$, the inequality (3.1) reduces to

$$n-k \geq \log_q\left\{1 + \sum_{i=1}^{e}\binom{t}{i}(q-1)^i\right\},$$

which coincides with the result of Theorem 4.5 of [11].

Theorem 3.3. [*Upper Bound*] *There exists an* (n, k) *linear code over* $GF(q)$ *sub-divided into* m *sub-blocks of length* t *each, that corrects* e *or less errors within a sub-block* $(t \geq 2e)$ *provided that*

$$q^{n-k} > 1+\sum_{i=1}^{2e-1}\binom{t-1}{i}(q-1)^i+(m-1)\left\{1+\sum_{i=1}^{e-1}\binom{t-1}{i}(q-1)^i\right\}\sum_{i=1}^{e}\binom{t}{i}(q-1)^i. \tag{3.2}$$

Proof. The existence of the code can be ensured by constructing a suitable $(n-k)\times n$ parity check matrix H for the desired code. Without loss of generality, we can assume that the columns of the first $m-1$ sub-blocks of H and the first $j-1$ columns $h_1, h_2, \ldots, h_{j-1}$ of the m^{th} sub-block have been appropriately added. The j^{th} column h_j of the m^{th} sub-block of the matrix H is needed to add according to conditions $(1)-(3)$. This is done in the following two cases:

Case I: First we consider condition (3), according to which the j^{th} column h_j should not be a linear combination of previous any $2e-1$ or less columns of the m^{th} column sub-block. In other words

$$h_j \neq u_{j-1}h_{j-1} + u_{j-2}h_{j-2} + \cdots + u_1h_1, \tag{3.3}$$

where $u_i \in GF(q)$ are any $2e-1$ or less nonzero coefficients.

The number of linear combinations on the R.H.S. of inequality (3.3), including the vector of all zeros, is

$$1 + \sum_{i=1}^{2e-1}\binom{j-1}{i}(q-1)^i. \tag{3.4}$$

(Note that the condition (1) is also taken care of in (3.4).)

Case II: In this case we consider condition (2), accordingly the j^{th} column h_j should not be a linear combination of previous any $e-1$ or less columns of the

m^{th} sub-block, together with any e or less columns within any other sub-block of the previous $m-1$ sub-blocks. In other words

$$h_j \neq (u_{j-1}h_{j-1} + u_{j-2}h_{j-2} + \cdots + u_1h_1) + (v_ih_i + v_{i+1}h_{i+1} + \cdots + v_{i+b-1}h_{i+b-1}), \tag{3.5}$$

where $u_i, v_i \in GF(q)$, u_i are any $e-1$ or less nonzero and h_i are any e or less columns of a sub-block from the previous $m-1$ sub-blocks.

The number of linear combinations on the R.H.S. of the inequality (3.5) is given by

$$(m-1)\left\{1+\sum_{i=1}^{e-1}\binom{j-1}{i}(q-1)^i\right\}\sum_{i=1}^{e}\binom{t}{i}(q-1)^i. \tag{3.6}$$

At worst, all the linear combinations computed in (3.4) and (3.6) might yield distinct sums. Therefore, h_j can be added to the m^{th} sub-block of H provided that

$$q^{n-k} > expr.(3.4) + expr.(3.6)$$

i.e.

$$q^{n-k} > 1+\sum_{i=1}^{2e-1}\binom{j-1}{i}(q-1)^i+(m-1)\left\{1+\sum_{i=1}^{e-1}\binom{j-1}{i}(q-1)^i\right\}\sum_{i=1}^{e}\binom{t}{i}(q-1)^i.$$

Replacing j by t gives the theorem. □

Remark 3.4. For $m=1$, the inequality (3.2) reduces to

$$q^{n-k} > 1+\sum_{i=1}^{2e-1}\binom{t-1}{i}(q-1)^i,$$

which coincides with the result of Theorem 4.7 of [11].

Example 3.5. Consider a $(12,5)$ binary code with the following parity check matrix 7×12 matrix P which has been constructed by the synthesis procedure given in the proof of Theorem 3.3 by taking $m=3, t=4, e=2, q=2$.

$$P = \left[\begin{array}{cccc|cccc|cccc} 1&0&0&1&0&0&0&0&1&0&0&0\\ 0&1&0&1&0&0&0&0&0&1&0&0\\ 0&0&1&1&0&0&0&0&0&0&1&0\\ 0&0&0&1&0&0&0&1&0&0&0&1\\ 0&0&0&0&1&0&0&1&0&0&1&0\\ 0&0&0&0&0&1&0&1&0&1&0&0\\ 0&0&0&0&0&0&1&1&1&0&0&0 \end{array}\right]$$

The null space of this matrix can be used to correct 2 or less errors within a sub-block of length 4. It may be verified from Error Pattern-Syndrome Table 3.1 that syndromes produced by 2 or less errors within a sub-block, whether in the same sub-block or in different sub-blocks are distinct.

Table 3.1
Error Pattern - Syndrome Table

Error-patterns	Syndromes
Errors within the first sub-block	
1000 0000 0000	1000000
0100 0000 0000	0100000
0010 0000 0000	0010000
0001 0000 0000	1111000
1100 0000 0000	1100000
1010 0000 0000	1010000
1001 0000 0000	0111000
0110 0000 0000	0110000
0101 0000 0000	1011000
0011 0000 0000	1101000
Errors within the second sub-block	
0000 1000 0000	0000100
0000 0100 0000	0000010
0000 0010 0000	0000001
0000 0001 0000	0001111
0000 1100 0000	0000110
0000 1010 0000	0000101
0000 1001 0000	0001011
0000 0110 0000	0000011
0000 0101 0000	0001101
0000 0011 0000	0001110
Errors within the third sub-block	
0000 0000 1000	1000001
0000 0000 0100	0100010
0000 0000 0010	0010100
0000 0000 0001	0001000
0000 0000 1100	1100011
0000 0000 1010	1010101
0000 0000 1001	1001001
0000 0000 0110	0110110
0000 0000 0101	0101010
0000 0000 0011	0011100

4. Blockwise-Tensor Product and EL-Codes

We start this section with the definition of Blockwise-Tensor Product of matrices.

Definition 4.1. If $A = (A_1\ A_2\ ....\ A_m)$, where A_i is a $k_1 \times k_2$ matrix and $B = (B_1\ B_2\ ....\ B_m)$ where B_i is a $s_1 \times s_2$ matrix, the *Blockwise-Tensor(Kronecker) Product* $A \otimes_b B$ is the $(k_1 s_1) \times (m k_2 s_2)$ matrix

$$A \otimes_b B = \left[\ A_1 \otimes B_1 \quad A_2 \otimes B_2 \quad .... \quad A_m \otimes B_m\ \right].$$

By using the blockwise-tensor product and codes discussed in Section 3, we propose new classes of EL-codes that can locate several sub-blocks.

Consider a code of length n consists of m sub-blocks, each sub-block containing t digits, then $n = tm$. Further if $m = m_1 m_2$, then the code of length n can be considered as consisting of m_2 sub-blocks, each sub-block containing tm_1 digits. This new sub-blocks of length tm_1 constitute of m_1 consecutive original sub-blocks of length t. We call the new sub-blocks as *cluster-sub-blocks* of length tm_1. First we present a new class of EL-codes which can locate several sub-blocks within a cluster-sub-block.

Theorem 4.2. *Let $C_1(n_1 = mt, n_1 - k)$ be a linear code over $GF(q)$ with parity check matrix H that detects e_1 or fewer errors within a sub-block of length t and $C_2(n_2 = ms, n_2 - \rho)$ be a linear code over $GF(q)$ with parity check matrix P that corrects e_2 or fewer errors within a sub-block of length s. Then the $(mts, mts - k\rho)$ code C obtained from the parity check matrix $P \otimes_b H$ can detect e_1 or fewer errors within a sub-block of length t and locate any e_2 or fewer such corrupted sub-blocks within a cluster-sub-block of length ts.*

Proof. The parity check matrix of the code C is given by

$$P \otimes_b H = \left[\ P_1 \otimes H_1 \quad P_2 \otimes H_2 \quad .... \quad P_m \otimes H_m\ \right]$$

$$= \begin{bmatrix} p^1_{11}H_1 & p^1_{12}H_1 & .. & p^1_{1s}H_1 & ... & p^m_{11}H_m & p^m_{12}H_m & .. & p^m_{1s}H_m \\ . & . & .. & . & .. & . & .. & . & . \\ . & . & .. & . & .. & . & .. & . & . \\ p^1_{\rho 1}H_1 & p^1_{\rho 2}H_1 & .. & p^1_{\rho s}H_1 & ... & p^m_{\rho 1}H_m & p^m_{\rho 2}H_m & .. & p^m_{\rho s}H_m \end{bmatrix}$$

where P_i and H_j are sub-blocks of length t of P and H, respectively and p^l_{ij} is the $(ij)^{th}$ element of sub matrix P_l $(1 \le l \le m)$.

Consider the situation where e_1 or fewer errors occur only in the λ^{th} sub-block of length t. Let us write $\lambda = (\tau - 1)s + j,\ 1 \le j \le s,\ 1 \le \tau \le m$. Then, the resulting syndrome is $S^\lambda = (S_1, S_2, ...S_\rho)$, where

$$S_i = \sum_{r=1}^{t} u_r p^\tau_{ij} h_{(\tau)_r} = p^\tau_{ij} \sum_{r=1}^{t} u_r h_{(\tau)_r} = p^\tau_{ij} A_\lambda,$$

$u_r \in GF(q)$ are any e_1 or less nonzero coefficients, $h_{(\tau)_r}$ are columns of the sub-block H_τ and $A_\lambda = \sum_{r=1}^{t} u_r h_{(\tau)_r}$. Furthermore, the sum A_λ can not be all zero k-tuple, because the code C_1 detects e_1 or fewer errors within a sub-block of length t. Also since, all p^τ_{ij} are nonzero, the syndrome S cannot be zero (which

satisfies condition (1)). This shows that the code obtained from the parity check matrix $P \otimes_b H$ can detect e_1 or fewer errors within a sub-block of length t.

In order to locate any e_2 or fewer sub-blocks of length t containing e_1 or fewer errors within a cluster-sub-block of length s, the syndrome resulting from detectable errors occurring within any e_2 or fewer sub-blocks within a cluster-sub-block is distinct from syndromes resulting from detectable errors occurring within any other e_2 or fewer sub-blocks within the same cluster-sub-block. Let S^{λ_i} be the syndrome corresponding to the λ_i^{th} sub-block of length t within a cluster-sub-block of length s, where $\lambda_i = (\tau_1 - 1)s + j_i,\ i = 1, 2, ..., 2e_2$. Each sub-block contains e_1 or fewer errors.

Assume that

$$\sum_{i=1}^{2e_2} x_i S^{\lambda_i} = 0, \quad x_i \in GF(q), \tag{4.1}$$

i.e.

$$\left(\sum_{i=1}^{2e_2} x_i p_{1j_i}^{\tau_i} A_{\lambda_i}, \sum_{i=1}^{2e_2} x_i p_{2j_i}^{\tau_i} A_{\lambda_i}, ..., \sum_{i=1}^{2e_2} x_i p_{\rho j_i}^{\tau_i} A_{\lambda_i} \right) = 0$$

Since C_2 corrects e_2 or fewer errors within a sub-block of length s, all sets of $2e_2$ or fewer syndromes resulting from detectable errors within a sub-block are linearly independent. So, the equation (4.1) has only trivial solution $x_i = 0$ for $i = 1, 2, \ldots, 2e_2$. This shows that condition (4) is true.

To show that the code C satisfies condition (5), suppose S^{β_i} is the syndrome corresponding to the β_i^{th} sub-block of length t within a cluster-sub-block of length s, where $\beta_i = \tau_2 s + f_i,\ i = 1, 2, ..., e_2$.

Let

$$\sum_{i=1}^{e_2} x_i S^{\lambda_i} + \sum_{i=1}^{e_2} y_i S^{\beta_i} = 0, \quad x_i, y_i \in GF(q) \tag{4.2}$$

i.e.

$$\left(\sum_{i=1}^{e_2} x_i p_{1j_i}^{\tau_1} A_{\lambda_i} + \sum_{i=1}^{e_2} y_i p_{1f_i}^{\tau_2} A_{\beta_i}, \sum_{i=1}^{e_2} x_i p_{2j_i}^{\tau_1} A_{\lambda_i} + \sum_{i=1}^{e_2} y_i p_{2f_i}^{\tau_2} A_{\beta_i}, ... \right.$$
$$\left. ..., \sum_{i=1}^{e_2} x_i p_{\rho j_i}^{\tau_1} A_{\lambda_i} + \sum_{i=1}^{e_2} y_i p_{\rho f_i}^{\tau_2} A_{\beta_i} \right) = 0.$$

Then

$$\sum_{i=1}^{e_2} x_i p_{rj_i}^{\tau_1} A_{\lambda_i} + \sum_{i=1}^{e_2} y_i p_{rf_i}^{\tau_2} A_{\beta_i} = 0,\ r = 1, 2, \ldots, \rho.$$

The code C_2 corrects e_2 or fewer errors within a sub-block of length s, so the syndromes of such errors within a sub-block are distinct from syndromes resulting from such errors within any other sub-block. Therefore, the equation (4.2) gives only trivial solution $x_i = 0$ and $y_i = 0$ for $i = 1, 2, \ldots, e_2$. Thus, condition (5) is satisfied. Hence the theorem is proved. □

Example 4.3. Consider a $(9 = 3 \times 3, 6)$ binary code that detects 3 or less errors within a sub-block of length 3 whose parity check matrix H is given by

$$H = \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

and the $(12 = 4 \times 3, 5)$ binary code that corrects 2 or less errors within a sub-block of length 4 whose parity check matrix P is given in Example 3.5. Then the blockwise-tensor product matrix $P \otimes_b H$ is given as follows.

$$P \otimes_b H = \left[\begin{array}{cccc|cccc|cccc} 100 & 000 & 000 & 100 & 000 & 000 & 000 & 000 & 100 & 000 & 000 & 000 \\ 010 & 000 & 000 & 010 & 000 & 000 & 000 & 000 & 010 & 000 & 000 & 000 \\ 001 & 000 & 000 & 001 & 000 & 000 & 000 & 000 & 001 & 000 & 000 & 000 \\ 000 & 100 & 000 & 100 & 000 & 000 & 000 & 000 & 000 & 100 & 000 & 000 \\ 000 & 010 & 000 & 010 & 000 & 000 & 000 & 000 & 000 & 010 & 000 & 000 \\ 000 & 001 & 000 & 001 & 000 & 000 & 000 & 000 & 000 & 001 & 000 & 000 \\ 000 & 000 & 100 & 100 & 000 & 000 & 000 & 000 & 000 & 000 & 100 & 000 \\ 000 & 000 & 010 & 010 & 000 & 000 & 000 & 000 & 000 & 000 & 010 & 000 \\ 000 & 000 & 001 & 001 & 000 & 000 & 000 & 000 & 000 & 000 & 001 & 000 \\ 000 & 000 & 000 & 100 & 000 & 000 & 000 & 100 & 000 & 000 & 000 & 100 \\ 000 & 000 & 000 & 010 & 000 & 000 & 000 & 010 & 000 & 000 & 000 & 010 \\ 000 & 000 & 000 & 001 & 000 & 000 & 000 & 001 & 000 & 000 & 000 & 001 \\ 000 & 000 & 000 & 000 & 100 & 000 & 000 & 100 & 000 & 000 & 100 & 000 \\ 000 & 000 & 000 & 000 & 010 & 000 & 000 & 010 & 000 & 000 & 010 & 000 \\ 000 & 000 & 000 & 000 & 001 & 000 & 000 & 001 & 000 & 000 & 001 & 000 \\ 000 & 000 & 000 & 000 & 000 & 100 & 000 & 100 & 000 & 100 & 000 & 000 \\ 000 & 000 & 000 & 000 & 000 & 010 & 000 & 010 & 000 & 010 & 000 & 000 \\ 000 & 000 & 000 & 000 & 000 & 001 & 000 & 001 & 000 & 001 & 000 & 000 \\ 000 & 000 & 000 & 000 & 000 & 000 & 100 & 100 & 100 & 000 & 000 & 000 \\ 000 & 000 & 000 & 000 & 000 & 000 & 010 & 010 & 010 & 000 & 000 & 000 \\ 000 & 000 & 000 & 000 & 000 & 000 & 001 & 001 & 001 & 000 & 000 & 000 \end{array}\right]$$

By Theorem 4.2, the null space of this matrix, i.e., the (36, 15) code not only detects 3 or less errors within a sub-block of length 3, but also locates any 2 or less sub-blocks within a cluster-sub-block of length 12. It can be verified from error pattern-syndrome table (as done in Example 3.5) that:
(i) the syndromes of 3 or less errors within a cluster-sub-block of length 3 are nonzero,
(ii) the syndromes of 2 or less such corrupted sub-blocks within a cluster-sub-block of length 12 are all distinct.

The following two classes of EL-codes can be obtained by modifying the nature of the code C_2 in Theorem 4.2.

Theorem 4.4. *Let $C_1(n_1 = mt, n_1 - k)$ be a linear code over $GF(q)$ with parity check matrix H that detects e_1 or fewer errors within a sub-block of length t and $C_2(n_2 = ms, n_2 - \rho)$ be a linear code over $GF(q)$ with parity check matrix P that corrects e_2 or fewer errors. Then the $(mts, mts - k\rho)$ code obtained from the*

parity check matrix $P \otimes_b H$ can detect e_1 or fewer errors within a sub-block of length t and locate any e_2 or fewer such corrupted sub-blocks.

Proof. Similar to proof of Theorem 4.2. □

Theorem 4.5. *Let $C_1(n_1 = mt, n_1 - k)$ be a linear code over $GF(q)$ with parity check matrix H that detects e_1 or fewer errors within a sub-block of length t and $C_2(n_2 = ms, n_2 - \rho)$ be a linear code over $GF(q)$ with parity check matrix P that corrects any burst of length b or less. Then the $(mts, mts - k\rho)$ code obtained from the parity check matrix $P \otimes_b H$ can detect e_1 or fewer errors within a sub-block of length t and locate any adjacent b or fewer such corrupted sub-blocks.*

Proof. Similar to proof of Theorem 4.2. □

To conclude the section, we observe that distance of a code obtained by blockwise-tensor product of the generator matrices of two blockwise linear codes coincides with Theorem 5.3 given by Peterson and Weldon [11]. This is given in the following theorem.

Theorem 4.6. *The distance of the linear code whose generator matrix is the blockwise-tensor product of generator matrices of the two blockwise linear codes, is the product of the distances of the two codes.*

Proof. Consider the linear code C_1 has distance $D = d_1 + d_2 + ... + d_m$ with generator matrix G_1 and the other linear code C_2 has distance $L = l_1 + l_2 + ... + l_m$ with generator matrix G_2. Then, a codevector of the code whose generator matrix is blockwise-tensor product $G_1 \otimes_b G_2$, must have weight at least

$$d_1(l_1 + l_2 + \cdots + l_m) + d_2(l_1 + l_2 + \cdots + l_m) + \cdots + d_m(l_1 + l_2 + \cdots + l_m) = DL.$$

Since D and L are the distances of the codes C_1 and C_2 respectively, there exists a codevector of weight D in C_1 and a codevector of weight L in C_2. As a result, there exist a codevector of weight DL in the code whose generator matrix is $G_1 \otimes_b G_2$. This completes the proof. □

5. Comparative Study

In the section, we make comparison between bounds on the number of parity check digits for codes obtained in Section 3 and Section 4 with the following bounds which already exist in literature.

Theorem 5.1. [11] *For any (n, k) linear code that corrects e or less errors satisfies the following inequality*

$$q^{n-k} \geq 1 + \sum_{i=1}^{e} \binom{n}{i} (q-1)^i.$$

Theorem 5.2. [11] *There exists an (n, k) linear code that corrects e or less errors provided that*

$$q^{n-k} > 1 + \sum_{i=1}^{2e-1} \binom{n}{i} (q-1)^i.$$

Theorem 5.3. [13] *Let $C_1(n_1 = mt, n_1 - k)$ be a binary linear code with parity check matrix H that detects e_1 or fewer errors within a sub-block of length t and $C_2(n_2 = ms, n_2 - \rho)$ be a nonbinary linear code over $GF(2^k)$ with parity check matrix P that corrects e_2 or less random errors (or any burst of length b or less). Then, the $(n_1n_2, n_1n_2 - k\rho)$ code obtained from the parity check matrix $P \otimes H$ can detect e_1 or fewer errors within a sub-block of length n_1 and locate any e_2 or fewer such corrupted sub-blocks (or any adjacent b or fewer such corrupted sub-blocks).*

Comparison (I): First we compare bounds on the number of parity check digits for the existence of linear codes discussed in Theorem 3.1 (Theorem 3.3) and Theorem 5.1 (Theorem 5.2). Since for $m \neq 1$, we have

$$\sum_{i=1}^{e} \binom{n}{i} (q-1)^i > m \sum_{i=1}^{e} \binom{t}{i} (q-1)^i$$

and

$$\sum_{i=1}^{2e-1} \binom{n}{i} (q-1)^i > \sum_{i=1}^{2e-1} \binom{t-1}{i} (q-1)^i + (m-1)\left\{1 + \sum_{i=1}^{e-1} \binom{t-1}{i} (q-1)^i\right\} \sum_{i=1}^{e} \binom{t}{i} (q-1)^i.$$

So, the necessary (sufficient) number of parity check digits for the existence of linear codes considered in Theorem 3.1 (Theorem 3.3) is less than the necessary (sufficient) number of parity check digits for (n, k) codes considered in Theorem 5.1 (Theorem 5.2).

Outcome: If errors are occurred blockwise, the codes discussed in Theorem 3.1 and Theorem 3.3 will be more efficient than the codes discussed in Theorem 5.1 and Theorem 5.2, respectively.

Comparison (II): We now compare the codes obtained in Theorem 4.4 and Theorem 4.5 with the codes mentioned in Theorem 5.3. In Theorem 4.4 and Theorem 4.5, we obtain linear codes that detect e_1 or fewer errors within a sub-block of length t and locate any e_2 or fewer such corrupted sub-blocks (or any adjacent b or fewer such corrupted sub-blocks) whereas in Theorem 5.3, there exists linear codes that can detect e_1 or fewer errors within a sub-block of length n_1 and locate any e_2 or fewer such corrupted sub-blocks (or any adjacent b or fewer such corrupted sub-blocks). In Theorem 5.3, the identified sub-blocks are of length n_1 whereas in our results (Theorem 4.4 and Theorem 4.5), the identified sub-blocks are of length t which may be made much smaller that n_1.

Outcome: The sender needs to retransmit smaller sub-block(s) instead of bigger sub-block(s) of length n_1. In addition, the rate of transmission in our results is also improved, this is because

$$1 - \frac{k\rho}{m^2ts} > 1 - \frac{k\rho}{mts} \text{ for } m \neq 1,$$

i.e., rate of transmission of the codes obtained in Theorem 4.4 and Theorem 4.5 is better than rate of transmission of the codes given in Theorem 5.3.

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TRACES OF HADAMARD AND KRONECKER PRODUCTS OF MATRICES

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Abstract. We present some inequality/equality for traces of Hadamard product and Kronecker product of matrices. Some numerical examples are given to support the results.

1. Introduction

The Hadamard (or Schur) and Kronecker products are widely studied and applied in matrix theory, statistics, system theory and other areas [2]. It was Schur who initially studied algebraic and analytic properties of Hadamard product. In 1990, Horn [1] presents a widespread information focusing on the Hadamard product. Magnus and Neudecker [4] give some basic results and statistical applications involving Hadamard or Kronecker products. For basics on these two matrix products, one may refer to [3,5–7]. In [6], the authors investigated traces of Hadamard and Kronecker products of matrices and obtained some inequalities for traces of products of matrices. In this note, we present some inequality for traces of Hadamard product, Kronecker product and mixed type product of matrices.

We recall the basic definitions and notations to make our presentation self-contained. The set of all positive real numbers is denoted by $\mathbb{R}^+$. By M_n we denote the family of n-by-n matrices over the real field $\mathbb{R}$. For $A = [a_{ij}] \in M_n$, the scalar $\sum_{i=1}^n a_{ii}$ is called the *trace* of A and is usually denoted by $\mathrm{tr}(A)$ or $\mathrm{trace}A$. The *Hadamard product* of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of identical size is just their element-wise product which is given by

$$A \circ B = [a_{ij}b_{ij}]_{m\times n}.$$

Let $A = [a_{ij}]$ be an m-by-n matrix and let $B = [b_{ij}]$ be a p-by-q matrix. The *Kronecker product* of A and B is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix}$$

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Lemma 1.1. [6] *Let $A \in M_n$ and $B \in M_m$. Then,*

$$tr(A \otimes B) = tr(B \otimes A) = tr(A)tr(B).$$

Theorem 1.2. [6] *If $A = [a_{ij}]$, $B = [b_{ij}] \in M_n$, then*

(1) $tr(A \otimes B) = n\ tr(A \circ B) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (a_{ii} - a_{jj})(b_{ii} - b_{jj})$.

(2) $tr(A \circ B) \leq tr\left(\left(\frac{A+B}{2}\right) \circ \left(\frac{A+B}{2}\right) \right)$.

2. Main results

We start with an inequality involving the trace of m times Hadamard product of a matrix and the trace of the given matrix .

Proposition 2.1. *Let $A \in M_n$ with positive real diagonal. Then,*

$$tr\Big(\underbrace{A \circ A \circ A \circ \cdots \circ A}_{m} \Big) \leq (tr(A))^m.$$

Proof. Let $A = [a_{ij}] \in M_n$ and let $a_{ii} \in \mathbb{R}^+$. Then, by the definitions of Hadamard product and trace of matrices, we have

$$tr\Big(\underbrace{A \circ A \circ A \circ \cdots \circ A}_{m} \Big) = \sum_{i=1}^{n} a_{ii}^m \leq \left(\sum_{i=1}^{m} a_{ii} \right)^m = (tr(A))^m.$$

The proposition is proved. □

Proposition 2.2. *If $A = [a_{ij}], B = [b_{ij}] \in M_n$, then*

$$n\ tr\Big((\underbrace{A \circ A \circ \cdots \circ A}_{n}) \circ (\underbrace{B \circ B \circ \cdots \circ B}_{n}) \Big) = \sum_{i=1}^{n} a_{ii}^n \sum_{i=1}^{n} b_{ii}^n - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (a_{ii}^n - a_{jj}^n)(b_{ii}^n - b_{jj}^n).$$

Proof. Since

$$tr\Big((\underbrace{A \circ A \circ \cdots \circ A}_{n}) \otimes (\underbrace{B \circ B \circ \cdots \circ B}_{n}) \Big) = \sum_{i=1}^{n} a_{ii}^n \sum_{i=1}^{n} b_{ii}^n,$$

applying Theorem 1 in [6], we get

$$n\ tr\Big((\underbrace{A \circ A \circ \cdots \circ A}_{n}) \circ (\underbrace{B \circ B \circ \cdots \circ B}_{n}) \Big) = \sum_{i=1}^{n} a_{ii}^n \sum_{i=1}^{n} b_{ii}^n - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (a_{ii}^n - a_{jj}^n)(b_{ii}^n - b_{jj}^n).$$

□

The following proposition provides a relation between the trace of a matrix and Kronecker product of the given matrix.

Proposition 2.3. *For any $A \in M_n$, we have*

$$tr\Big(\underbrace{A \otimes A \otimes A \otimes \cdots \otimes A}_{m}\Big) = (tr(A))^m.$$

Proof. We prove the proposition by induction. By Lemma 1.1, the equation is true for $n = 2$. Assume that it is true $n = k$, i.e.,

$$tr\Big(\underbrace{A \otimes A \otimes A \otimes \cdots \otimes A}_{k}\Big) = (tr(A))^k. \tag{2.1}$$

Again using Lemma 1.1, we compute

$$\begin{aligned} tr\Big(\underbrace{A \otimes A \otimes A \otimes \cdots \otimes A}_{k+1}\Big) &= tr\Big(\underbrace{A \otimes A \otimes A \otimes \cdots \otimes A}_{k}\Big) tr(A) \\ &= (tr(A))^k tr(A) \qquad \text{(by (2.1))} \\ &= (tr(A))^{k+1}. \end{aligned}$$

The proposition is proved. □

Example 2.4. Let $A = \begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}$.

Then,

$$A \circ A = \begin{bmatrix} 9 & 1 \\ 16 & 1 \end{bmatrix}, \quad A \otimes A \otimes A = \begin{bmatrix} 27 & 9 & 9 & 3 & 9 & 3 & 3 & 1 \\ 36 & 9 & 12 & 3 & 12 & 3 & 4 & 1 \\ 36 & 12 & 9 & 3 & 12 & 4 & 3 & 1 \\ 48 & 12 & 12 & 3 & 16 & 4 & 4 & 1 \\ 36 & 12 & 12 & 4 & 9 & 3 & 3 & 1 \\ 48 & 12 & 16 & 4 & 12 & 3 & 4 & 1 \\ 48 & 16 & 12 & 4 & 12 & 4 & 3 & 1 \\ 64 & 16 & 16 & 4 & 16 & 4 & 4 & 1 \end{bmatrix}$$

and $tr(A) = 4$. Now we have $tr(A \circ A) = 10 \leq 4^2 = (tr(A))^3$. So, Proposition 2.1 is true. Also, $tr(A \otimes A \otimes A) = 64 = (tr(A))^3$. This verifies Proposition 2.2.

Proposition 2.5. *For any $A \in M_n$, we have*

$$tr\Big((A + A^t) \circ (A + A^t)\Big) = 4\ tr(A \circ A).$$

Proof. Let $A = [a_{ij}] \in M_n$. Then

$$tr\Big((A + A^t) \circ (A + A^t)\Big) = \sum_{i=1}^{n} (a_{ii} + a_{ii})(a_{ii} + a_{ii}) = 4 \sum_{i=1}^{n} a_{ii}^2 = 4\ tr(A \circ A).$$

The proposition is proved. □

Proposition 2.6. *For any $A, B \in M_n$, we have*

$$tr\Big((A + B) \circ (A - B)\Big) = tr(A \circ A) - tr(B \circ B).$$

Proof. Let $A = [a_{ij}],\ B = [b_{ij}] \in M_n$ be arbitrary.
We compute

$$\begin{aligned} tr\Big((A+B)\circ(A-B)\Big) &= \sum_{i=1}^{n}(a_{ii}+b_{ii})(a_{ii}-b_{ii}) \\ &= \sum_{i=1}^{n}\Big((a_{ii})^2-(b_{ii})^2\Big) \\ &= tr(A\circ A)-tr(B\circ B). \end{aligned}$$

□

Proposition 2.7. *For any $A, B \in M_n$, we have*

$$tr\Big((A+B)\circ(A+B)\Big) = tr(A\circ A)+2tr(A\circ B)+tr(B\circ B).$$

Proof. Similar to proof of Proposition 2.6. □

The following proposition gives the relationship between trace of a matrix obtained as Kronecker product of a finite sum of matrices and the traces of the matrices. We prove the result for two matrices.

Proposition 2.8. *Let $A, B \in M_n$. Then*

$$tr\Big(\underbrace{(A+B)\otimes(A+B)\otimes(A+B)\otimes\cdots\otimes(A+B)}_{k}\Big) = \sum_{i=1}^{k}\binom{k}{i}\Big(tr(A)\Big)^{k-i}\Big(tr(B)\Big)^{i}.$$

Proof. Using Proposition 2.3, we compute

$$\begin{aligned} &tr\Big(\underbrace{(A+B)\otimes(A+B)\otimes(A+B)\otimes\cdots\otimes(A+B)}_{k}\Big) \\ =&\Big(tr(A+B)\Big)^{k} = \Big(tr(A)+tr(B)\Big)^{k} \\ =&\sum_{i=1}^{k}\binom{k}{i}\Big(tr(A)\Big)^{k-i}\Big(tr(B)\Big)^{i}. \end{aligned}$$

The result is proved. □

Remark 2.9. Using Proposition 2.1, for any $A, B \in M_n$ with positive diagonal, we can show that

$$tr\Big(\underbrace{(A+B)\circ(A+B)\circ(A+B)\circ\cdots\circ(A+B)}_{n}\Big) \le \sum_{i=1}^{n}\binom{n}{i}\Big(tr(A)\Big)^{n-i}\Big(tr(B)\Big)^{i}.$$

The next proposition gives a trace inequality for the Hadamard product of matrix sums.

Proposition 2.10. *If $A, B \in M_n$, then*

$$(tr(A \circ B))^2 \leq \frac{1}{16}\, tr\Big((A+B)\circ(A+B)\circ(A+B)\circ(A+B)\Big).$$

Proof. Using Theorem 1.2 and Proposition 2.3, we compute

$$\begin{aligned}(tr(A \circ B))^2 &\leq tr\left(\left(\frac{A+B}{2}\right)\circ\left(\frac{A+B}{2}\right)\right) tr\left(\left(\frac{A+B}{2}\right)\circ\left(\frac{A+B}{2}\right)\right)\\ &= tr\left(\left(\frac{A+B}{2}\right)\circ\left(\frac{A+B}{2}\right)\circ\left(\frac{A+B}{2}\right)\circ\left(\frac{A+B}{2}\right)\right)\\ &= \frac{1}{16}\, tr\Big((A+B)\circ(A+B)\circ(A+B)\circ(A+B)\Big).\end{aligned}$$

The proposition is proved. □

The following theorem provides a relation between the trace of matrices generated by Kronecker and Hadamard product of matrices.

Theorem 2.11. *If $A, B, C, D \in M_n$, then*

(1) $(A \otimes B) \circ (C \otimes D) = (A \circ C) \otimes (B \circ D)$.

(2) $tr(A \otimes (B \circ C)) = n\, tr(A \circ B \circ C) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (a_{ii} - a_{jj})(b_{ii}c_{ii} - b_{jj}c_{jj})$.

Proof. Let us write $A = [a_{ij}]$ and $C = [c_{ij}]$. Then, we compute

$$\begin{aligned}&(A \otimes B) \circ (C \otimes D)\\ &= \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \dots & a_{nn}B \end{bmatrix} \circ \begin{bmatrix} c_{11}D & c_{12}D & \dots & c_{1n}D \\ c_{21}D & c_{22}D & \dots & c_{2n}D \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1}D & c_{n2}D & \dots & c_{nn}D \end{bmatrix}\\ &= \begin{bmatrix} a_{11}c_{11}B \circ D & a_{12}c_{12}B \circ D & \dots & a_{1n}c_{1n}B \circ D \\ a_{21}c_{21}B \circ D & a_{22}c_{22}B \circ D & \dots & a_{2n}c_{2n}B \circ D \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}c_{n1}B \circ D & a_{n2}c_{n2}B \circ D & \dots & a_{nn}c_{nn}B \circ D \end{bmatrix} = (A \circ C) \otimes (B \circ D).\end{aligned}$$

Hence, (1) is proved.

Let us write $B = [b_{ij}]$. Then, by Theorem 1.2, we have

$$\begin{aligned}tr(A \otimes (B \circ C)) &= n\, tr(A \circ (B \circ C)) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (a_{ii} - a_{jj})(b_{ii}c_{ii} - b_{jj}c_{jj})\\ &= n\, tr(A \circ B \circ C) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (a_{ii} - a_{jj})(b_{ii}c_{ii} - b_{jj}c_{jj}).\end{aligned}$$

Thus, (2) is proved. □

Corollary 2.12. *If $A, B, C, D \in M_n$, then*

$$tr\Big((A \otimes B) \circ (C \otimes D)\Big) = n\ tr(A \circ B \circ C \circ D) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (a_{ii}c_{ii} - a_{jj}c_{jj})(b_{ii}d_{ii} - b_{jj}d_{jj}).$$

Indeed, by Theorem 2.11, we can write

$$tr\Big((A \otimes B) \circ (C \otimes D)\Big) = tr\Big((A \circ C) \otimes (B \circ D)\Big).$$

Thus, by Theorem 1.2*, we have*

$$tr\Big((A \otimes B) \circ (C \otimes D)\Big) = n\ tr(A \circ C \circ B \circ D) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (a_{ii}c_{ii} - a_{jj}c_{jj})(b_{ii}d_{ii} - b_{jj}d_{jj}).$$

Remark 2.13. If $A, C \in M_n$ and $B, D \in M_k$ $(k \neq n)$, then the equations given in Theorem 2.11 hold too.

Example 2.14. Let $A = \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -1 \\ 3 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} -2 & 1 \\ -1 & -3 \end{bmatrix}$.
Then

$$A \otimes B = \begin{pmatrix} 12 & -3 & 4 & -1 \\ 9 & 6 & 3 & 2 \\ 16 & -4 & -4 & 1 \\ 12 & 8 & -3 & -2 \end{pmatrix}, \quad C \otimes D = \begin{pmatrix} -4 & 2 & -6 & 3 \\ -2 & -6 & -3 & -9 \\ -2 & 1 & 2 & -1 \\ -1 & -3 & 1 & 3 \end{pmatrix};$$

$$A \circ C = \begin{bmatrix} 6 & 3 \\ 4 & 1 \end{bmatrix}, \quad B \circ D = \begin{bmatrix} -8 & -1 \\ -3 & -6 \end{bmatrix};$$

and

$$A \otimes (B \circ D) = \begin{pmatrix} -24 & -3 & -8 & -1 \\ -9 & -18 & -3 & -6 \\ -32 & -4 & 8 & 1 \\ -12 & -24 & 3 & 6 \end{pmatrix}, \quad A \circ B \circ D = \begin{bmatrix} -24 & -1 \\ -12 & 6 \end{bmatrix}.$$

It is easy to see that

$$(A \otimes B) \circ (C \otimes D) = \begin{pmatrix} -48 & -6 & -24 & -3 \\ -18 & -36 & -9 & -18 \\ -32 & -4 & -8 & -1 \\ -12 & -24 & -3 & -6 \end{pmatrix} = (A \circ C) \otimes (B \circ D).$$

We compute

$$n\ tr(A \circ B \circ D) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (a_{ii} - a_{jj})(b_{ii}d_{ii} - b_{jj}d_{jj})$$

$$= 2(-18) - (-8) \text{ (here } n = 2) = -28 = tr\ (A \otimes (B \circ D)).$$

Thus, Theorem 2.11 is verified.

To conclude the paper, we give a result that connects the traces of matrices obtained by Kronecker and Hadamard product of matrices in terms of the trace of Hadamard product of matrices.

Theorem 2.15. *If $A_i, B_i \in M_n$ for $1 \leq i \leq n$, then*

$$tr\Big((A_1 \otimes A_2 \otimes \cdots \otimes A_n) \circ (B_1 \otimes B_2 \otimes \cdots \otimes B_n)\Big) = \prod_{i=1}^{n} tr(A_i \circ B_i).$$

Proof. We prove the theorem by induction. Clearly, the equation is true for $n = 1$. Assume that it is true for $n = k$, i.e.,

$$tr\Big((A_1 \otimes A_2 \otimes \cdots \otimes A_k) \circ (B_1 \otimes B_2 \otimes \cdots \otimes B_k)\Big) = \prod_{i=1}^{k} tr(A_i \circ B_i).$$

We compute

$$\begin{aligned}
&tr\Big((A_1 \otimes A_2 \otimes \cdots \otimes A_{k+1}) \circ (B_1 \otimes B_2 \otimes \cdots \otimes B_{k+1})\Big) \\
&= tr\Big\{\Big((A_1 \otimes A_2 \otimes \cdots \otimes A_k) \circ (B_1 \otimes B_2 \otimes \cdots \otimes B_k)\Big) \otimes (A_{k+1} \circ B_{k+1})\Big\} \\
&\quad \text{(by Theorem 2.11)} \\
&= tr\Big((A_1 \otimes A_2 \otimes \cdots \otimes A_k) \circ (B_1 \otimes B_2 \otimes \cdots \otimes B_k)\Big) tr(A_{k+1} \circ B_{k+1}) \\
&\quad \text{(by Lemma 1.1)} \\
&= \prod_{i=1}^{k} tr(A_i \circ B_i) tr(A_{k+1} \circ B_{k+1}) \\
&= \prod_{i=1}^{k+1} tr(A_i \circ B_i).
\end{aligned}$$

Thus, the equation is true for $n = k + 1$. The theorem is proved. □

Remark 2.16. Theorem 2.15 is useful because the computation of $A_i \circ B_i$ $(1 \leq i \leq n)$ is much easier than $(A_1 \otimes A_2 \otimes \cdots \otimes A_n) \circ (B_1 \otimes B_2 \otimes \cdots \otimes B_n)$.

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ON THE NUMBER OF SOLUTIONS OF A GENERALIZED COMMUTATOR EQUATION IN FINITE GROUPS

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Abstract. We consider some generalized commutator equations in a finite group and show that the number of solutions of such equations are characters of that group. We also obtain explicit formula for this character, considering the equation $[\dots[[[x_1,x_2],x_3],\ x_4],\dots,x_n]=g$, for some well-known classes of finite groups in terms of orders of the group, its center and its commutator subgroup.

1. Introduction

Let $F(x_1,x_2,\dots,x_n)$ denote the free group on n generators $x_1,x_2,\dots,x_n$. Let $w=w(x_1,\cdots,x_n)$ be a non-trivial reduced word in $F(x_1,x_2,\dots,x_n)$. For any finite group G the word $w(x_1,x_2,\dots,x_n)$ defines a map $w\colon G^n\to G$ given by $(g_1,g_2,\dots,g_n)\mapsto w(g_1,g_2,\dots,g_n)$ called a word map induced by the word $w(x_1,x_2,\dots,x_n)$. For $g\in G$ we have

$$|w^{-1}(g)|=\big|\big\{(a_1,a_2,\dots,a_n)\in G^n : w(a_1,a_2,\dots,a_n)=g\big\}\big|.$$

Consider the map $\zeta_G^w\colon G\to\mathbb{N}\cup\{0\}$ given by $g\mapsto|w^{-1}(g)|$. Note that $\zeta_G^w(g)$ is the number of solutions of the equation $w(x_1,x_2,\dots,x_n)=g$. For any word w, the map ζ_G^w is a class function on G but not necessarily a character. Equations in finite groups have been studied by many authors since the days of Frobenius (see for example [2,5,22,27,28,30,31] etc.). If

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$w(x_1, x_2) = [x_1, x_2] := x_1^{-1}x_2^{-1}x_1x_2$ then Frobenius [10] showed that

$$\zeta_G^w = \sum_{\chi \in \mathrm{Irr}(G)} \frac{|G|}{\chi(1)}\chi, \tag{1.1}$$

which is a character of G. This result has been used as one of the main ingredients to settle down the longstanding Ore's Conjecture (see [16]). It is also used in order to study generalized commutativity degrees of finite groups (see [6,8,24]). In [32], Tambour derived explicit formulas for ζ_G^w considering $w(x_1, x_2, \ldots, x_n) = [x_1, x_2] \cdots [x_{n-1}, x_n]$ and $w(x_1, x_2, \ldots, x_n) = x_1x_2 \cdots x_nx_1^{-1}x_2^{-1} \cdots x_n^{-1}$. In particular, he deduced that ζ_G^w is a character of G for these words. Extending the results of Tambour, Das and Nath [5] showed that ζ_G^w is a character of G if w is an admissible word. Further, Parzanchevski and Schul [22] have considered some classes of words and extended the results of Das and Nath [5]. In Section 2 of this paper, we consider the word $w(x_1, x_2, \ldots, x_m) = [w_1(x_1, \ldots, x_n), w_2(x_{n+1}, \ldots, x_m)]$ and show that ζ_G^w is a character of G under some condition. In particular, we prove the following result.

THEOREM 1.1. *Let G be a finite group and*

$$w(x_1, \ldots, x_m) := \big[w_1(x_1, \ldots, x_n), w_2(x_{n+1}, \ldots, x_m)\big].$$

If $\zeta_G^{w_1}$ is a character of G and $\zeta_G^{w_2}$ is a constant map then ζ_G^w is a character of G and

$$\zeta_G^w = \sum_{\chi \in \mathrm{Irr}(G)} \frac{|G|^{m-n}}{\chi(1)} \langle \zeta_G^{w_1}\chi, \chi\rangle\, \chi.$$

This extends a classical result of Frobenius mentioned above. Note that the word $w(x_1, x_2, \ldots, x_m) = [w_1(x_1, \ldots, x_n), w_2(x_{n+1}, \ldots, x_m)]$ also includes words that are not considered in [22]. In Section 3, we consider the word $w_n(x_1, \ldots, x_n) = [\ldots[[[x_1, x_2], x_3], x_4], \ldots, x_n] = [w_{n-1}(x_1, \ldots, x_{n-1}), x_n]$ for any n and using Theorem 1.1 we obtain exact formulas for $\zeta_G^{w_n}$ in case of nilpotent Camina groups and its generalizations. In the next paragraph we recall the definition and the main properties of these groups.

Two groups G and H are said to be n-isoclinic, if there exist isomorphisms ϕ and ψ such that

(1) ϕ is an isomorphism from $G/Z_n(G)$ to $H/Z_n(H)$;
(2) ψ is an isomorphism from $\gamma_{n+1}(G)$ to $\gamma_{n+1}(H)$;
(3) ψ is compatible with ϕ, i.e.

$$\psi([\ldots[[g_1, g_2], g_3], \ldots, g_{n+1}]) = \big[\ldots[[h_1, h_2], h_3], \ldots, h_{n+1}\big],$$

for any $h_i \in \phi(g_i Z_n(G))$, $g_i \in G$, $1 \leq i \leq n+1$. In other words, we have the commutative diagram

$$\begin{array}{ccc} \frac{G}{Z_n(G)} \times \cdots \times \frac{G}{Z_n(G)} & \xrightarrow{\phi \times \cdots \times \phi} & \frac{H}{Z_n(H)} \times \cdots \times \frac{H}{Z_n(H)} \\ \downarrow a_{(n+1,G)} & & \downarrow a_{(n+1,H)} \\ \gamma_{n+1}(G) & \xrightarrow{\psi} & \gamma_{n+1}(H) \end{array}$$

where

$$a_{(n+1,G)}\big(\big(g_1 Z_n(G), \ldots, g_{n+1} Z_n(G)\big)\big) = \big[\ldots[g_1, g_2], \ldots, g_{n+1}\big]$$

and

$$a_{(n+1,H)}\big(\big(h_1 Z_n(H), \ldots, h_{n+1} Z_n(H)\big)\big) = \big[\ldots[h_1, h_2], \ldots, h_{n+1}\big].$$

Such a pair (ϕ, ψ) is called an n-isoclinism between G and H. If $n = 1$, then (ϕ, ψ) is called an isoclinism between G and H and the groups are called isoclinic. The notion of isoclinism of groups was introduced by P. Hall [9]. In Section 3.1, we prove the following generalization of [24, Lemma 3.5] and derive an explicit formula for $\zeta_G^{w_n}(g)$ if G is a finite nilpotent generalized Camina group.

THEOREM 1.2. *Let G and H be two finite groups and (ϕ, ψ) be an n-isoclinism from G to H. If $g \in \gamma_{n+1}(G)$, then*

$$\zeta_G^{w_{n+1}}(g) = (|G|/|H|)^{n+1} \zeta_H^{w_{n+1}}(\psi(g)).$$

A group G is called a *Camina group* if $gG' = \mathrm{Cl}(g)$ for all $g \in G \setminus G'$. Note that the fact $gG' = \mathrm{Cl}(g)$ for all $g \in G \setminus G'$ is equivalent with the fact $\chi(g) = 0$ for all $\chi \in \mathrm{nl}(G)$ and $g \notin G'$, where $\mathrm{nl}(G)$ is the set of all nonlinear irreducible characters of G. A pair (G, N) is said to be a *Camina pair* if N is a proper non-trivial normal subgroup of G such that $gN \subseteq \mathrm{Cl}(g)$ for all $g \in G \setminus N$. We remark that $Z(G) < N < G'$ if (G, N) is a Camina pair and G is a Camina group if and only if (G, G') is a Camina pair. A group G is said to be a *generalized Camina group* if $gG' = \mathrm{Cl}(g)$ for all $g \in G \setminus G'Z(G)$. A pair (G, N) is called a *generalized Camina pair* (abbreviated as GCP) if N is a normal subgroup of the group G and $gG' = \mathrm{Cl}(g)$ for all $g \notin N$ equivalently $\chi(g) = 0$ for all $\chi \in \mathrm{nl}(G)$ and $g \notin N$ (see [12]). Note that G is a generalized Camina group if and only if $(G, G'Z(G))$ is a generalized Camina pair. The groups with $(G, Z(G))$, a GCP were studied under the name of VZ-groups by Lewis in [13]. In Section 3.2, we derive an explicit formula for $\zeta_G^{w_n}(g)$ considering groups for which $(G, Z(G))$ is a Camina pair and $(G/Z(G), Z(G/Z(G)))$ is a generalized Camina pair. In the last section,

we derive an explicit formula for $\zeta_G^{w_n}(g)$ for groups having unique nonlinear irreducible character.

Let $P_w^{(n)}$ be the probability distribution associated to the word map w induced by $w(x_1, x_2, \ldots, x_n)$ on G. Then $P_w^{(n)}(g) = \frac{\zeta_G^w(g)}{|G|^n}$. In recent years, many authors have studied the probability distribution associated to the word maps induced by different kinds of words (see [1,7,18,19,21] etc.). As an application of our results, one may obtain the probability distribution associated to the word map induced by the word $w_n(x_1, x_2, \ldots, x_n) = [[\ldots[[x_1, x_2], x_3]\ldots], x_n]$ for the different families of finite groups considered in Section 3. It may be mentioned here that $P_{w_n}^{(n)}(g)$ is also considered in [8] (see Theorem 7.4). The case when $g = 1$, the identity element of G, is studied by Moghaddam et al. [17] in the name of nth nilpotency degree of G. If $n = 2$ then we have $P_{w_n}^{(2)}(g) = P_g(G)$, a notion introduced and studied by Pournaki and Sobhani [24].

Throughout this paper G denotes a finite non-abelian group, $\mathrm{Irr}(G)$ and $\mathrm{lin}(G)$ denote the set of all irreducible characters and linear irreducible characters of G, respectively. We write $\mathrm{cd}(G)$ to denote the set of all irreducible character degrees of G. For any normal subgroup N of G, $\mathrm{Irr}(G|N)$ denotes $\mathrm{Irr}(G) \setminus \mathrm{Irr}(G/N)$. Also, for any character χ of G we write $\chi\downarrow_N$ to denote the restriction of χ on N. If χ is a character of any subgroup H then we write $\chi\uparrow_H^G$ to denote the induced character of G induced by χ. The inner product of two characters ϕ and ψ of G is given by $\langle\phi, \psi\rangle = \frac{1}{|G|}\sum_{g\in G}\phi(g)\overline{\psi(g)}$. For any integer $i \geq 1$ define inductively $Z_0(G) = \{1\}$, $Z_1(G) = Z(G)$ the center of G, $Z_i(G)/Z_{i-1}(G) =$ the center of $G/Z_{i-1}(G)$. Also $\gamma_1(G) = G$, $\gamma_2(G) = [G, G] = G'$ the commutator subgroup of G and $\gamma_{i+1}(G) = [\gamma_i(G), G]$.

2. Proof of Theorem 1.1

Suppose $\chi \in \mathrm{Irr}(G)$ is afforded by the irreducible representation ρ_χ of G. Consider the element

$$z = \sum_{\substack{x_1,\ldots,x_n,\\ x_{n+1},\ldots,x_m\in G}} \big[w_1(x_1, \ldots, x_n), w_2(x_{n+1}, \ldots, x_m)\big].$$

Then, we have

$$\rho_\chi(z) = \sum_{x_1,\ldots,x_n\in G} \rho_\chi\big(w_1(x_1, \cdots, x_n)^{-1}\big)\, A_\chi(x_1, \ldots, x_n) \tag{2.1}$$

where

$$A_\chi(x_1, \cdots, x_n) \tag{2.2}$$

$$= \sum_{x_{n+1},\ldots,x_m \in G} \rho_\chi\big(w_2(x_{n+1},\ldots,x_m)^{-1} w_1(x_1,\ldots,x_n) w_2(x_{n+1},\ldots,x_m)\big).$$

Since $\zeta_G^{w_2}$ is a constant function, $A_\chi(x_1,\cdots,x_n)$ commutes with $\rho_\chi(g)$ for all $g \in G$. Hence, by Schur's lemma (see [11, Lemma 2.25]), we have

$$A_\chi(x_1,\ldots,x_n) = \alpha I \tag{2.3}$$

for some $\alpha \in \mathbb{C}$. Now taking trace on the both sides of (2.3) and using (2.2) we get

$$\alpha = \frac{|G|^{m-n}}{\chi(1)} \chi(w_1(x_1,\cdots,x_n)). \tag{2.4}$$

Again taking trace on both side of (2.1) and using (2.3), (2.4) we get

$$\chi(z) = \frac{|G|^{m-n}}{\chi(1)} \sum_{h\in G} \zeta_G^{w_1}(h)\chi(h)\chi(h^{-1}) = \frac{|G|^{m-n+1}}{\chi(1)} \langle \zeta_G^{w_1}\overline{\chi}, \overline{\chi}\rangle. \tag{2.5}$$

By definition of ζ_G^w, we have

$$\chi(z) = |G|\langle \zeta_G^w, \overline{\chi}\rangle. \tag{2.6}$$

Therefore, by (2.5) and (2.6), we have

$$\zeta_G^w = \sum_{\chi\in \mathrm{Irr}(G)} \frac{|G|^{m-n}}{\chi(1)} \langle \zeta_G^{w_1}\chi, \chi\rangle\ \chi. \tag{2.7}$$

Since $\zeta_G^{w_1}$ is a character of G, the coefficient $\frac{|G|^{m-n}}{\chi(1)}\langle \zeta_G^{w_1}\chi, \chi\rangle$ is a non-negative integer. Hence ζ_G^w is a character of G. □

3. Explicit formula for $\zeta_G^{w_n}$

In this section, we consider the word

$$w_n(x_1,\ldots,x_n) := [\ldots[[[x_1,x_2],x_3],x_4],\ldots,x_n]$$

for $n \geq 2$. Then $\zeta_G^{w_n}(g)$ is the number of solutions of the equation

$$[\ldots[[[x_1,x_2],x_3],x_4],\ldots,x_n] = g.$$

Note that the word w_n can be defined recursively as

$$w_2(x_1,x_2) = [x_1,x_2] = x_1^{-1}x_2^{-1}x_1x_2,$$

$$w_3(x_1, x_2, x_3) = [w_2(x_1, x_2), x_3],$$
$$\dots$$
$$w_n(x_1, \dots, x_n) = [w_{n-1}(x_1, \dots, x_{n-1}), x_n].$$

Thus w_n is a particular type of the words considered in Section 2. By Theorem 1.1, it follows that $\zeta_G^{w_n}$ is a character of G and

$$\zeta_G^{w_n} = |G| \sum_{\chi \in \mathrm{Irr}(G)} \frac{C^{w_n}(\chi)\chi}{\chi(1)} \tag{3.1}$$

where $C^{w_n}(\chi) = \langle \zeta_G^{w_{n-1}}\chi, \chi\rangle$. Note that $C^{w_{m+1}}(\chi) = \chi(1)^2|G|^{m-1}$ whenever m is greater than the nilpotency class of G. For $n = 2$, $C^{w_2}(\chi) = 1$ and so equation (3.1) reduces to (1.1); which is the classical result of Frobenius. For $n \geq 3$ no explicit formula is known for $C^{w_n}(\chi)$. However, the following proposition shows that $C^{w_n}(\chi) = |G|^{n-2}$ if $\chi \in \mathrm{lin}(G)$.

PROPOSITION 3.1. *Let G be a finite group and $w_n(x_1, \dots, x_n) := [\dots[[x_1, x_2], x_3], \dots, x_n]$ with $n \geq 3$. Then we have the following.*
(1) *If $\chi \in \mathrm{lin}(G)$, then $C^{w_n}(\chi) = \langle \zeta_G^{w_{n-1}}, 1_G\rangle$.*
(2) *$\langle \zeta_G^{w_{n-1}}, 1_G\rangle = |G|\langle \zeta_G^{w_{n-2}}, 1_G\rangle$. In particular, $\langle \zeta_G^{w_{n-1}}, 1_G\rangle = |G|^{n-2}$.*

PROOF. (1) Let $\chi \in \mathrm{lin}(G)$. Then

$$C^{w_n}(\chi) = \langle \zeta_G^{w_{n-1}}\chi, \chi\rangle = \frac{1}{|G|}\sum_{g\in G} \zeta_G^{w_{n-1}}(g)|\chi(g)|^2 = \langle \zeta_G^{w_{n-1}}, 1_G\rangle.$$

(2) Using (3.1), we have

$$\langle \zeta_G^{w_{n-1}}, 1_G\rangle = \frac{1}{|G|}\sum_{g\in G} \zeta_G^{w_{n-1}}(g) = \frac{1}{|G|}\sum_{\chi\in \mathrm{Irr}(G)} \left(\sum_{g\in G} \frac{|G|}{\chi(1)} \langle \zeta_G^{w_{n-2}}\chi, \chi\rangle \chi(g)\right)$$
$$= \frac{1}{|G|}\sum_{\chi\in \mathrm{Irr}(G)} \left(\frac{|G|}{\chi(1)} \langle \zeta_G^{w_{n-2}}\chi, \chi\rangle |G| \langle \chi, 1_G\rangle\right) = |G|\langle \zeta_G^{w_{n-2}}, 1_G\rangle.$$

Therefore, $\langle \zeta_G^{w_{n-1}}, 1_G\rangle = |G|^{n-2}$. □

In view of the above proposition, (3.1) becomes

$$\zeta_G^{w_n} = |G|^{n-1} \sum_{\chi\in \mathrm{lin}(G)} \chi + |G| \sum_{\chi\in \mathrm{nl}(G)} \frac{C^{w_n}(\chi)\ \chi}{\chi(1)}\,. \tag{3.2}$$

In the following subsections we compute $C^{w_n}(\chi)$ where $\chi \in \mathrm{nl}(G)$ for $n \geq 3$ and hence derive an explicit formula for $\zeta_G^{w_n}(g)$ for some classes of finite groups.

3.1. n-isoclinism and generalized Camina group. We begin with the proof of Theorem 1.2. Let the pair (ϕ, ψ) be an n-isoclinism between G and H. Consider the group $\overline{G} = G/Z_n(G)$ and $\overline{H} = H/Z_n(H)$. Then

$$\frac{1}{|Z_n(G)|^{n+1}}\zeta_G^{w_{n+1}}(g)$$

$$= \frac{1}{|Z_n(G)|^{n+1}}\big|\big\{(g_1, g_2, \dots, g_{n+1}) \in G^{n+1} \mid [\dots[[g_1, g_2], g_3], \dots, g_{n+1}] = g\big\}\big|$$

$$= \frac{1}{|Z_n(G)|^{n+1}}\big|\big\{(g_1, g_2, \dots, g_{n+1}) \in G^{n+1} \mid$$

$$a_{(n+1,G)}\big([\dots[[\bar{g}_1, \bar{g}_2], \bar{g}_3], \dots, \bar{g}_{n+1}]\big) = g\big\}\big|$$

$$= \big|\big\{(\bar{g}_1, \bar{g}_2, \dots, \bar{g}_{n+1}) \in \overline{G}^{n+1} \mid a_{(n+1,G)}\big([\dots[[\bar{g}_1, \bar{g}_2], \bar{g}_3], \dots, \bar{g}_{n+1}]\big) = g\big\}\big|$$

$$= \big|\big\{(\bar{g}_1, \bar{g}_2, \dots, \bar{g}_{n+1}) \in \overline{G}^{n+1} \mid$$

$$\psi\big(a_{(n+1,G)}\big([\dots[[\bar{g}_1, \bar{g}_2], \bar{g}_3], \dots, \bar{g}_{n+1}]\big)\big) = \psi(g)\big\}\big|$$

(since ψ is an isomorphism)

$$= \big|\big\{(\bar{g}_1, \bar{g}_2, \dots, \bar{g}_{n+1}) \in \overline{G}^{n+1} \mid$$

$$a_{(n+1,H)}\big([\dots[[\phi(\bar{g}_1), \phi(\bar{g}_2)], \phi(\bar{g}_3)], \dots, \phi(\bar{g}_{n+1})]\big) = \psi(g)\big\}\big|$$

(using property (3) of n-isoclinism)

$$= \big|\big\{(\bar{h}_1, \bar{h}_2, \dots, \bar{h}_{n+1}) \in \overline{H}^{n+1} \mid$$

$$a_{(n+1,H)}\big([\dots[[\bar{h}_1, \bar{h}_2], \bar{h}_3], \dots, \bar{h}_{n+1}]\big) = \psi(g)\big\}\big|$$

(since ϕ is an isomorphism)

$$= \frac{1}{|Z_n(H)|^{n+1}}\big|\big\{(h_1, h_2, \dots, h_{n+1}) \in H^{n+1} \mid$$

$$a_{(n+1,H)}\big([\dots[[\bar{h}_1, \bar{h}_2], \bar{h}_3], \dots, \bar{h}_{n+1}]\big) = \psi(g)\big\}\big|$$

$$= \frac{1}{|Z_n(H)|^{n+1}}\big|\big\{(h_1, h_2, \dots, h_{n+1}) \in H^{n+1} \mid$$

$$[\dots[[h_1, h_2], h_3], \dots, h_{n+1}] = \psi(g)\big\}\big| = \frac{1}{|Z_n(H)|^{n+1}}\,\zeta_H^{w_{n+1}}(\psi(g)).$$

Hence the result follows. □

Now we derive an explicit formula for $\zeta_G^{w_n}(g)$ if G is a finite nilpotent generalized Camina group. Dark and Scoppola [4] completed the classification of Camina groups. Indeed, they proved the following theorem.

THEOREM 3.2 [4]. *Let G be a finite group. Then G is Camina group if and only if one of the following holds*:

(1) *G is a Camina p-group of nilpotence class 2 or 3.*

(2) *G is a Frobenius group with a cyclic Frobenius complement.*

(3) *G is a Frobenius group whose Frobenius complement is isomorphic to the quaternions.*

In [14], Lewis showed that a finite group G is a generalized Camina group if and only if G is isoclinic to a Camina group. Hence by the above theorem, if G is a finite nilpotent generalized Camina group, then G is isoclinic to a Camina p-group for some prime p. So, in view of Theorem 1.2, it is enough to compute $\zeta_G^{w_n}(g)$ for a Camina p-group. If G is a finite Camina p-group of nilpotency class 2, then $(G, Z(G))$ is a GCP. The following result gives some properties of groups such that $(G, Z(G))$ is a GCP.

THEOREM 3.3 [26, Section 3]. *Let G be a finite group such that $(G, Z(G))$ is a GCP. Then we have the following.*

(1) $\mathrm{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$.

(2) *The number of nonlinear irreducible characters of G is $|Z(G)| - |Z(G)/G'|$.*

(3) *There is a bijection $\widehat{\Phi}\colon \mathrm{Irr}(Z(G)|G') \to \mathrm{nl}(G)$ defined by*

$$\widehat{\Phi}(\lambda)(g) := \begin{cases} |G : Z(G)|^{1/2}\lambda(g) & \text{if } g \in Z(G) \\ 0 & \text{otherwise.} \end{cases}$$

If G is a finite Camina p-group of nilpotency class 2 then $\zeta_G^{w_n}(g)$ can be obtained by the following theorem.

THEOREM 3.4. *Let G be a finite group such that $(G, Z(G))$ is a GCP. If $g \in G'$ then*

$$\zeta_G^{w_2}(g) = \begin{cases} \frac{|G|^2}{|G'|}\big(1 - \frac{1}{|G:Z(G)|}\big) & \text{if } g \neq 1 \\ \frac{|G|^2}{|G'|}\big(1 + \frac{|G'|-1}{|G:Z(G)|}\big) & \text{if } g = 1 \end{cases}$$

and

$$\zeta_G^{w_n}(g) = \begin{cases} 0 & \text{if } g \neq 1 \\ |G|^n & \text{if } g = 1 \end{cases} \quad \text{if } n \geq 3.$$

PROOF. If $(G, Z(G))$ is a GCP then by [12, Lemma 2.4] we have that G' is a subgroup of $Z(G)$. That is, the nilpotency class of G is 2. Hence the case when $n \geq 3$ follows.

The case when $n = 2$ follows from [24, Corollary 2.3] and the fact that $\mathrm{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$ (which follows from Theorem 3.3(1)). □

Now we compute $\zeta_G^{w_n}(g)$ if G is a finite Camina p-group of nilpotency class 3. We need the following result which follows from [25, Section 3].

THEOREM 3.5. *Let G be a finite Camina p-group of nilpotency class* 3. *Then*

(1) *$Z(G) \leq G'$; G/G', $G'/Z(G)$ and $\gamma_3(G) = [G, G'] = Z(G)$ are elementary abelian p-groups with $|G : G'| = p^{2m}$, $|G' : Z(G)| = p^m$ and $|G : Z(G)| = p^{3m}$, where m is even.*

(2) $\mathrm{nl}(G) = \mathrm{Irr}(G|Z(G)) \sqcup \mathrm{nl}(G/Z(G))$, $\mathrm{cd}(G) = \{1, p^m, p^{3m/2}\}$,

$$|\mathrm{Irr}(G|Z(G))| = |Z(G)| - 1$$

and $|\mathrm{nl}(G/Z(G))| = \frac{|G'|}{|Z(G)|} - 1$.

(3) *If $\chi \in \mathrm{Irr}(G|Z(G))$ then $\chi(G \setminus Z(G)) = 0$ and $\chi{\downarrow_{Z(G)}} = p^{3m/2}\lambda$, where $\lambda \in \mathrm{Irr}(Z(G))$, not the principal character.*

(4) *If $\chi \in \mathrm{nl}(G/Z(G))$ then $\chi(G \setminus G') = 0$ and $\chi(g) = p^m(\lambda \circ \eta)(g)$, where $g \in G'$, $\lambda \in \mathrm{Irr}(G'/Z(G))$, not the principal character, and $\eta\colon G \to G/\gamma_3(G)$ is the natural homomorphism.*

Using Theorem 3.5 part (3) and (4), we get the following result.

PROPOSITION 3.6. *Let G be a finite Camina p-group of nilpotency class* 3. *Then*

$$C^{w_3}(\chi) = \begin{cases} \langle \zeta_G^{w_2} \downarrow_{Z(G)}, 1_{Z(G)} \rangle & \text{if } \chi \in \mathrm{Irr}(G|Z(G)), \\ \langle \zeta_G^{w_2} \downarrow_{G'}, 1_{G'} \rangle & \text{if } \chi \in \mathrm{nl}(G/Z(G)). \end{cases}$$

The next theorem gives expression for $\zeta_G^{w_3}$ if G is a finite Camina p-group of nilpotency class 3.

THEOREM 3.7. *Let G be a finite Camina p-group of nilpotency class* 3. *Then*

$$\zeta_G^{w_3} = |G|^2 \sum_{\chi \in \mathrm{lin}(G)} \chi + \left(\frac{|G|^3}{|G'|} + \frac{|G|^2(|G'| - |Z(G)|)}{|Z(G)|} \right) \sum_{\chi \in \mathrm{Irr}(G|Z(G))} \frac{\chi}{\chi(1)} + \frac{|G|^3}{|G'|} \sum_{\chi \in \mathrm{nl}(G/Z(G))} \frac{\chi}{\chi(1)}.$$

PROOF. By Theorem 3.5(2) we have $\mathrm{nl}(G) = \mathrm{Irr}(G|Z(G)) \sqcup \mathrm{nl}(G/Z(G))$. Therefore, using (3.2), we have

$$\zeta_G^{w_3} = |G|^2 \sum_{\chi\in\mathrm{lin}(G)} \chi + |G| \sum_{\chi\in\mathrm{Irr}(G|Z(G))} \frac{C^{w_3}(\chi)\,\chi}{\chi(1)} + |G| \sum_{\chi\in\mathrm{nl}(G/Z(G))} \frac{C^{w_3}(\chi)\,\chi}{\chi(1)}\,.$$

By Proposition 3.6 the above expression becomes

$$\zeta_G^{w_3} = |G|^2 \sum_{\chi\in\mathrm{lin}(G)} \chi + \langle \zeta_G^{w_2}\downarrow_{Z(G)}, 1_{Z(G)}\rangle |G| \sum_{\chi\in\mathrm{Irr}(G|Z(G))} \frac{\chi}{\chi(1)} + \langle \zeta_G^{w_2}\downarrow_{G'}, 1_{G'}\rangle |G| \sum_{\chi\in\mathrm{nl}(G/Z(G))} \frac{\chi}{\chi(1)}\,. \tag{3.3}$$

Now, using Theorem 3.5 part (3) and (4), and the the expression of $\zeta_G^{w_2}$, we get

$$\langle \zeta_G^{w_2}\downarrow_{Z(G)}, 1_{Z(G)}\rangle = |G|\langle \varphi\downarrow_{Z(G)}, 1_{Z(G)}\rangle + |G| \sum_{\lambda\in\mathrm{Irr}(Z(G))\setminus\{1_{Z(G)}\}} \langle \lambda, 1_{Z(G)}\rangle + |G||\mathrm{nl}(G/Z(G))|.$$

where $\varphi = \sum_{\chi\in\mathrm{lin}(G)} \chi$. Since $Z(G) \subseteq G'$ and φ is the regular character of G/G', $\langle \varphi\downarrow_{Z(G)}, 1_{Z(G)}\rangle = |G|/|G'|$. Also, $\langle \lambda, 1_{Z(G)}\rangle = 0$ for any $1_{Z(G)} \neq \lambda \in \mathrm{Irr}(Z(G))$. Hence,

$$\langle \zeta_G^{w_2}\downarrow_{Z(G)}, 1_{Z(G)}\rangle = \frac{|G|^2}{|G'|} + |G||\mathrm{nl}(G/Z(G))| = \frac{|G|^2}{|G'|} + \frac{|G|(|G'| - |Z(G)|)}{|Z(G)|}\,.$$

We have

$$\zeta_G^{w_2}\downarrow_{G'} = |G| \sum_{\chi\in\mathrm{lin}(G)} \chi\downarrow_{G'} + |G| \sum_{\chi\in\mathrm{Irr}(G|Z(G))} \frac{\chi\downarrow_{G'}}{\chi(1)} + |G| \sum_{\chi\in\mathrm{nl}(G/Z(G))} \frac{\chi\downarrow_{G'}}{\chi(1)}\,.$$

It is easy to see that $\langle \chi\downarrow_{G'}, 1_{G'}\rangle = \langle \phi\downarrow_{G'}, 1_{G'}\rangle = 0$ for $\chi \in \mathrm{Irr}(G|Z(G))$ and $\phi \in \mathrm{nl}(G/Z(G))$. Therefore, $\langle \zeta_G^{w_2}\downarrow_{G'}, 1_{G'}\rangle = |G|^2/|G'|$. Now putting the values of $\langle \zeta_G^{w_2}\downarrow_{Z(G)}, 1_{Z(G)}\rangle$ and $\langle \zeta_G^{w_2}\downarrow_{G'}, 1_{G'}\rangle$ in (3.3) we get the required expression for $\zeta_G^{w_3}$. □

The following lemma is useful in proving the final result of this subsection.

LEMMA 3.8. *Let G be a finite Camina p-group of nilpotency class* 3. *Then*

$$\sum_{\chi\in \mathrm{Irr}(G|Z(G))} \frac{\chi(g)}{\chi(1)} = \begin{cases} -1 & \text{if } 1\neq g\in Z(G) \\ 0 & \text{if } g\in G'\setminus Z(G) \end{cases}$$

and

$$\sum_{\chi\in \mathrm{nl}(G/Z(G))} \frac{\chi(g)}{\chi(1)} = \begin{cases} \frac{|G'|-|Z(G)|}{|Z(G)|} & \text{if } 1\neq g\in Z(G) \\ -1 & \text{if } g\in G'\setminus Z(G). \end{cases}$$

PROOF. Follows from Theorem 3.5 part (3) and (4). □

We conclude this subsection with the following result.

THEOREM 3.9. *Let G be a finite Camina p-group of nilpotency class* 3. *If $g\in G'$ then*

$$\zeta_G^{w_2}(g) = \begin{cases} \frac{|G|(|G|-|G'|)}{|G'|} + \frac{|G|(|G'|-|Z(G)|)}{|Z(G)|} & \text{if } 1\neq g\in Z(G) \\ \frac{|G|(|G|-|G'|)}{|G'|} & \text{if } g\in G'\setminus Z(G) \\ \frac{|G|^2}{|G'|} + \frac{|G||G'|}{|Z(G)|} + |G|(|Z(G)|-2) & \text{if } g=1 \end{cases}$$

and

$$\zeta_G^{w_3}(g) = \begin{cases} \frac{|G|^2(|G'|-|Z(G)|)(|G|-|G'|)}{|G'||Z(G)|} & \text{if } 1\neq g\in Z(G) \\ 0 & \text{if } g\in G'\setminus Z(G) \\ \frac{|G|^3(|Z(G)|^2+|G'|-|Z(G)|)}{|Z(G)||G'|} + \frac{|G|^2(|G'|-|Z(G)|)(|Z(G)|-1)}{|Z(G)|} & \text{if } g=1. \end{cases}$$

PROOF. If $g=1$ then (1.1) gives $\zeta_G^{w_2}(1) = |G||\operatorname{Irr}(G)|$. Hence, the result follows from Theorem 3.5(2). The case when $g\neq 1$ follows from [20, Proposition 5.4].

Again, if $g=1$ then by Theorem 3.7 we have

$$\begin{aligned} \zeta_G^{w_3}(1) &= |G|^2\frac{|G|}{|G'|} \\ &+ \left(\frac{|G|^3}{|G'|} + \frac{|G|^2(|G'|-|Z(G)|)}{|Z(G)|}\right)|\operatorname{Irr}(G|Z(G))| + \frac{|G|^3}{|G'|}|\mathrm{nl}(G/Z(G))| \\ &= \frac{|G|^3(|Z(G)|^2+|G'|-|Z(G)|)}{|Z(G)||G'|} + \frac{|G|^2(|G'|-|Z(G)|)(|Z(G)|-1)}{|Z(G)|}, \end{aligned}$$

using Theorem 3.5(2). If $1 \neq g \in G'$ then $\sum_{\chi \in \mathrm{lin}(G)} \chi(g) = |G|/|G'|$. Therefore, by Theorem 3.7, we have

$$\zeta_G^{w_3}(g) = \frac{|G|^3}{|G'|} + \left(\frac{|G|^3}{|G'|} + \frac{|G|^2(|G'| - |Z(G)|)}{|Z(G)|}\right) \sum_{\chi \in \mathrm{Irr}(G|Z(G))} \frac{\chi(g)}{\chi(1)} + \frac{|G|^3}{|G'|} \sum_{\chi \in \mathrm{nl}(G/Z(G))} \frac{\chi(g)}{\chi(1)}. \tag{3.4}$$

If $1 \neq g \in Z(G)$ then by (3.4) and Lemma 3.8, we have

$$\zeta_G^{w_3}(g) = \frac{|G|^3}{|G'|} + \left(\frac{|G|^3}{|G'|} + \frac{|G|^2(|G'| - |Z(G)|)}{|Z(G)|}\right)(-1) + \frac{|G|^3(|G'| - |Z(G)|)}{|G'||Z(G)|} = \frac{|G|^2(|G'| - |Z(G)|)(|G| - |G'|)}{|G'||Z(G)|}.$$

If $g \in G' \setminus Z(G)$ then by (3.4) and Lemma 3.8, we have $\zeta_G^{w_3}(g) = 0$. □

3.2. Groups for which $(G, Z(G))$ is a Camina pair and $(G/Z(G), Z(G/Z(G)))$ is a GCP. In [15], Lewis began the study of those groups G for which $(G, Z(G))$ is a Camina pair and, proved that such a group G must be a p-group for some prime p. Note that if G is a finite group such that $(G, Z(G))$ is a Camina pair and $(\frac{G}{Z(G)}, Z(\frac{G}{Z(G)}))$ is a GCP then the nilpotency class of G is 3. In the following theorem, we quote some facts about the groups for which $(G, Z(G))$ is a Camina pair.

THEOREM 3.10 [25, Section 4]. *Let G be a finite group such that $(G, Z(G))$ is a Camina pair. Then we have the following.*

(1) *If $\chi \in \mathrm{Irr}(G|Z(G))$, then $\chi(G \setminus Z(G)) = 0$ and $\chi(1) = |G/Z(G)|^{1/2}$.*

(2) *There is a bijection $\Phi : \mathrm{Irr}(Z(G)) \setminus \{1_{Z(G)}\} \to \mathrm{Irr}(G|Z(G))$ such that*

$$\Phi(\lambda)(g) := \begin{cases} |G/Z(G)|^{1/2}\lambda(g) & \text{if } g \in Z(G) \\ 0 & \text{otherwise.} \end{cases}$$

(3) $|\mathrm{Irr}(G|Z(G))| = |Z(G)| - 1$.

If $(G, Z(G))$ is a Camina pair and $(G/Z(G), Z(G/Z(G)))$ is a GCP, then $Z(G) \subseteq G'$ and $(G/Z(G))' = G'/Z(G) \subseteq Z(G/Z(G))$. Therefore, as a corollary of Theorem 3.3, we have the following result.

COROLLARY 3.11. *Let G be a finite group such that $(G, Z(G))$ is a Camina pair and $(\frac{G}{Z(G)}, Z(\frac{G}{Z(G)}))$ is a GCP. Then we have the following.*

(1) *There is a bijection* $\Psi : \mathrm{Irr}(Z(G/Z(G)) \mid G'/Z(G)) \to \mathrm{nl}(G/Z(G))$ *such that*

$$\Psi(\lambda)(g) := \begin{cases} |G : Z_2(G)|^{1/2}\lambda(g) & \text{if } g \in Z_2(G)/Z(G) \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the degree of every nonlinear irreducible characters of $G/Z(G)$ *is equal to* $|G : Z_2(G)|^{1/2}$.

(2) $|\mathrm{nl}(G/Z(G))| = \frac{|Z_2(G)|(|G'|-|Z(G)|)}{|G'||Z(G)|}$ *and*

$$\mathrm{nl}(G) = \mathrm{Irr}(G|Z(G)) \sqcup \mathrm{nl}(G/Z(G)).$$

Proposition 3.12. *Let* G *be a finite group such that* $(G, Z(G))$ *is a Camina pair and* $\big(\frac{G}{Z(G)}, Z(\frac{G}{Z(G)})\big)$ *a GCP. Then*

$$C^{w_3}(\chi) = \begin{cases} \langle \zeta_G^{w_2} \downarrow_{Z(G)}, 1_{Z(G)} \rangle & \text{if } \chi \in \mathrm{Irr}(G|Z(G)) \\ \langle \zeta_G^{w_2} \downarrow_{Z_2(G)}, 1_{Z_2(G)} \rangle & \text{if } \chi \in \mathrm{nl}(G/Z(G)). \end{cases}$$

Proof. By Theorem 3.10(1) one can easily compute that $\langle \zeta_G^{w_2}\chi, \chi\rangle = \langle \zeta_G^{w_2} \downarrow_{Z(G)}, 1_{Z(G)}\rangle$, where $\chi \in \mathrm{Irr}(G|Z(G))$. Again, by Corollary 3.11, we have $\langle \zeta_G^{w_2}\chi, \chi\rangle = \langle \zeta_G^{w_2} \downarrow_{Z_2(G)}, 1_{Z_2(G)}\rangle$, where $\chi \in \mathrm{nl}(G/Z(G))$. □

The following theorem gives an expression for $\zeta_G^{w_3}$ if G is a finite group such that $(G, Z(G))$ is a Camina pair and $(G/Z(G), Z(G/Z(G)))$ a GCP.

Theorem 3.13. *Let* G *be a finite group such that* $(G, Z(G))$ *is a Camina pair and* $\big(\frac{G}{Z(G)}, Z(\frac{G}{Z(G)})\big)$ *a GCP. Then*

$$\zeta_G^{w_3} = |G|^2 \sum_{\chi\in \mathrm{lin}(G)} \chi + \left(\frac{|G|^3}{|G'|} + \frac{|G|^2|Z_2(G)|(|G'|-|Z(G)|)}{|G'||Z(G)|}\right) \sum_{\chi\in \mathrm{Irr}(G|Z(G))} \frac{\chi}{\chi(1)} + \frac{|G|^3}{|Z_2(G)|} \sum_{\chi\in \mathrm{nl}(G/Z(G))} \frac{\chi}{\chi(1)}.$$

Proof. By Corollary 3.11(2) we have $\mathrm{nl}(G) = \mathrm{Irr}(G|Z(G)) \sqcup \mathrm{nl}(G/Z(G))$. Therefore, using (3.2), we get

$$\zeta_G^{w_3} = |G|^2 \sum_{\chi\in \mathrm{lin}(G)} \chi + |G| \sum_{\chi\in \mathrm{Irr}(G|Z(G))} \frac{C^{w_3}(\chi)\chi}{\chi(1)} + |G| \sum_{\chi\in \mathrm{nl}(G/Z(G))} \frac{C^{w_3}(\chi)\chi}{\chi(1)}.$$

By Proposition 3.12 the above expression becomes

$$\zeta_G^{w_3} = |G|^2 \sum_{\chi\in \mathrm{lin}(G)} \chi + \langle \zeta_G^{w_2} \downarrow_{Z(G)}, 1_{Z(G)}\rangle |G| \sum_{\chi\in \mathrm{Irr}(G|Z(G))} \frac{\chi}{\chi(1)} + \langle \zeta_G^{w_2} \downarrow_{Z_2(G)}, 1_{Z_2(G)}\rangle |G| \sum_{\chi\in \mathrm{nl}(G/Z(G))} \frac{\chi}{\chi(1)}. \tag{3.5}$$

Let $\phi = \sum_{\chi\in \mathrm{lin}(G)} \chi$. Then $\langle \phi \downarrow_{Z_2(G)}, 1_{Z_2(G)}\rangle = |G|/|Z_2(G)|$, since $G' \subseteq Z_2(G)$. Therefore

$$\left\langle \zeta_G^{w_2} \downarrow_{Z_2(G)}, 1_{Z_2(G)} \right\rangle = |G| \langle \phi \downarrow_{Z_2(G)}, 1_{Z_2(G)}\rangle = \frac{|G|^2}{|Z_2(G)|},$$

observing that $\langle \chi \downarrow_{Z_2(G)}, 1_{Z_2(G)}\rangle = 0$ for $\chi \in \mathrm{Irr}(G|Z(G)) \sqcup \mathrm{nl}(G/Z(G))$.

Also, $\langle \phi \downarrow_{Z(G)}, 1_{Z(G)}\rangle = |G|/|G'|$, since $Z(G) \subseteq G'$ and ϕ is the regular character of G/G'. Observe that $\langle \chi \downarrow_{Z(G)}, 1_{Z(G)}\rangle = 0$ for $\chi \in \mathrm{Irr}(G|Z(G))$ and $\langle \chi \downarrow_{Z(G)}, 1_{Z(G)}\rangle = \chi(1)$ for $\chi \in \mathrm{nl}(G/Z(G))$. Therefore,

$$\begin{aligned}\left\langle \zeta_G^{w_2} \downarrow_{Z(G)}, 1_{Z(G)} \right\rangle &= \frac{|G|^2}{|G'|} + |G||\mathrm{nl}(G/Z(G))| \\ &= \frac{|G|^2}{|G'|} + \frac{|G||Z_2(G)|(|G'| - |Z(G)|)}{|G'||Z(G)|},\end{aligned}$$

using Corollary 3.11(2). Now putting the values of $\langle \zeta_G^{w_2} \downarrow_{Z(G)}, 1_{Z(G)}\rangle$ and $\langle \zeta_G^{w_2} \downarrow_{Z(G)}, 1_{Z(G)}\rangle$ in (3.5) we get the required expression for $\zeta_G^{w_3}$. □

The following lemma is useful in proving the final result of this subsection.

LEMMA 3.14. *Let G be a finite group such that $(G, Z(G))$ is a Camina pair and $(\frac{G}{Z(G)}, Z(\frac{G}{Z(G)}))$ a GCP. Then*

$$\sum_{\chi\in \mathrm{Irr}(G|Z(G))} \frac{\chi(g)}{\chi(1)} = \begin{cases} -1 & \text{if } 1 \neq g \in Z(G) \subseteq G' \\ 0 & \text{if } g \in G' \setminus Z(G) \end{cases}$$

and

$$\sum_{\chi\in \mathrm{nl}(G/Z(G))} \frac{\chi(g)}{\chi(1)} = \begin{cases} \frac{|Z_2(G)|(|G'|-|Z(G)|)}{|G'||Z(G)|} & \text{if } 1 \neq g \in Z(G) \subseteq G' \\ -\frac{|Z_2(G)|}{|G')|} & \text{if } g \in G' \setminus Z(G). \end{cases}$$

PROOF. Follows from Theorem 3.10(1), (2) and Corollary 3.11. □

We conclude this subsection by the following result.

THEOREM 3.15. *Let G be a finite group such that $(G, Z(G))$ is a Camina pair and $(\frac{G}{Z(G)}, Z(\frac{G}{Z(G)}))$ a GCP. If $g \in G'$ then*

$$\zeta_G^{w_2}(g) = \begin{cases} \frac{|G|[|Z(G)|(|G|+|G'|(|Z(G)|-1))+|Z_2(G)|(|G'|-|Z(G)|)]}{|G'||Z(G)|} & \text{if } g = 1 \\ \frac{|G|[|Z(G)|(|G|-|G'|)+|Z_2(G)|(|G'|-|Z(G)|)]}{|G'||Z(G)|} & \text{if } 1 \neq g \in Z(G) \subseteq G' \\ \frac{|G|(|G|-|Z_2(G)|)}{|G'|} & \text{if } g \in G' \setminus Z(G) \end{cases}$$

and

$$\zeta_G^{w_3}(g) = \begin{cases} \frac{|G|^2[|G||Z(G)|^2+(|G'|-|Z(G)|)(|G|+|Z_2(G)|(|Z(G)|-1))]}{|G'||Z(G)|} & \text{if } g = 1 \\ \frac{|G|^2(|G'|-|Z(G)|)(|G|-|Z_2(G)|)}{|G'||Z(G)|} & \text{if } 1 \neq g \in Z(G) \subseteq G' \\ 0 & \text{if } g \in G' \setminus Z(G). \end{cases}$$

PROOF. If $g = 1$, then by (1.1) we have

$$\zeta_G^{w_2}(1) = |G||\operatorname{Irr}(G)|$$
$$= \frac{|G|[|Z(G)|(|G| + |G'|(|Z(G)| - 1)) + |Z_2(G)|(|G'| - |Z(G)|)]}{|G'||Z(G)|},$$

noting that $\operatorname{Irr}(G) = \operatorname{lin}(G) \sqcup \operatorname{Irr}(G|Z(G)) \sqcup \operatorname{nl}(G/Z(G))$. If $1 \neq g \in G'$ then by (1.1) we have

$$\zeta_G^{w_2}(g) = \frac{|G|^2}{|G'|} + |G| \sum_{\chi \in \operatorname{Irr}(G|Z(G))} \frac{\chi(g)}{\chi(1)} + |G| \sum_{\chi \in \operatorname{nl}(G/Z(G))} \frac{\chi(g)}{\chi(1)}. \tag{3.6}$$

Now, if $1 \neq g \in Z(G) \subseteq G'$, then by (3.6) and Lemma 3.14 we have

$$\zeta_G^{w_2}(g) = \frac{|G|^2}{|G'|} + |G|(-1) + \frac{|G||Z_2(G)|(|G'| - |Z(G)|)}{|G'||Z(G)|}$$
$$= \frac{|G|[|Z(G)|(|G| - |G'|) + |Z_2(G)|(|G'| - |Z(G)|)]}{|G'||Z(G)|}.$$

If $g \in G' \setminus Z(G)$, then by (3.6) and Lemma 3.14 we have

$$\zeta_G^{w_2}(g) = \frac{|G|^2}{|G'|} + \frac{-|G||Z_2(G)|}{|G'|} = \frac{|G|(|G| - |Z_2(G)|)}{|G'|}.$$

For the second part, if $g = 1$, then by Theorem 3.13, we have

$$\zeta_G^{w_3}(1) = \frac{|G|^3}{|G'|}$$

$$+\left(\frac{|G|^3}{|G'|}+\frac{|G|^2|Z_2(G)|(|G'|-|Z_2(G)|)}{|G'||Z(G)|}\right)|\operatorname{Irr}(G|Z(G))|+\frac{|G|^3|\operatorname{nl}(G/Z(G))|}{|Z_2(G)|}$$
$$=\frac{|G|^2[|G||Z(G)|^2+(|G'|-|Z(G)|)(|G|+|Z_2(G)|(|Z(G)|-1))]}{|G'||Z(G)|},$$

using Theorem 3.10(3) and Corollary 3.11(2).

If $1\neq g\in G'$, then by Theorem 3.13 we have

(3.7)
$$\zeta_G^{w_3}(g)=\frac{|G|^3}{|G'|}+\left(\frac{|G|^3}{|G'|}+\frac{|G|^2|Z_2(G)|(|G'|-|Z(G)|)}{|G'||Z(G)|}\right)\sum_{\chi\in\operatorname{Irr}(G|Z(G))}\frac{\chi(g)}{\chi(1)}$$
$$+\frac{|G|^3}{|Z_2(G)|}\sum_{\chi\in\operatorname{nl}(G/Z(G))}\frac{\chi(g)}{\chi(1)}.$$

If $1\neq g\in Z(G)\subseteq G'$, then by (3.7) and Lemma 3.14 we have

$$\zeta_G^{w_3}(g)=\frac{|G|^3}{|G'|}+\left(\frac{|G|^3}{|G'|}+\frac{|G|^2|Z_2(G)|(|G'|-|Z(G)|)}{|G'||Z(G)|}\right)(-1)$$
$$+\frac{|G|^3(|G'|-|Z(G)|)}{|G'||Z(G)|}=\frac{|G|^2(|G'|-|Z(G)|)(|G|-|Z_2(G)|)}{|G'||Z(G)|}.$$

If $g\in G'\setminus Z(G)$, then by (3.7) and Lemma 3.14, we have $\zeta_G^{w_3}(g)=0$. □

3.3. Groups with unique nonlinear irreducible character. The class of groups considered in this subsection is a subclass of the groups G with $|\operatorname{cd}(G)|=2$, where $\operatorname{cd}(G)=\{\chi(1):\chi\in\operatorname{Irr}(G)\}$. Groups having $|\operatorname{cd}(G)|=2$ are considered in [24] for the word $w_2(x_1,x_2)=[x_1,x_2]$. Note that an explicit formula for $\zeta_G^{w_2}(g)$ can be obtained from [24, Theorem 2.2] for these groups. In this subsection, we derive an explicit formula for $\zeta_G^{w_n}(g)$ where $n\geq 3$ for groups having a unique nonlinear irreducible character. We start with the following theorem which is a consequence of the main theorem in [3].

THEOREM 3.16. *Let G be a finite group having a unique nonlinear irreducible character. Then one of the following holds*:

(1) *G is an extra-special 2-group.*

(2) *G is a Frobenius group of order $p^m(p^m-1)$ for some prime power p^m with an abelian Frobenius kernel of order p^m, a cyclic Frobenius complement and $Z(G)=\{1\}$. Furthermore, the Frobenius kernel is equal to G' and G acts transitively, by conjugation, on $G'\setminus\{1\}$.*

Theorem 3.16 is also proved independently in [29] and [23]. Note that the groups in this theorem are Camina groups. Further, groups of type (1) have nilpotency class 2 whereas groups of type (2) are not nilpotent. For groups of type (1), an explicit formula of $\zeta_G^{w_n}$ can be obtained by Theorem 3.4. In this section, we derive an explicit formula of $\zeta_G^{w_n}$ for groups of type (2) considering $n \geq 3$.

We need the following lemma.

LEMMA 3.17. *Let G be a finite group with unique nonlinear irreducible character ϕ with $Z(G) = \{1\}$. Then we have the following.*

(1) *ϕ vanishes outside G' and $\phi(1) = p^m - 1$.*
(2) *$\phi^2 = \sum_{\lambda \in \mathrm{lin}(G)} \lambda + (p^m - 2)\phi$.*
(3) *$\phi(g) = -1$ for all $g \in G' \setminus \{1\}$.*

PROOF. By Theorem 3.16, G is a Frobenius group of order $p^m(p^m - 1)$ for some prime power p^m.

(1) Since G is a Camina group, ϕ vanishes outside G'. The second part follows from the relation $|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2$.

(2) Since ϕ^2 is a character of G, $\phi^2 = \sum_{\chi \in \mathrm{Irr}(G)} c_\chi \chi$, where $c_\chi = \langle \phi^2, \chi \rangle \in \mathbb{N} \cup \{0\}$. Let $\lambda \in \mathrm{lin}(G)$. Then $c_\lambda = \langle \phi^2, \lambda \rangle = \langle \phi, \overline{\phi}\lambda \rangle = \langle \phi, \phi\lambda \rangle = \langle \phi, \phi \rangle = 1$. Therefore,

$$\phi^2 = \sum_{\lambda \in \mathrm{lin}(G)} \lambda + c_\phi \phi.$$

Now, by degree computation of both sides of the above expression, we get $c_\phi = p^m - 2$. Hence, the result follows.

(3) Since G is a Frobenius group with Frobenius kernel G', any nonlinear irreducible character of G is induced from a non-trivial irreducible character of G' (see [11, Theorem 6.34 (a)]). But G has a unique nonlinear irreducible character ϕ. Therefore, $\phi \downarrow_{G'} = \sum_{\gamma \in (\mathrm{Irr}(G') \setminus \{1_{G'}\})} \gamma$, where $1_{G'}$ is the trivial character of G'. Now take $g \neq 1 \in G'$. Then $\left(\sum_{\gamma \in \mathrm{Irr}(G')} \gamma \right)(g) = 0$ and hence $\phi(g) = -1$. □

THEOREM 3.18. *Let G be a finite group with unique nonlinear irreducible character ϕ and $Z(G) = \{1\}$. If $n \geq 3$ then*

$$\zeta_G^{w_n} = |G|^{n-1} \sum_{\lambda \in \mathrm{lin}(G)} \lambda + \frac{|G|}{p^m - 1} C^{w_n}(\phi)\phi$$

and $C^{w_n}(\phi) = \frac{|G|^{n-1}}{|G'|} + \frac{|G|(p^m-2)}{p^m-1} C^{w_{n-1}}(\phi)$.

PROOF. The expression for $\zeta_G^{w_n}$ follows from (3.2) applying Lemma 3.17(1). Using this we have

$$C^{w_n}(\phi) = \langle \zeta_G^{w_{n-1}}\phi, \phi\rangle = \left\langle \left(|G|^{n-2} \sum_{\lambda\in \mathrm{lin}(G)} \lambda + \frac{|G|}{p^m-1} C^{w_{n-1}}(\phi)\phi \right)\phi, \phi \right\rangle$$

$$= |G|^{n-2} \sum_{\lambda\in \mathrm{lin}(G)} \langle \phi, \phi\rangle + \frac{|G|}{p^m-1} C^{w_{n-1}}(\phi)\langle \phi^2, \phi\rangle.$$

Hence, using Lemma 3.17(2) and the fact that $|\mathrm{lin}(G)| = |G : G'|$, the result follows. □

THEOREM 3.19. *Let G be a finite group with a unique nonlinear irreducible character ϕ with $Z(G) = \{1\}$. If $g \in G'$ and $n \geq 3$ then*

$$\zeta_G^{w_n}(g) = \begin{cases} \frac{|G|^n}{|G'|} + \frac{|G|^n((p^m-1)^{n-2} - (p^m-2)^{n-2})}{|G'|(p^m-1)^{n-3}} + \frac{|G|^{n-1}(p^m-2)^{n-2}}{(p^m-1)^{n-2}} & \text{if } g = 1 \\ \frac{|G|^n}{|G'|} - \frac{|G|^n((p^m-1)^{n-2} - (p^m-2)^{n-2})}{|G'|(p^m-1)^{n-2}} - \frac{|G|^{n-1}(p^m-2)^{n-2}}{(p^m-1)^{n-1}} & \text{if } g \neq 1. \end{cases}$$

PROOF. Let $g \in G'$ and $n \geq 3$. Then, using Theorem 3.18 and Lemma 3.17(3), we have

$$\zeta_G^{w_n}(g) = \begin{cases} \frac{|G|^n}{|G'|} + |G| C^{w_n}(\phi) & \text{if } g = 1 \\ \frac{|G|^n}{|G'|} - \frac{|G|}{(p^m-1)} C^{w_n}(\phi) & \text{if } g \neq 1. \end{cases}$$

By the recursive formula for $C^{w_n}(\phi)$, given in Theorem 3.18, we have

$$C^{w_n}(\phi) = \frac{|G|^{n-1}}{|G'|}\Big(1 + \frac{p^m-2}{p^m-1} + \frac{(p^m-2)^2}{(p^m-1)^2} + \cdots + \frac{(p^m-2)^{n-3}}{(p^m-1)^{n-3}}\Big) + \frac{|G|^{n-2}(p^m-2)^{n-2}}{(p^m-1)^{n-2}}$$

$$= \frac{|G|^{n-1}((p^m-1)^{n-2} - (p^m-2)^{n-2})}{|G'|(p^m-1)^{n-3}} + \frac{|G|^{n-2}(p^m-2)^{n-2}}{(p^m-1)^{n-2}}.$$

Putting $C^{w_n}(\phi)$ in this formula for $\zeta_G^{w_n}(g)$ we get the required result. □

COROLLARY 3.20. *Let G be a finite group with unique nonlinear irreducible character ϕ with $Z(G) = \{1\}$. If $g \in G'$ then*

$$\zeta_G^{w_3}(g) = \begin{cases} \frac{2|G|^3}{|G'|} + \frac{|G|^2(p^m-2)}{p^m-1} & \text{if } g = 1 \\ \frac{|G|^3}{|G'|} - \frac{|G|^3}{|G'|(p^m-1)} - \frac{|G|^2(p^m-2)}{(p^m-1)^2} & \text{if } g \neq 1. \end{cases}$$

It is easy to see that the nonlinear irreducible character comes by induction of a linear character of the abelian commutator subgroup for the group of type (2) in Theorem 3.16. So it is natural to ask for some bounds of $C^{w_n}(\chi)$ if all the nonlinear irreducible characters come by induction of some linear character of a fixed abelian normal subgroup N of G. We conclude this paper by the following upper bound of $C^{w_n}(\chi)$.

PROPOSITION 3.21. *Let G be a finite group with $\mathrm{cd}(G) = \{1, m\}$ and let G have an abelian normal subgroup N of index m such that every nonlinear irreducible character of G is induced from some irreducible character of N. Then for $n \geq 3$, $C^{w_n}(\phi) \leq \frac{|G|^n}{|N|^2}$ whenever $\phi \in \mathrm{nl}(G)$.*

PROOF. Let N be a normal subgroup of index m. Then G/N has no nonlinear irreducible character and hence G/N is abelian. Therefore $G' \leq N$.

Let $\phi \in \mathrm{nl}(G)$ then $\psi \uparrow_N^G = \phi$ for some $\psi \in \mathrm{Irr}(N)$. Therefore $\phi(g) = 0$ for all $g \in G \setminus N$. Thus

$$C^{w_n}(\phi) = \langle \zeta_G^{w_{n-1}}\phi, \phi\rangle = \frac{1}{|G|}\sum_{g\in N} \zeta_G^{w_{n-1}}(g)|\phi(g)|^2 \leq \frac{\phi(1)^2}{|G|}\sum_{g\in N} \zeta_G^{w_{n-1}}(g).$$

Since $n \geq 3$, the image of w_{n-1} is contained in $G' \leq N$. Therefore,

$$\sum_{g\in N} \zeta_G^{w_{n-1}}(g) = \sum_{g\in G} \zeta_G^{w_{n-1}}(g) = |G|^{n-1}.$$

Hence, the result follows noting that $\phi(1) = m = |G : N|$. □

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A fast and robust hybrid method for block-structured mesh deformation with emphasis on FSI-LES applications

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SUMMARY

The present work introduces an efficient technique for the deformation of block-structured grids occurring in simulations of fluid-structure interaction (FSI) problems relying on large-eddy simulations (LES). The proposed hybrid approach combines the advantages of the inverse distance weighting (IDW) interpolation with the simplicity and low computational effort of transfinite interpolations (TFI), while preserving the mesh quality in boundary layers. It is an improvement over the state-of-the-art currently in use.
To reach this objective, in a first step three elementary mesh deformation methods (TFI, IDW and radial basis functions) are investigated based on several test cases of different complexities analyzing their capabilities but also their computational costs. That allows to point out the advantages of each method but also demonstrates their drawbacks. Based on these specific properties of the different methods, a hybrid methodology is suggested which splits the entire grid deformation into two steps: First the movement of the block-boundaries of the block-structured grid and second the deformation of each block of the grid. Both steps rely on different methodologies, which allows to work out the most appropriate method for each step leading to a reasonable compromise between the grid quality achieved and the computational effort required. Finally, a hybrid IDW-TFI methodology is suggested which best fits to the specific requirements of coupled FSI-LES applications. This hybrid procedure is then applied to a real-life FSI-LES case. Copyright © 2016 John Wiley & Sons, Ltd.

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KEY WORDS: Mesh deformation; Radial basis function; Inverse distance weighting function; Greedy algorithm; Transfinite interpolation; Fluid-structure interaction; Large-eddy simulation

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1. INTRODUCTION

To compile an optimal methodology for the simulation of dynamically coupled fluid-structure interactions (FSI) of lightweight flexible structures in turbulent flow, a robust and efficient method combining eddy-resolving methods such as large-eddy simulation (LES) with flexible structures was developed and validated [1]. It relies on a partitioned Arbitrary Lagrangian-Eulerian (ALE) method applying block-structured grids. The FSI-LES applications tested were three-dimensional, however, with nominally two-dimensional geometries [2, 3]. In these FSI-LES cases a simple mesh deformation method based on transfinite interpolation (TFI) was sufficient to obtain a reasonable quality of the deformed mesh. However, for practically relevant FSI applications with fully three-dimensional geometries and large deformations, this mesh deformation method is not usable anymore. Moreover, improvements of the FSI-LES method have been achieved concerning the quality of the description of the interface geometry and the kinematics by applying the Iso-Geometric Analysis (IGA) and the Exact Coupling Layer (ECL) [4, 5]. Therefore, an appropriate grid deformation algorithm has to be developed, which allows to transfer the newly achieved high grid quality of the surface mesh at the FSI interface (fluid/structure) to the CFD volume grid. Another crucial postulate is that this mesh deformation algorithm should preserve the essential quality standard of the original mesh as far as possible. For LES the orthogonality of the grid is beside the smoothness one of the most important properties, especially in the vicinity of solid boundaries, where the flow gradients are large. Finally, for the developed FSI-LES methodology it is essential to find the best compromise between the attainable mesh quality and the required CPU-time.

In the literature various approaches to generate high-quality mesh deformations can be found. Historically for structured meshes efficient algebraic interpolation functions have been proposed to move the volume mesh points solely based on the surface motion. Such methods, known as transfinite interpolation (TFI) [6, 7], are very fast since the computational effort is proportional to the number of volume grid points N_v. TFI was used with great success even for some cases with large deformations [8, 9]. However, for more complex geometries applying block-structured grids, TFI can not be used directly. The block faces and edges have to be appropriately adapted first before TFI can be applied in each block separately. It is also important to notice that these TFI methods are not suitable for unstructured meshes.

Another classical mesh deformation algorithm applicable for all kinds of grids was originally proposed by Batina [10]. This method is based on torsional springs between all neighboring mesh points. The mesh deformation is controlled by the stiffness coefficient of the fictitious springs. A substantial improvement of this method was achieved by Farhat et al. [11, 12]. The authors introduced a safeguard against invalid cross-overs preventing neighboring triangles from inter-penetrating during large deformations. Furthermore, a mixture of normal and torsional springs was used to preserve the high orthogonality of the grid in the near-wall region. Since this method is based on an analogy with torsional springs, it is often termed as an analogy method. Another analogy-based method was proposed by Lynch and O'Neill [13]. The computational domain was seen as an elastic structure undergoing stress deformation with time. Here, the governing equations of elastostatics are solved with prescribed displacements on the boundary. This so-called *pseudo-structural* method was improved by Stein et al. [14]. The authors made a systematic choice of the

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stiffening parameters and thus were able to substantially improve the deformed mesh quality near solid boundaries even for cases involving large boundary displacements.

Considering the problem of mesh deformation for ALE simulations as a process, in which all grid points are typically assigned to some varying velocity, Löhner and Yang [15] used Laplace equations with variable diffusivity to deform unstructured grids. This so-called process of Laplacian smoothing of the mesh velocities leads to smooth meshes but the orthogonality is not preserved. Helenbrook [16] pointed out limitations of such methods relying on second-order partial differential equations (PDE) and proposed a biharmonic operator-based method for deforming the mesh. His algorithm, which is a generalization of Laplacian-based methods, allows to control both position and wall-normal spacing of the grid points along the boundaries. These analogy-based methods have the main disadvantage that they eventually lead to a system of equations of the order $(N_v \times N_v)$. Thus, these methods are too expensive for a recurring grid deformation within three-dimensional unsteady flow simulations using a large number of mesh points.

Liu et al. [17] proposed an interesting method using *Delaunay triangulations*. In this method boundary displacements are interpolated into the whole domain using a barycentric interpolation algorithm. This method gives good results and is fast. However, for cases involving rotations the orthogonality is adversely affected .

The interpolation of data using *radial basis functions* (RBFs) is an established procedure. De Boer et al. [18] used various RBFs to interpolate known displacements at the boundary nodes to interior volume mesh points and showed that the method can handle large deformations for both two-dimensional and three-dimensional grids. A comprehensive comparison by the authors reveals that five RBFs out of fourteen investigated are capable of generating high-quality meshes consistent with the boundary deformation. According to the authors Wendland's C^2 function with compact support is the best choice and is closely followed by the thin plate spline (TPS) function having global support. Subsequently, Zuijlen et al. [19] were successful in using a RBF-based mesh deformation for three-dimensional FSI simulations. Rendall and Allen [20] demonstrated that the mesh deformation strategy based on RBF is appropriate to preserve the global grid quality. In particular they observed that the orthogonality remains undisturbed at or near moving surfaces. Using Wendland's C^2 function they were also able to adopt this method for coupled FSI problems. The real computational costs in using RBFs arise due to the inversion of a system of equations having the dimension $(N_s \times N_s)$, where N_s is the total number of surface grid points [18, 20]. In addition, the update of the volume mesh requires a further effort of the order $(N_v \times N_s)$. Nevertheless, these computational costs are a fraction of the effort required for analogy-based methods, but it is still prohibitive for simulating three-dimensional flows in complex and realistic geometries using the FSI-LES methodology.

An algorithmic error-driven method to reduce the number of boundary data points was proposed by Rendall and Allen [21]. This so-called *greedy* algorithm was used to identify a reduced set of sample surface points in a manner that minimizes the error of the interpolated data. Sheng and Allen [22] investigated two different *greedy* algorithms in the context of RBFs on unstructured meshes, one using the exact instantaneous surface motion at each time step and the other applying a constant displacement function. For the latter the surface points are selected once before the calculation, which makes the method very efficient. Recently, the *greedy* algorithm has been further improved and a double-edged strategy was proposed by Wang et al. [23]. Significant contributions

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and subsequent applications of RBFs can be found in the studies of Rendall and Allen [24, 25, 26] and Bos et al. [27].

An explicit and robust alternative to the implicit interpolation procedures involved in RBFs was presented by Witteveen and Bijl [28] and Witteveen [29]. This technique denoted *inverse distance weighting* (IDW) method utilizes reciprocal distance weighted sums of the boundary node displacements to the volume vertices. Witteveen demonstrated that the IDW mesh deformation applied to FSI leads to a reduction of the computational costs compared to RBF, but reaching a comparable mesh quality and accuracy. The method does not depend on the nature of the grid, but requires a weighting function composed of several parameters which have to be estimated. Significant improvements of the IDW method were achieved by Luke et al. [30] by assuming that displacements of the boundary surface nodes can be approximated by a combination of a rigid-body rotation and a translation vector. The rigid-body motion at each boundary point is estimated by fitting a quaternion and calculating the relative displacements of the neighboring surface points. Ultimately, the displacements of the volume grid points are described by a weighted average of all boundary nodes. The weighting function is carefully designed based on a combination of parameters, which need to be estimated only once at the beginning of the interpolation process. By including the rotations in the IDW method, Luke et al. [30] significantly improved the orthogonality of the deformed mesh near boundaries. Contrary to RBF the IDW technique does not require any matrix inversion, however, the computational costs still scale with $(N_v \times N_s)$.

The idea of using quaternions for the mesh deformation dates back to the work of Samareh [31]. Maruyama et al. [32] were able to extend this idea and used it for estimating the rotation of the bounding surface. They applied quaternions for orthogonality preserving mesh deformations. Each surface node was associated with a quaternion and a translation vector and two methods were applied to obtain the deformed mesh, viz., a spherical linear interpolation and a Lie-algebra linear interpolation.

Recently, a few other novel point-by-point mesh deformation schemes have been proposed. Among those the works of Stadler et al. [33] basing on neural networks and Zhou and Li [34] relying on the disk relaxation idea deserve special attention. The main advantage of the method proposed by Stadler et al. [33] is that the volume point interpolation does not require a summation over all boundary nodes and hence is computationally efficient. The mesh deformation procedure outlined by Zhou and Li [34] starts with an initially rough distribution of the mesh points, which is updated by the disk relaxation algorithm using repulsive and attractive interactions to give rise to a high-quality mesh.

Apart from the above-mentioned methods, hybrid procedures combining different techniques are also available in the literature. The so-called *moving sub-mesh approach* (MSA) is a mixture between a physical analogy-based pseudo-structure method carried out on a coarser grid and an interpolation from this coarse grid onto the original grid [35]. A similar idea based on a *spring* method was successfully applied by Xuan et al. [36]. Another hybrid algorithm proposed by Landry et al. [37] moves the coarse mesh by the IDW method and then interpolates it to the fine mesh using a linear interpolation. In general, hybrid methods have the advantage that an optimal compromise between computational speed and quality of the grid can be found.

In the present work a new hybrid method is introduced with special emphasis on FSI-LES applications. The aim is to arrive at an appropriate grid deformation algorithm, which is the

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best compromise between high mesh quality and low CPU-time for block-structured grids with orthogonal cells in the vicinity of walls. To reach this goal, three elementary mesh deformation methods are selected, evaluated and then combined. TFI is chosen for its fast speed and its high quality of the deformed mesh obtained in a single block. RBF is taken into account, because of its reputation of robustness and the very high grid quality, which can be achieved. Finally, IDW is also selected for its simplicity, speed and good control of the mesh deformation in the near-wall region preserving the grid orthogonality.

The paper is organized as follows: Section 2 mathematically discusses the TFI, RBF and IDW methods. Section 3 deals with the numerical evaluation and comparison of these elementary methods. In Section 4, the hybrid algorithm is proposed. Section 5 is concerned with the numerical validation of the proposed hybrid algorithm based on a real-life FSI-LES application.

2. MATHEMATICAL DESCRIPTION AND IMPLEMENTATION

2.1. *Transfinite interpolation (TFI)*

The transfinite interpolation was introduced by Gordon and Hall [38]. They used the term "transfinite" to describe the general class of interpolation methods, which match the original primitive function to be interpolated at a non-denumerable, i.e., transfinite number of points. Later Eriksson [6] used TFI for grid generation in CFD. When used for this purpose, TFI has the advantage to provide complete conformity to surface boundaries of the domain to be meshed. Mathematically TFI is the Boolean sum of the tensor products formed using univariate interpolations. In general, these univariate interpolation functions are restricted to the linear Lagrangian combination of surface displacements based on the arc-length from the boundary points to the node.

For a known surface displacement, the mesh deformation $\boldsymbol{d}_{i,j,k} = (d^x_{i,j,k}, d^y_{i,j,k}, d^z_{i,j,k})$ at an arbitrary volume point (i, j, k), $1 \le i \le I$, $1 \le j \le J$, $1 \le k \le K$ in three dimensions is given by

$$\boldsymbol{d}_{i,j,k} = \boldsymbol{F_1} \oplus \boldsymbol{F_2} \oplus \boldsymbol{F_3} = \boldsymbol{F_1} + \boldsymbol{F_2} + \boldsymbol{F_3} - \boldsymbol{F_1F_2} - \boldsymbol{F_2F_3} - \boldsymbol{F_3F_1} + \boldsymbol{F_1F_2F_3}, \tag{1}$$

where $\boldsymbol{F}_i$ are univariate projectors. For two-dimensional cases the interpolation is only done along two directions and thus $\boldsymbol{d}_{i,j} = (d^x_{i,j}, d^y_{i,j})$ reads

$$\boldsymbol{d}_{i,j} = \boldsymbol{F_1} \oplus \boldsymbol{F_2} = \boldsymbol{F_1} + \boldsymbol{F_2} - \boldsymbol{F_1F_2}. \tag{2}$$

The linear interpolation and the tensor products of univariates are given by [39, 40]:

$$\begin{aligned}
\boldsymbol{F}_1 &= (1-\xi_{i,j,k})\,\boldsymbol{d}_{1,j,k} + \xi_{i,j,k}\,\boldsymbol{d}_{I,j,k}, && (3)\\
\boldsymbol{F}_2 &= (1-\eta_{i,j,k})\,\boldsymbol{d}_{i,1,k} + \eta_{i,j,k}\,\boldsymbol{d}_{i,J,k}, && (4)\\
\boldsymbol{F}_3 &= (1-\zeta_{i,j,k})\,\boldsymbol{d}_{i,j,1} + \zeta_{i,j,k}\,\boldsymbol{d}_{i,j,K}, && (5)\\
\boldsymbol{F_1F_2} &= (1-\xi_{i,j,k})(1-\eta_{i,j,k})\,\boldsymbol{d}_{1,1,k} + \xi_{i,j,k}(1-\eta_{i,j,k})\,\boldsymbol{d}_{I,1,k}\\
&\quad +(1-\xi_{i,j,k})\eta_{i,j,k}\,\boldsymbol{d}_{1,J,k} + \xi_{i,j,k}\eta_{i,j,k}\,\boldsymbol{d}_{I,J,k}, && (6)\\
\boldsymbol{F_2F_3} &= (1-\eta_{i,j,k})(1-\zeta_{i,j,k})\,\boldsymbol{d}_{i,1,1} + \eta_{i,j,k}(1-\zeta_{i,j,k})\,\boldsymbol{d}_{i,J,1}
\end{aligned}$$

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$$+(1-\eta_{i,j,k})\zeta_{i,j,k}\ \boldsymbol{d}_{i,1,K}+\eta_{i,j,k}\zeta_{i,j,k}\ \boldsymbol{d}_{i,J,K}, \tag{7}$$

$$\boldsymbol{F}_3\boldsymbol{F}_1 = (1-\zeta_{i,j,k})(1-\xi_{i,j,k})\ \boldsymbol{d}_{1,j,1}+\zeta_{i,j,k}(1-\xi_{i,j,k})\ \boldsymbol{d}_{1,j,K}$$
$$+(1-\zeta_{i,j,k})\xi_{i,j,k}\ \boldsymbol{d}_{I,j,1}+\zeta_{i,j,k}\xi_{i,j,k}\ \boldsymbol{d}_{I,j,K}, \tag{8}$$

$$\boldsymbol{F}_1\boldsymbol{F}_2\boldsymbol{F}_3 = (1-\xi_{i,j,k})(1-\eta_{i,j,k})(1-\zeta_{i,j,k})\ \boldsymbol{d}_{1,1,1}+\xi_{i,j,k}(1-\eta_{i,j,k})(1-\zeta_{i,j,k})\ \boldsymbol{d}_{I,1,1}$$
$$+(1-\xi_{i,j,k})\eta_{i,j,k}(1-\zeta_{i,j,k})\ \boldsymbol{d}_{1,J,1}+\xi_{i,j,k}\eta_{i,j,k}(1-\zeta_{i,j,k})\ \boldsymbol{d}_{I,J,1}$$
$$+(1-\xi_{i,j,k})(1-\eta_{i,j,k})\zeta_{i,j,k}\ \boldsymbol{d}_{1,1,K}+\xi_{i,j,k}(1-\eta_{i,j,k})\zeta_{i,j,k}\ \boldsymbol{d}_{I,1,K}$$
$$+(1-\xi_{i,j,k})\eta_{i,j,k}\zeta_{i,j,k}\ \boldsymbol{d}_{1,J,K}+\xi_{i,j,k}\eta_{i,j,k}\zeta_{i,j,k}\ \boldsymbol{d}_{I,J,K}. \tag{9}$$

The arc-length based computational coordinates (ξ,η,ζ) used to maintain the grid distribution laws of the initial grid are

$$\xi_{1,j,k}=0,\qquad \eta_{i,1,k}=0,\qquad \zeta_{i,j,1}=0,$$
$$\xi_{i,j,k}=\frac{\sum_{n=2}^{i}\parallel \boldsymbol{r}_{n,j,k}-\boldsymbol{r}_{n-1,j,k}\parallel}{\sum_{n=2}^{I}\parallel \boldsymbol{r}_{n,j,k}-\boldsymbol{r}_{n-1,j,k}\parallel},\quad \eta_{i,j,k}=\frac{\sum_{n=2}^{j}\parallel \boldsymbol{r}_{i,n,k}-\boldsymbol{r}_{i,n-1,k}\parallel}{\sum_{n=2}^{J}\parallel \boldsymbol{r}_{i,n,k}-\boldsymbol{r}_{i,n-1,k}\parallel},\quad \zeta_{i,j,k}=\frac{\sum_{n=2}^{k}\parallel \boldsymbol{r}_{i,j,n}-\boldsymbol{r}_{i,j,n-1}\parallel}{\sum_{n=2}^{K}\parallel \boldsymbol{r}_{i,j,n}-\boldsymbol{r}_{i,j,n-1}\parallel}, \tag{10}$$

where $\boldsymbol{r}_{i,j,k}$ denotes the coordinate at (i,j,k) and $\parallel . \parallel$ describes the Euclidian norm.

In order to reduce the CPU-time consumption of a CFD simulation, the coefficients $(\xi_{i,j,k},\eta_{i,j,k},\zeta_{i,j,k})$ can be computed and stored at the beginning. Indeed, they are only based on the initially undeformed grid and do not vary during the time marching. TFI is a very fast method since its computational complexity linearly scales with the number of volume grid points N_v. However, it requires a certain amount of memory for storing all coefficients.

2.2. *Radial basis function (RBF) interpolation*

Applying RBFs displacements at the coordinate points denoted by $\boldsymbol{d}(\boldsymbol{r})$ are interpolated using the following function

$$\boldsymbol{d}(\boldsymbol{r})=\sum_{n=1}^{N_s}\boldsymbol{\alpha}_n\phi(\parallel \boldsymbol{r}-\boldsymbol{r}_{s_n}\parallel)+\boldsymbol{P}(\boldsymbol{r}). \tag{11}$$

Here, $\boldsymbol{r}_{s_n}$ are the surface nodes at which the displacements are known, ϕ is the form of the RBF being used and $\boldsymbol{P}(\boldsymbol{r})$ is a set of three polynomials. Following the work of De Boer et al. [18] $\boldsymbol{P}(\boldsymbol{r})$ is restricted to linear polynomials so that rigid-body displacements can be recovered. Given a surface displacement vector $\boldsymbol{d}_{s_n}$, $n\in\{1,2,...,N_s\}$ for the three-dimensional case, three different sets of coefficients $\boldsymbol{\alpha}_n$ appearing in Eq. (11) and three linear polynomials of the form

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$\boldsymbol{P} = \boldsymbol{\beta}_0 + \boldsymbol{\beta}_1 x + \boldsymbol{\beta}_2 y + \boldsymbol{\beta}_3 z$ can be obtained by solving the linear symmetric indefinite system

$$\begin{bmatrix} \boldsymbol{d}_{s_1} \\ \boldsymbol{d}_{s_2} \\ \vdots \\ \boldsymbol{d}_{s_{N_s}} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \phi_{s_1 s_1} & \phi_{s_1 s_2} & \cdots & \phi_{s_1 s_{N_s}} & 1 & x_{s_1} & y_{s_1} & z_{s_1} \\ \phi_{s_2 s_1} & \phi_{s_2 s_2} & \cdots & \phi_{s_2 s_{N_s}} & 1 & x_{s_2} & y_{s_2} & z_{s_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_{s_{N_s} s_1} & \phi_{s_{N_s} s_2} & \cdots & \phi_{s_{N_s} s_{N_s}} & 1 & x_{s_{N_s}} & y_{s_{N_s}} & z_{s_{N_s}} \\ 1 & 1 & \cdots & 1 & 0 & 0 & 0 & 0 \\ x_{s_1} & x_{s_2} & \cdots & x_{s_{N_s}} & 0 & 0 & 0 & 0 \\ y_{s_1} & y_{s_2} & \cdots & y_{s_{N_s}} & 0 & 0 & 0 & 0 \\ z_{s_1} & z_{s_2} & \cdots & z_{s_{N_s}} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \vdots \\ \boldsymbol{\alpha}_{N_s} \\ \boldsymbol{\beta}_0 \\ \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \\ \boldsymbol{\beta}_3 \end{bmatrix}. \tag{12}$$

The first N_s vector equations of the above system correspond to N_s surface nodes whose displacements are known. The additional four vector equations originate from the compatibility requirement of a zero displacement vector at the far field given by

$$\sum_{n=1}^{N_s} \boldsymbol{\alpha}_n f(\boldsymbol{r}_{s_n}) = \boldsymbol{0} \tag{13}$$

with $f(\boldsymbol{r}) = \{1, x, y, z\}$. Hence, the system finally is of the order $(N_s + 4) \times (N_s + 4)$ with $\phi_{s_i s_j} = \phi(\| \boldsymbol{r}_{s_i} - \boldsymbol{r}_{s_j} \|)$. Once the values of $\boldsymbol{\alpha}_n$ and $\boldsymbol{P}$ are obtained, Eq. (11) can be used to estimate displacements at the interior nodal points.

In the literature various RBFs have been used to deform a mesh [18, 20, 23, 26, 27]. These basis functions can be divided into two categories: (a) RBFs with compact support and (b) RBFs with global support. It has been noticed that the approximation properties of RBFs with global support are generally better than those with compact support [18, 23]. On the other hand, RBFs with compact support lead to a well-conditioned sparse system and are computationally more efficient [20, 23, 26]. Some popular RBFs are listed in Table I for reference. Note that RBFs with compact support are scaled with a support radius R and are set to zero outside R.

Table I. Radial basis functions.

Name	Abbreviation	$\phi(\boldsymbol{r})$	Support
Thin plate spline	TPS	$\| \boldsymbol{r} \|^2 \ln(\| \boldsymbol{r} \|)$	Global
Wendland's C^0	W C^0	$(1- \| \boldsymbol{r} \|)^2$	Compact
Wendland's C^2	W C^2	$(1- \| \boldsymbol{r} \|)^4(4 \| \boldsymbol{r} \| +1)$	Compact
Continuous TPS C^1	CTPS C^1	$1 + \frac{80}{3} \| \boldsymbol{r} \|^2 -40 \| \boldsymbol{r} \|^3 +15 \| \boldsymbol{r} \|^4 -\frac{8}{3} \| \boldsymbol{r} \|^5 +20 \| \boldsymbol{r} \|^2 \ln(\| \boldsymbol{r} \|)$	Compact
Continuous TPS C^2_a	CTPS C^2_a	$1 - 30 \| \boldsymbol{r} \|^2 -10 \| \boldsymbol{r} \|^3 +45 \| \boldsymbol{r} \|^4 -6 \| \boldsymbol{r} \|^5 -60 \| \boldsymbol{r} \|^3 \ln(\| \boldsymbol{r} \|)$	Compact
Continuous TPS C^2_b	CTPS C^2_b	$1 - 20 \| \boldsymbol{r} \|^2 +80 \| \boldsymbol{r} \|^3 -45 \| \boldsymbol{r} \|^4 -16 \| \boldsymbol{r} \|^5 +60 \| \boldsymbol{r} \|^4 \ln(\| \boldsymbol{r} \|)$	Compact

For solving the resulting system of equations an iterative Krylov subspace based solver called *minimum residual* (MINRES) [41] is preferred. MINRES is particularly suitable since the associated system of equations is symmetric. The system is assumed to be converged if the residual falls below a predefined tolerance ε_c. Note that for small problems (see Section 3.1) a direct solver such as LU

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decomposition is advantageous in terms of required CPU-time. Furthermore, no dependency on the matrix characteristics of different RBFs is observed. However, for practical applications involving large systems of equations, direct solvers become unfeasible.

The solution of the system of equations (12) is computationally intensive. To reduce the computational effort Rendall and Allen [21] introduced a data reduction strategy called the *greedy* algorithm. They discussed three different implementation perspectives for this algorithm and were able to significantly reduce the computational cost. The single point *greedy* algorithm [21] starts with a single boundary point randomly chosen and calculating the solution of this system based on prescribed displacements. It then calculates the largest error between the exact and the approximate system. Based on this largest error and a predefined tolerance ε_g the algorithm updates the coefficients of the reduced system, which may lead to the addition of further points in the already chosen list of so-called *greedy* points. Note that a *greedy* point is added when the corresponding coefficient is updated for the first time. However, the main deficiency of the *greedy* algorithm is that it fails to take the consequence of correcting a single equation of the system on the remaining equations of this system into account.

The *greedy* algorithm was speed up by Wang et al. [23] using a double-edged strategy, where a second search for the largest error is required in an approximately conjugate direction. In the present work a *hybrid greedy* algorithm [21] is implemented. This algorithm starts by carrying out a *greedy* point selection procedure for m times, where m is some predefined number. As noted by Rendall and Allen [21] the *greedy* algorithm visits a point many times and hence the number of executions is not equal to the total number of points chosen. Next the system consisting of so-called active *greedy* points is inverted. The process continues iteratively until the maximum absolute error reduces to the predefined tolerance ε_g.

Here it should be noted that the *greedy* algorithm does not work with some global RBFs, in particular the thin plate spline (TPS). A point reduction algorithm for TPS, if it exists, is not known.

2.3. *Inverse distance weighting (IDW) interpolation*

IDW interpolation, first introduced by Shepard [42], was adopted for mesh deformation by Witteveen [29]. In its generality this method interpolates a function at a given point $\boldsymbol{r}$ based on the weighted average of known values of scattered data points. Witteveen proposed that for a given boundary displacement vector $\boldsymbol{d}_{s_n}$, $n \in \{1, 2, ..., N_s\}$. The displacements at the interior nodes can then be calculated as

$$\boldsymbol{d}(\boldsymbol{r}) = \sum_{n=1}^{N_s} w_n(\boldsymbol{r})\, \boldsymbol{d}_{s_n} \Big/ \sum_{n=1}^{N_s} w_n(\boldsymbol{r}) \ , \tag{14}$$

where $w_n(\boldsymbol{r})$ is the weighting function often expressed as

$$w_n(\boldsymbol{r}) = \| \ \boldsymbol{r} - \boldsymbol{r}_{s_n} \ \|^{-p} \ . \tag{15}$$

Witteveen used different values of $p \in \{1, 2, 3, 4\}$ depending on the nature of the boundary node, i.e., whether the boundary node is dynamic or static as well as translating or rotating.

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Luke et al. [30] modified the IDW method by applying the improved weighting function

$$w_n(\boldsymbol{r}) = A_n \left[\left(\frac{L_{\text{def}}}{\parallel \boldsymbol{r} - \boldsymbol{r}_{s_n} \parallel} \right)^a + \left(\frac{\alpha L_{\text{def}}}{\parallel \boldsymbol{r} - \boldsymbol{r}_{s_n} \parallel} \right)^b \right], \tag{16}$$

where A_n is the area weight of the boundary node n. They also suggested the best values for the parameters a and b as 3 and 5, respectively. L_{def} and α are defined as

$$L_{\text{def}} = \max_{n=1}^{N_s} \parallel \boldsymbol{r}_{s_n} - \boldsymbol{r}_{\text{centroid}} \parallel \qquad \text{where} \qquad \boldsymbol{r}_{\text{centroid}} = \frac{1}{N_s + N_v} \sum_{m=1}^{N_s+N_v} \boldsymbol{r}_m \ , \tag{17}$$

$$\alpha = \frac{\eta}{L_{\text{def}}} \max_{n=1}^{N_s} \parallel \boldsymbol{d}_{s_n}(\boldsymbol{r}) - \boldsymbol{d}_{\text{mean}} \parallel \quad \text{with} \quad \boldsymbol{d}_{\text{mean}} = \sum_{n=1}^{N_s} a_n \, \boldsymbol{d}_{s_n}(\boldsymbol{r}) \quad \text{and} \quad a_n = \frac{A_n}{\sum_{m=1}^{N_s} A_m}. \tag{18}$$

Following Luke et al. [30] the value of the parameter is set to $\eta = 5$. L_{def} denotes the maximum distance of any mesh node from the mesh centroid and α is a parameter, which provides the relative weight of the nearby nodes in comparison to the distant ones. Equation (18) is only one option and the user is free to set a fixed value of α based on his need.

Furthermore, Luke et al. [30] improves the quality of their deformed mesh by interpolating not the displacements, but the rotations and the translations resulting from the splitting of the displacements. However, it is worthwhile to point out that the IDW method loses its *extreme conserving* property once rotations are included. Denoting the introduced rotation matrix $\boldsymbol{M}_s$ and the associated translation vector as $\boldsymbol{t}_s$ for a surface boundary point having the coordinate $\boldsymbol{r}$ in the original mesh, the displacement field $\boldsymbol{d}_s(\boldsymbol{r})$ is defined as

$$\boldsymbol{r} + \boldsymbol{d}_s(\boldsymbol{r}) = \boldsymbol{M}_s \boldsymbol{r} + \boldsymbol{t}_s \ . \tag{19}$$

Following the work of Maruyama et al. [32] an ideal choice to estimate the rotation matrix is to define quaternions at each boundary mesh point. In terms of a quaternion Q defined at a boundary node, Eq. (19) can be rewritten as

$$R_d = Q \, R_u \, Q^* + T \ , \tag{20}$$

where $R_u = [0, \boldsymbol{r}]$, $R_d = [0, \boldsymbol{r} + \boldsymbol{d}]$ and $T = [0, \boldsymbol{t}]$ are the quaternions associated with the undeformed position vector, the deformed position vector and the translation vector, respectively. The subscripts u and d are used to indicate the undeformed and deformed states, respectively. The subscript s used to denote surface boundary points is dropped for the sake of brevity.

Q and T at each boundary node can be determined by using the steps outlined by Maruyama et al. [32] and briefly reproduced in Appendix A. Note that T is coordinate dependent since the rotation is calculated with respect to a local coordinate system (see example in Appendix B). To eliminate this coordinate dependence, a strategy called *transformation division* algorithm was proposed by Maruyama et al. [32] (see Appendix A). For large deformations this *transformation division* process is an absolute necessity as seen in Appendix B. However, if the IDW procedure relies on the last deformed mesh and the new displacements at the boundary nodes are small, this procedure can be omitted (see Appendix B). For example, in case of unsteady FSI-LES computations, very small time

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steps are typically used and thus marginal deformations appear at the boundaries of the structure between two time steps, so that the *transformation division steps* can be skipped.

Once the rotation quaternions and the translation vectors are known at the surface nodes, they can be interpolated to the volume nodes by using the IDW interpolation methodology given by Eq. (14). Note that the components of the rotation quaternions at the volume grid points have to be normalized before finally estimating the displacement vectors.

In the context of block-structured grids and a parallelization strategy based on domain decomposition, the application of IDW is done in the following steps: At the beginning of the simulation all fixed and moving boundary points are marked and saved in two different arrays for optimization purpose. These arrays have to be known by all processors. Then, at each time step the quaternions and translations of all boundary points N_s are evaluated as explained above and shared by all processors. Finally, each processor executes a loop over all volume grid points of its own block evaluating the displacements based on Eq. (14). This explains the fact that the computational effort of IDW scales according to $N_s \times N_v$, since the sum of all volume grid points contained in all blocks is N_v.

3. EVALUATION OF THE ELEMENTARY METHODS BASED ON TEST CASES

In this section the elementary mesh deformation methods described above are evaluated on a few test cases of rising complexity. To provide a global measure of the mesh quality, the harmonic mean value

$$\langle f_{\text{skew}} \rangle = \begin{cases} \left(\frac{1}{N_{cv}} \sum_{k=0}^{N_{cv}-1} \frac{1}{f_{\text{skew}_k}} \right)^{-1} & \text{if } 0 < f_{\text{skew}_k} \leq 1 \text{ for all } k = 0, 1, ..., N_{cv} \\ 0 & \text{if } f_{\text{skew}_k} = 0 \text{ for any } k = 0, 1, ..., N_{cv} \end{cases}$$

is defined following the work of Knupp [43] where f_{skew_k} is the *skew quality metric* of the k-th control volume of the mesh having N_{cv} control volumes. $\langle f_{\text{skew}} \rangle = 1$ will denote perfect orthogonality of the mesh.

For comparison of the computational effort of the different methods the floating-point operations and CPU-times are measured on the same empty machine with identical compiler options. The floating-point operations are evaluated by the open-source tool likwid[†] [44] and given in Giga-FLOP (GFLOP). The CPU-times reported below are scaled by the CPU-time of the fastest successful method leading to relative computational costs nearly independent of the machine.

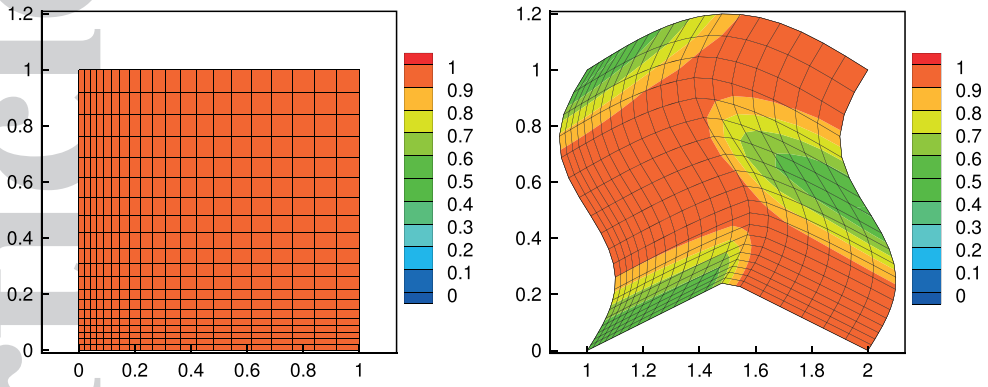
3.1. Problem 1: Deformation of a two-dimensional square region

3.1.1. Description of the problem As depicted in Fig. 1 a square with the center at (0.5, 0.5) and a prescribed deformation on all sides of the domain is considered. Obviously, a sharp angle is generated at the bottom. On the other three sides the deformations are according to trigonometric functions in order to vary the angles and the overall shape. On the left and right boundaries identical

[†]https://github.com/RRZE-HPC/likwid/

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(a) Undeformed mesh. (b) Deformed mesh and f_{skew} contour.

Figure 1. *Problem 1:* Undeformed mesh and TFI deformation.

displacements given by $0.1 \sin(2\pi y)$ are applied, whereas on the top a deformation according to $0.2 \sin(\pi x)$ is used. Moreover, on all boundary points a unit translation along the x-direction is prescribed. The original grid consisting of 21×21 grid points is clustered at the bottom and left wall. The undeformed grid is shown in Fig. 1(a). Computations are carried out for this coarse grid and a finer one with 301×301 grid points. The main objective is to evaluate the performance of TFI, RBF and IDW in a situation, which involves both contraction and relaxation. Also of interest is to see the effect on grid clustering. Furthermore, the orthogonality preserving nature of the schemes and its effect on the interior nodes is analyzed.

3.1.2. TFI mesh deformation For this type of deformation involving a convex undeformed domain, TFI is expected to perform well. The deformed grid obtained by TFI is depicted in Fig. 1(b). The color contours show the *skew quality metric* of the grid f_{skew}. Values of unity define the optimum and decreasing values represent a deterioration of the orthogonality. Apparently, the translation is recovered with ease. However, the orthogonality is deteriorated, which is particularly evident in the lower left portion. Although high *skew metric* values are preserved at a large portion of the interior nodes, overall it is clear that TFI does not maintain the high level of orthogonality of the original grid. Nevertheless, the clustering of the undeformed mesh is preserved. A benefit of the extreme preserving property of TFI is that all grid points are retained inside the boundary without any cross-over. The main advantage of TFI is its very low computational effort (see Table II).

3.1.3. RBF mesh deformation Since the main interest is to compare the number of floating-point operations and the computational costs of various RBFs having both global and local support, a uniform radius of influence of R = 1.5 is chosen for all RBFs with local support. This choice covers the entire domain. For all cases considered, convergence is assumed when the residual falls below $\varepsilon_c = 10^{-12}$ within the MINRES solver.

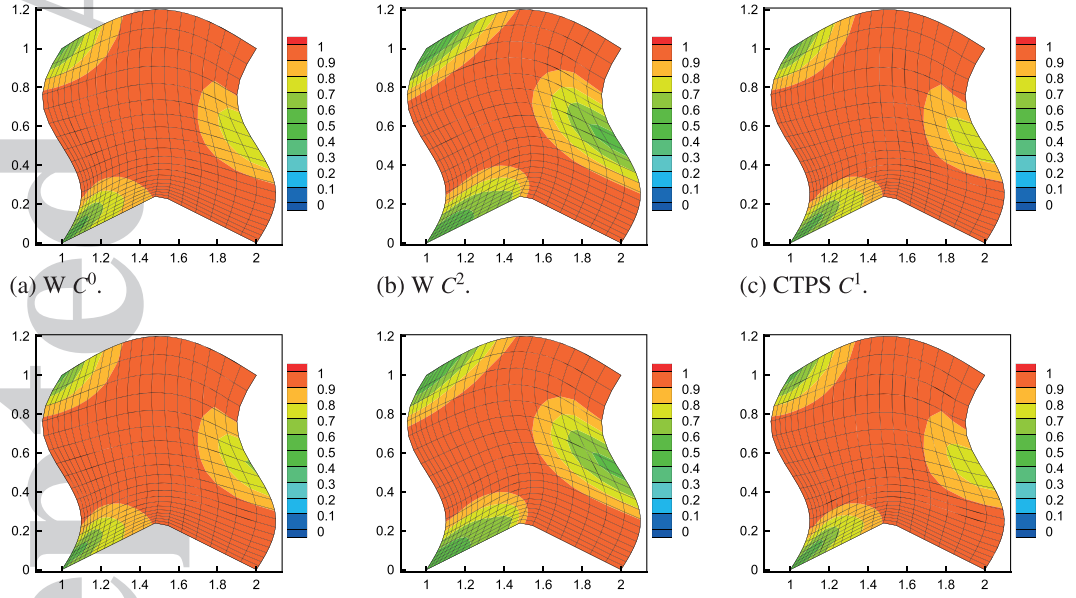
The grid deformed by Wendland's C^0 (W C^0) function is depicted in Fig. 2(a), while Fig. 2(d) uses the same basis function but with the *hybrid greedy* algorithm. Similarly, Figs. 2(b) and 2(e) show the results of Wendland's C^2 (W C^2) function without and with the *greedy* algorithm, respectively. The grids deformed by the continuous thin plate spline (CTPS) C^1 function are displayed in Figs. 2(c) and 2(f), where the *greedy* algorithm is applied in the second case. All RBFs

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Table II. *Problem 1:* Comparison of the floating-point operations (GFLOP), the relative computational costs and global measure of the mesh quality $\langle f_{\text{skew}} \rangle$. All computations done using the fine grid with 301×301 grid points.

Method		GFLOP	Rel. comp. costs	Remark	$\langle f_{\text{skew}} \rangle$
TFI	TFI-2D	0.014	1.00		0.76
RBF	W C^0	10.42	20.04		0.86
	W C^0 with *greedy* (484)	1.54	5.52		0.86
	W C^2	1207.18	1598.56		0.80
	W C^2 with *greedy* (268)	10.22	15.39		0.80
	CTPS C^1	88.95	121.45		0.86
	CTPS C^1 with *greedy* (329)	4.51	9.02		0.86
	TPS	108.92	144.61		0.82
IDW	Without rotation computation	2.42	4.45		0.89
($\alpha = 0.1$)	With 25 *transf. div.* steps	59.34	88.97	Ortho. preserved	0.84
	With 500 *transf. div.* steps	1182.79	1759.77	near boundaries	0.86



(a) W C^0. (b) W C^2. (c) CTPS C^1.

(d) W C^0 with *hybrid greedy* algorithm. (e) W C^2 with *hybrid greedy* algorithm. (f) CTPS C^1 with *hybrid greedy* algorithm.

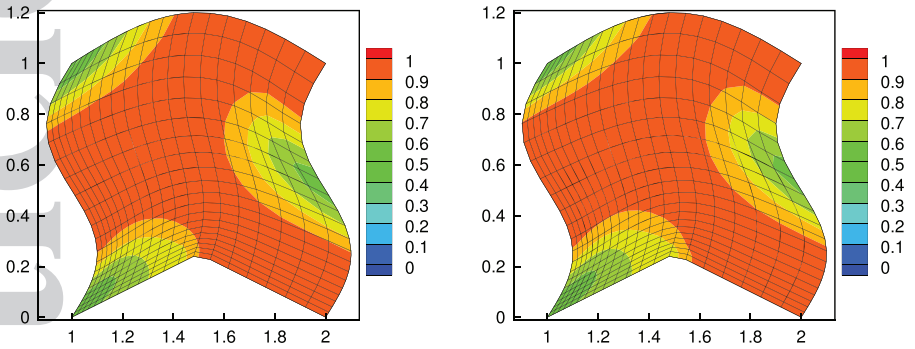
Figure 2. *Problem 1:* RBF mesh deformation with compact support (21×21 grid) and f_{skew} contour.

used here fully recover the translation. Concerning the resulting skewness f_{skew}, Fig. 2 shows that the W C^2 function leads to slightly worse values than for the two other RBFs. That is also visible in Table II based on the global measure of the mesh quality $\langle f_{\text{skew}} \rangle$. This observation is in contradiction to De Boer et al. [18] who identified W C^2 as the best choice for RBFs with local support.

For the *hybrid greedy* algorithm the value of m is set to the number of boundary points and the tolerance limit ε_g defining the maximum absolute error to 10^{-2}. The *hybrid greedy* method works well for all three RBFs with local support. A strong reduction of the floating-point operations and of the CPU-time is observed in Table II. Furthermore, based on Fig. 2 and the global quality measure $\langle f_{\text{skew}} \rangle$ listed in Table II it is also evident that the quality of the mesh deformation only marginally differs between the cases with or without *greedy* selection. To give an idea about the efficiency of

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(a) 21×21 grid. (b) 301×301 grid.

Figure 3. *Problem 1:* RBF mesh deformation with global support (TPS) and f_{skew} contour. (For the fine grid only every 15^{th} grid point in both directions is shown).

this algorithm, the number of points selected by the *greedy* procedure is listed in parentheses in Table II. This has to be compared to the entire number of boundary points $N_s = 1200$ for the fine grid. Among the three local RBFs the hybrid node selection algorithm works best with W C^2.

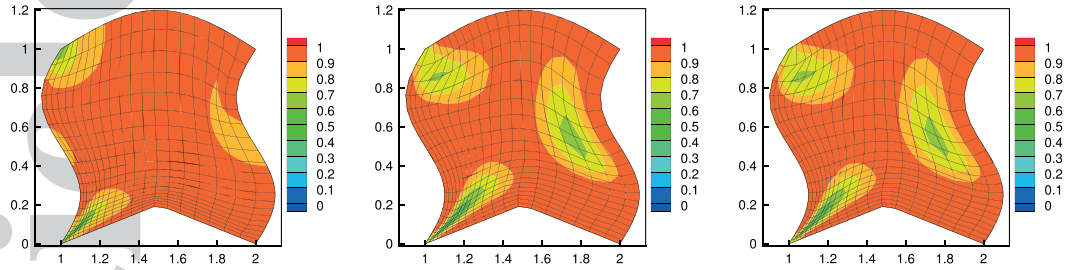
Figure 3 represents the grid deformed using the global TPS basis function providing a comparison of the coarse (21×21) and the fine grid (301×301). Note that in Fig. 3(b) only every 15^{th} grid point in both directions is shown. The quality of mesh deformations based on RBFs is theoretically independent of the number of nodes and their clustering. Thus, the deformed meshes are nearly identical. The same is true for the other RBFs not shown here. Figs. 2–3 and Table II reveal that all four RBFs are overall leading to a reasonable mesh deformation but none is able to maintain the orthogonality particularly at the challenging bottom side. The grid clustering is best preserved by W C^2 and TPS. W C^0 and CTPS C^1 tend to shrink the region of clustered points under severe deformation.

Another interesting issue is the computational effort required by different RBFs. W C^2 takes much more time, whereas W C^0 is computationally the fastest. It is also seen that the effort required by CTPS C^1 and TPS is comparable. The matrices associated with different RBFs have a higher condition number along with more clustering of the eigenvalues near zero for the matrix corresponding to W C^2. This explains its higher computational effort using the MINRES solver. Replacing the iterative solver by a direct LU decomposition the CPU-time required for the solution of the system does not depend on the chosen RBF anymore. The relative computational costs (compare values in Table II) are 12.96, 13.08, 21.07 and 16.98, for W C^0, W C^2, CTPS C^1 and TPS, respectively. Thus, for the present two-dimensional problem the direct solver leads to a drastic gain in CPU-time for the same grid quality.

3.1.4. IDW mesh deformation Finally, the IDW method is applied. Three different cases are considered to understand various effects of IDW. The first case depicted in Fig. 4(a) is obtained by directly interpolating the displacements with the weighting function of Luke et al. [30]. The two other cases (Figs. 4(b) and 4(c)) are based on the interpolation of the translations and rotations (represented by quaternions) with the same weighting function. Here the parameter α is kept fixed

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(a) IDW without rotation. (b) IDW using 25 *transf. div.* steps. (c) IDW using 500 *transf. div.* steps.

Figure 4. *Problem 1:* IDW mesh deformations ($\alpha = 0.1$) and f_{skew} contours.

and set to 0.1 which is especially suited for producing grid deformations of high quality according to Luke et al. [37]. Similar results are obtained with α dynamically predicted based on Eq. (18).

Obviously, IDW is able to recover the translation without any difficulty. Although the clustered nature of the original grid is apparently preserved in Fig. 4(a), it has to be noticed that the orthogonality near the boundaries is not conserved. Computation leading to Figs. 4(b) uses 25 *transformation division* steps. Figure 4(c) is obtained using even 500 *transformation division* steps in order to check the stability of the *transformation division* algorithm. It is found that the algorithm performs in a sound manner with minor improvements of the mesh rendering the algorithm reliable. Both grids make an effort to preserve the orthogonality near all boundaries. That is an advantage in comparison with the case without rotation. Although at some interior points, for example at the lower left side, the orthogonality is not preserved, such a situation is inevitable due to the singular character of the deformation. Thus, the fundamental property of the IDW procedure with rotation is the preservation of the orthogonality and the clustered nature of the grid near boundaries even under strained conditions.

As expected, a substantial increase of the computational effort is noted for the application of a large number of *transformation division* steps in Table II. It is found that the number of floating-point operations for IDW is directly proportional to the number of *transformation division* steps employed.

3.1.5. Summary All methods applied recover the translation perfectly. A comparison of the global skewness $\langle f_{\text{skew}} \rangle$ obtained by TFI, RBF and IDW reveals that all four RBFs deform the mesh slightly more appropriate than TFI. IDW possesses a good orthogonality preserving property near boundaries, better than TFI and RBF. TFI is faster than IDW, which is faster than pure RBFs. However, it is worthwhile to point out that by usage of the *greedy* algorithm the interpolations using compact RBFs become computationally much more efficient and reach a similar level as IDW. For the present small problem size a direct solver speeds up the RBF method significantly, but practical applications necessitate an iterative solver. The IDW computation based on rotation requires many steps of the *transformation division* technique, which decelerates the mesh deformation significantly. However, as discussed in Section 2.3 and shown in Appendix B, this drawback is eliminated for time-resolved predictions relevant for the present area of interest, i.e., LES of turbulent flows.

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A remark concerning the two measures in Table II to evaluate the computational effort has to be added. The number of floating-point operations has the advantage that it is independent of the machine architecture. A clear disadvantage is that it is fully unclear what the compiler makes out of the code and whether some operations may be vectorized or running in parallel. The relative CPU-time, however, comprises the floating-point operations but also other important issues such as the memory access leading to the overall performance of the algorithm. For example, comparing TFI and W C^0 yields a ratio of floating-point operations of about 720. The relative computational costs, however, increase only by a factor of 20, since the RBF algorithm runs more efficiently with a significantly increased number of floating-point operations per second. Therefore for practical purposes, it represents a more reliable measure of the computational effort and thus is solely used below.

3.2. Problem 2: Rotation and deformation of an airfoil

3.2.1. Description of the problem To evaluate the three elementary mesh deformation strategies in situations involving large rotation of a body with a sharp angle, a $90°$ rotation of an elliptic airfoil around the center of the profile is considered. The major and minor axes of the profile have a length of 1 and 0.1, respectively. When this airfoil rotates about the origin of the coordinate system in a unit step, it deforms into a NACA 0012 airfoil of the same chord length c. The initial undeformed O-grid and a reference final grid are both generated using conformal transformations [45]. The grid points at the outer radius ($r = 12\,c$) remain fixed during the deformation process. The grid consists of 401 points equally distributed around the airfoil and 241 points in the wall-normal direction clustered towards the body leading to 802 boundary points.

3.2.2. TFI mesh deformation From a theoretical point of view this type of rotational grid deformation is a major challenge for the TFI interpolation method. Although the grid depicted in Fig. 5(a) looks promising, a closer view in Fig. 5(b) reveals the failure of TFI regarding almost all aspects of this specific test case. Although TFI possesses the extreme preserving property, the grid deformation yields degenerated cells with cross-overs at the two extremities of the airfoil. The orthogonality is destroyed also at boundary points, where no degenerated cells appear. Furthermore, an artificial clustering of grid points on the surface of the airfoil appears, which is unacceptable for any numerical simulation. Hence as expected, TFI is not suitable in case of a large rotation of a structure with sharp edges.

3.2.3. RBF mesh deformation The computations are carried out using six different RBFs. The five compact RBFs have a radius of support R = 15 in order to cover a major part of the domain under investigation. This value offers the best compromise between grid quality and computational effort. For four of these six RBFs further computations using the *hybrid greedy* algorithm are performed.

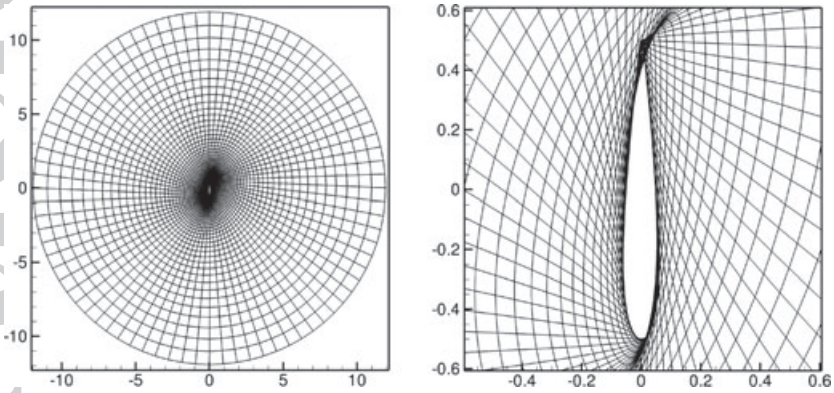
Table III provides an overview of all cases considered including the computational effort required and the global measure of the mesh quality $\langle f_{\text{skew}} \rangle$. The costs are provided as relative values to the corresponding time required for the mesh deformation using IDW[‡]. Obviously, Wendland's

[‡]IDW is taken as reference in this problem, because TFI fails to generate an appropriate grid.

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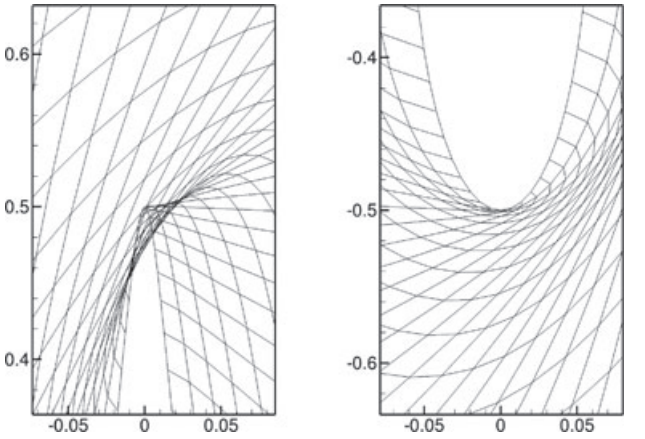
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(a) Complete view. (b) Close view.

Figure 5. *Problem 2:* TFI grid deformation (every 5^{th} grid point in each direction is displayed).



(a) View of trailing edge. (b) View of leading edge.

Figure 6. *Problem 2:* Mesh deformation using W C^0.

C^0 function fails to deform the mesh successfully leading to degenerated cells with cross-overs. A variation of the support radius R between 10 and 20 does not solve this problem. In a further test documented in Table III the convergence criterion for MINRES is decreased from $\varepsilon_c = 10^{-5}$ to 10^{-10}. Obviously, that has no effect and the deformation algorithm continues to produce degenerated cells. To the best of the authors' knowledge such a failure of W C^0 has not been reported in the literature. As visible in Fig. 6 the degenerated cells mainly appear in the vicinity of the trailing edge. This may be attributed to the combination of the rotation and the sudden change of the shape of the trailing edge.

W C^2 is in general able to deform the mesh in a correct manner. The deformed mesh possesses the best quality in the vicinity of the airfoil in comparison with the other RBFs depicted in Fig. 7. It also closely resembles the grid around the NACA 0012 generated by the conformal transformation. However, in regions away from the airfoil the function W C^2 does not reach the same level of grid quality as the other RBFs. As clearly visible in Figs. 7, 8 and 9, the function W C^2 produces more skewed cells compared with the other basis functions investigated. That explains the lowest values of $\langle f_{\text{skew}} \rangle$ found for W C^2.

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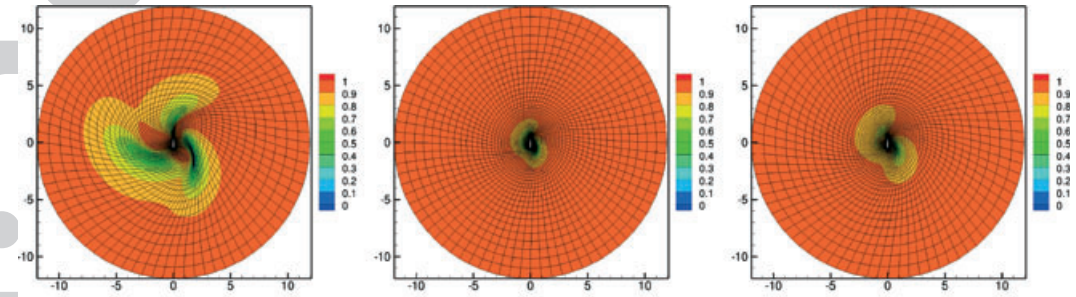
Table III. *Problem 2:* Comparison of relative computational costs (IDW as reference), global measure of the mesh quality $\langle f_{\text{skew}} \rangle$ and remarks concerning the grid quality using RBFs (support radius set to 15) and IDW (*Degenerated cells* correspond to cross-overs of the cells and *bad first cell* to a highly and unacceptably deformed first cell on the airfoil).

Method	ε_g	*Greedy points*	ε_c	Rel. comp. costs Total	*Greedy* part	Remark	$\langle f_{\text{skew}} \rangle$
RBF (W C^0)	-	-	10^{-5}	0.97	-	Degen. cells	0
	-	-	10^{-7}	1.24	-	Degen. cells	0
	-	-	10^{-10}	1.94	-	Degen. cells	0
RBF (W C^2)	-	-	10^{-5}	8.34	-		0.78
	-	-	10^{-7}	100.59	-		0.78
	-	-	10^{-10}	992.50	-		0.78
	10^{-2}	236	10^{-5}	1.70	79.2%	Bad first cell	0
		262	10^{-7}	438.29	99.9%	Degen. cells	0
		273	10^{-10}	4081.59	99.9%	Degen. cells	0
	10^{-3}	252	10^{-5}	6.53	94.2%		0.79
		271	10^{-7}	530.18	99.9%	Degen. cells	0
		291	10^{-10}	6718.52	99.9%	Bad first cell	0
RBF (CTPS C^1)	-	-	10^{-5}	3.84	-		0.86
	-	-	10^{-7}	8.89	-		0.86
	-	-	10^{-10}	28.91	-		0.86
	10^{-2}	430	10^{-5}	4.54	73.1%	Degen. cells	0
		430	10^{-7}	13.20	90.7%	Degen. cells	0
		430	10^{-10}	25.31	95.2%	Degen. cells	0
	10^{-3}	446	10^{-5}	11.42	88.9%		0.86
		447	10^{-7}	46.95	97.3%		0.86
		447	10^{-10}	99.86	98.7%		0.86
RBF (CTPS C^2_a)	-	-	10^{-5}	7.29	-		0.85
	-	-	10^{-7}	45.35	-		0.85
	-	-	10^{-10}	333.39	-		0.85
	10^{-2}	284	10^{-5}	2.79	70.3%	Bad first cell	0
		286	10^{-7}	20.62	95.9%	Degen. cells	0
		275	10^{-10}	36.20	97.8%	Bad first cell	0
	10^{-3}	326	10^{-5}	11.33	91.6%		0.85
		325	10^{-7}	128.03	99.2%		0.85
		318	10^{-10}	508.33	99.8%		0.85
RBF (CTPS C^2_b)	-	-	10^{-5}	8.98	-		0.82
	-	-	10^{-7}	73.96	-		0.82
	-	-	10^{-10}	679.70	-		0.82
	10^{-2}	275	10^{-5}	2.82	71.2%	Bad first cell	0
		307	10^{-7}	113.49	99.2%	Degen. cells	0
		305	10^{-10}	505.83	99.8%	Degen. cells	0
	10^{-3}	301	10^{-5}	15.03	94.0%		0.82
		313	10^{-7}	349.66	99.7%	Degen. cells	0
		328	10^{-10}	4968.41	99.9%	Bad first cell	0
RBF (TPS)	-	-	10^{-5}	4.45	-		0.89
	-	-	10^{-7}	13.29	-		0.89
	-	-	10^{-10}	52.78	-		0.89
IDW (no transf. div.)	-	-	-	1.01	-	α set by Eq. (18),	0.81
IDW (no transf. div.)	-	-	-	1	-	$\alpha = 0.1$,	0.81

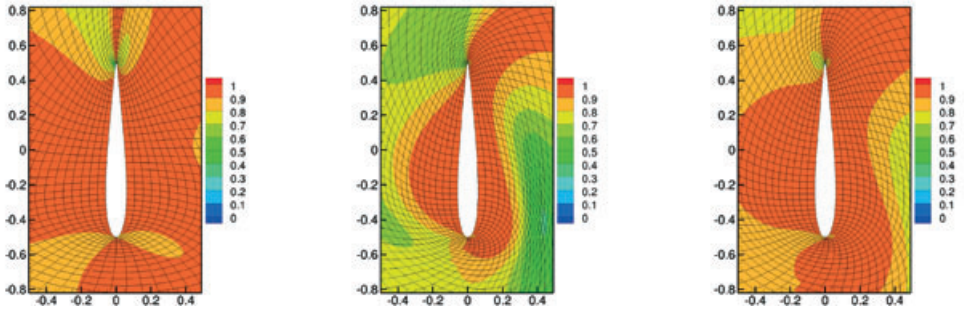
CTPS C^1, CTPS C^2_a, CTPS C^2_b and TPS perform almost in a similar manner and the outcome perceptibly differs from the deformations prescribed by W C^2. For the latter the effect of rotation on the grid has been successfully kept away from the vicinity of the airfoil, whereas in case of the various types of thin plate spline functions the effect of rotation is somehow distributed starting at the wall-nearest grid points and resulting in a skewed grid in the vicinity of the airfoil. It is noteworthy that all basis functions are able to cope with the near-singularity situation close to the trailing edge. In the regions away from the airfoil the CTPS C^1 and TPS basis functions are found

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(a) W C^2 full view. (b) CTPS C^1 full view. (c) TPS over view.



(d) W C^2 close view. (e) CTPS C^1 close view. (f) TPS close view.

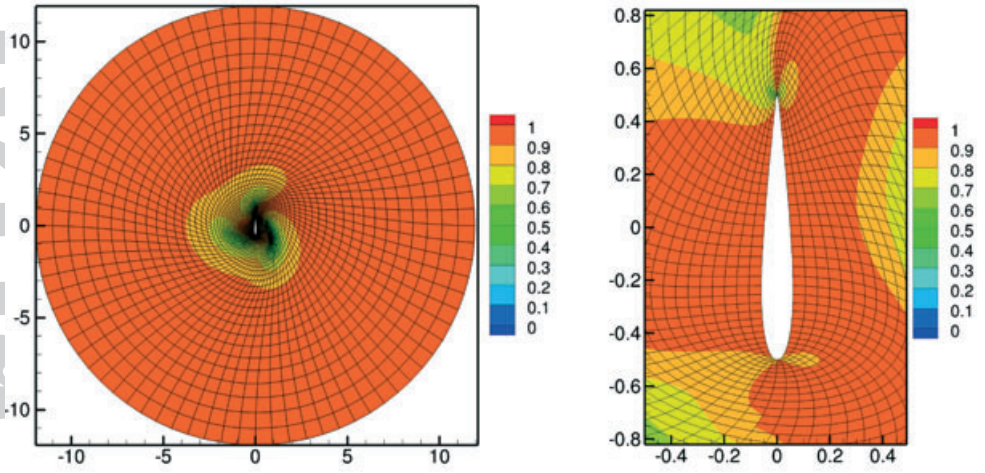
Figure 7. *Problem 2:* Mesh deformation using RBFs (only every 5^{th} grid point along each direction shown).

to perform best as visible based on the *skew quality metric* contours in Figs. 7 to 9 and the values of the global skewness $\langle f_{\text{skew}} \rangle$ in Table III.

To summarize, W C^0 fails to appropriately deform the grid in case of a sharp-edged rotating structure. W C^2 leads to a quasi rigid-body movement of the grid points near the rotating airfoil, whereas the major grid deformation is shifted in the direction of the fixed boundaries. Thus, the quality of the deformed grid suffers in this region leading to more skewed cells. The four RBFs (CTPS C^1, CTPS C^2_a, CTPS C^2_b and TPS) offer the best grid quality. The computational costs of CTPS C^1 and TPS are much lower than for CTPS C^2_a and CTPS C^2_b. W C^2 is the most expensive one. This trend closely resembles the relative costs reported for *problem 1* in Table II and is explained by the ill-conditioned W C^2 matrix. Note that the computational effort with compact RBFs obviously depends on the support radius. Indeed, for RBFs without any data reduction technique the computational effort can be reduced by decreasing the support radius. Since this also deteriorates the grid quality, a compromise on R has to be found.

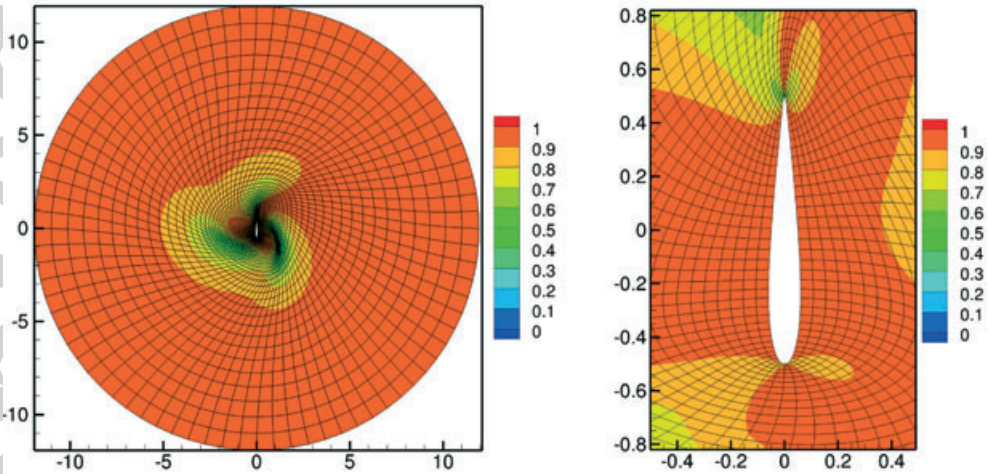
Next, it is of interest to investigate the performance of the *hybrid greedy* algorithm concerning data reduction and computational effort. If ε_g is set to a high value (here $\varepsilon_g = 10^{-2}$), the *greedy* algorithm generates an useless grid with degenerated cells for all RBFs tested. Applying a lower value of $\varepsilon_g = 10^{-3}$, CTPS C^1 and CTPS C^2_a yield an appropriate mesh. However, for W C^2 and CTPS C^2_b only a carefully chosen value of ε_c leads to appropriate deformations without degenerated cells. In most cases of failure the deformed mesh is degenerated at the trailing edge. From a geometrical point of view it is mandatory for an efficient data reduction strategy to choose *greedy* boundary points in the vicinity of this trailing edge. In all computations the initial guess needed for the *greedy* algorithm is on purpose taken at a node of the leading edge. This particularly harsh choice is made in order to see the response of the *greedy* algorithm to situations involving FSI-LES

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(a) Full view. (b) Close view.

Figure 8. *Problem 2:* Mesh deformation using CTPS C_a^2 without and with the *hybrid greedy* algorithm (Since the results in both cases are nearly identical, only one set is shown here. Note that only every 5^{th} grid point along each direction is depicted).



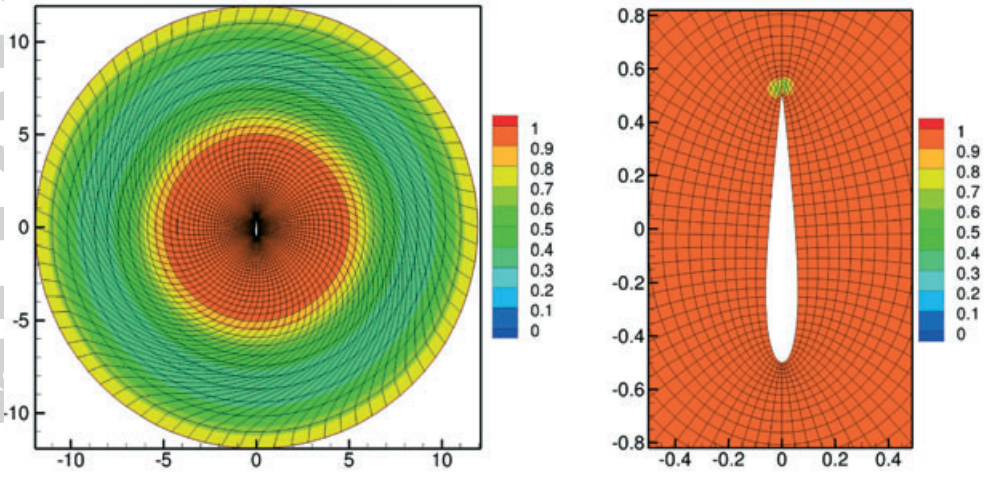
(a) Full view. (b) Close view.

Figure 9. *Problem 2:* Mesh deformation using CTPS C_b^2 without and with the *hybrid greedy* algorithm (only every 5^{th} grid point along each direction shown).

predictions, where the appearance of a sudden singularity can not be ruled out. Nevertheless, if the mesh is not degenerated, it is almost indistinguishable from the deformed mesh obtained without the data reduction technique. Therefore, the corresponding counterparts to Figs. 8 and 9 using the RBFs CTPS C_a^2 and CTPS C_b^2 are omitted here for the sake of brevity.

As can be seen in Table III the *hybrid greedy* algorithm is very time consuming. The procedure iterates many times and is more expensive than the algorithm without any data reduction in most cases. Therefore, the use of a *greedy* procedure often makes sense only for problems with several successive mesh deformations of a nearly identical grid such as encountered in time-dependent FSI problems. In this case the *greedy* algorithm is applied for a certain time step. The chosen *greedy*

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(a) Full view. (b) Close view.

Figure 10. *Problem 2:* Mesh deformation using IDW and α calculated by Eq. (18) (only every 5^{th} grid point along each direction shown).

boundary points are stored and used for many successive deformation cycles without renewal of the chosen boundary points, so that the CPU-time is globally reduced. However, if the grid is strongly deformed, it might be necessary to renew the choice of the boundary points after a while.

3.2.4. IDW mesh deformation Finally, *problem 2* is evaluated using the IDW mesh deformation technique with the value of α calculated by Eq. (18). The design of this interpolation strategy by Luke et al. [30] is specifically targeted towards a method, which can interpolate the rotation with high accuracy. Therefore, IDW theoretically should be able to provide a high-quality grid for this problem. Since the origin is located at the center of the airfoil and the rotation is carried out with respect to this origin, no transformation division technique is employed.

The results for the same initial grid as used before are depicted in Fig. 10. Note that a nearly identical distribution of the *skew quality metric* is found on a much coarser grid evidencing the grid independence of the algorithm. Based on the work of Luke et al. [30] the computations are repeated using a fixed value of $\alpha = 0.1$. Again, no significant variation of the grid properties can be noticed. Furthermore, the computational effort for a fixed or a predicted value of α are almost identical (see Table III). In comparison to the different RBFs without *greedy* algorithm, IDW is at least a factor of 3 faster even in the case of the worst tolerance $\varepsilon_c = 10^{-5}$. For stricter tolerance values ε_c the CPU-time ratio turns out to be even more clearly in favor of IDW. Here it is worthwhile to mention that the computational complexity of both RBF and IDW scales with $N_v \times N_s$ as noted earlier and hence for sufficiently large systems the CPU-times for both RBF and IDW may be isomorphic.

The mesh deformation using IDW is nearly perfect, at least in the vicinity of the airfoil. A comparison with the grid generated by the conformal transformation reveals the effectiveness and significance of IDW for grid deformations involving significant rotation. The deformed grid is nearly orthogonal at all points close to the boundary of the airfoil, including the trailing edge. In regions away from the airfoil the skewness level increases. Thus, it is obvious that the IDW grid deformation

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is well-suited for pushing skewed grid cells away from the rotating body even with sharp edges. This is an important advantage for FSI-LES computations.

3.2.5. Summary Based on the performance of the three methods (TFI, RBF and IDW) it can be concluded that RBF and IDW perform significantly better than TFI. IDW delivers a nearly perfect mesh in the vicinity of the airfoil, whereas all RBFs perform better at a certain distance from the body. Among all RBFs the performance of W C^2 is the best as far as the orthogonality preservation in the vicinity of the body is concerned. Thus, for this test case the outcome is in close agreement with the observations of De Boer et al. [18]. It is worthwhile to mention that the IDW technique without the transformation division procedure takes much less time than any RBF. However, in case of very large systems the computational costs for both may be comparable. Globally, the *greedy* algorithm does not perform very well for all RBFs applied to *problem 2* and shows its limits. This is due to the presence of a sharp edge on the moving body combined with the severe rotation. The use of a limited number of *greedy* boundary nodes implies that the imposed deformation is only correctly represented at these selected boundary points. In order to get an appropriate grid deformation, a sufficient number of boundary points describing the sharp geometry has to be selected. In the original *greedy* algorithm that is not assured. Hence, a possible improvement of the *greedy* algorithm might be to take the first cell height in the error analysis into account, so that more boundary points are taken in highly resolved near-wall regions. Moreover, the *greedy* algorithm is more expensive than the mesh deformation without any data reduction technique and thus only makes sense if the choice of the boundary points can be used several times.

4. HYBRID MESH DEFORMATION METHODS

In order to find a reasonable compromise between the quality of the deformed grid and the computational effort required for the deformation, block-structured grids offer the opportunity to combine the favorable properties of different methods while avoiding their drawbacks as far as possible. The hybrid mesh deformation methodology relies on this block structure and consists of a combination of two elementary mesh deformation algorithms: One for the deformation of the block interfaces and one for the inner points. In the following, the applied hybrid methods are denoted at first by the name of the technique applied for the block interface and in the second place by the name of the method used for the inner points. Since the number of grid points on the block interfaces only scales with $N_v^{2/3}$, major savings can be achieved if expensive techniques such as RBF or IDW are solely applied to the block interfaces and not to the entire grid. These hybrid grid deformation strategies are investigated based on the experience gained by the evaluation of the three elementary strategies. Furthermore, they are designed in such a manner that important advantages, i.e., low computational effort and high *skew quality metric* preservation are combined.

Since in FSI-LES predictions the CPU-time consumption is of paramount importance, it is anticipated that grid deformations based on RBFs applied to the entire domain are not feasible. However, taking the favorable mesh quality preservation property of RBF into account, it can be concluded that RBF may be of utmost importance when applied in a sub-domain. In such a situation a much smaller system of equations has to be inverted and therefore a huge amount of CPU-time can

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be saved. Furthermore, it has been noticed that for convex domains without substantial rotation TFI performs well and is very fast. The strength of IDW is its capacity to accurately capture the rotation and to preserve the orthogonality in the vicinity of the body, whereas it is found to be fall behind the others for a clustered grid. Based on these observations the following two hybrid methodologies are proposed:

- **IDW-RBF:** IDW is applied at the block interfaces to preserve more or less the orthogonality. For the inner points of each block it is exploited that RBF achieves a high grid quality. Furthermore, since the sub-domains are processed in parallel by different processors, the system of equations related to the RBF implementation of the inner points may not turn out to be very costly.
- **IDW-TFI:** Again IDW is applied at the block interfaces to retain the advantages of IDW, whereas TFI is used for the displacements of the inner points. Once the orthogonal character of the mesh is maintained at the interfaces, it is exploited that TFI produces high-quality grid deformations of the inner nodes very fast.

Both hybrid variants bypass the disadvantages of IDW for clustered meshes (see *problem 1*) to a large extent, since it is only used at the block interfaces and not for the nodes in the vicinity of the interface.

Additionally, a hybrid RBF-TFI methodology is used for comparison purposes. Since the solution of a huge system of equations involving all boundary nodes is required, this method is certainly not very interesting due to the required CPU-time.

For the sake of brevity and based on the experiences on the previous test cases considered, the application of the RBF mesh deformation technique is restricted to the TPS basis function with global support not taking the *greedy* algorithm into account. The fact that TPS leads to reasonable mesh deformations for *problem 1* and *2* which are among the best motivates this choice. The predictions based on IDW are carried out using quaternions but without transformation division steps. These are not required here since the origin of the coordinate system lies at the center of rotation and all boundary points of the structure undergo the same rotation.

4.1. Description of the problem

The proposed hybrid algorithms and their elementary counterparts are tested in a situation with substantial rotation and deformation. Furthermore, the analysis is extended to the three-dimensional case and for deformations, which resemble the most important features of a standard test case for FSI-LES simulations [1, 2, 3]. In analogy to most FSI-LES cases the integration domain is divided into several sub-domains allowing to parallelize the computation based on domain decomposition with explicit message passing.

The undeformed block-structured grid along with the partitioning into ten blocks is depicted in Fig. 11. The geometrical setup is as follows: A square cylinder with a unit edge is placed in a plane rectangular channel with a height of 13 units, a length of 12 units and a width of 0.5 units. A splitter plate with a length of 5 units and a thickness of 0.25 units is attached to the square cylinder. The center of the cylinder coincides with the origin of the coordinate system. Note that the associated flow problem is a three-dimensional generalization of the nominally two-dimensional geometric configuration.

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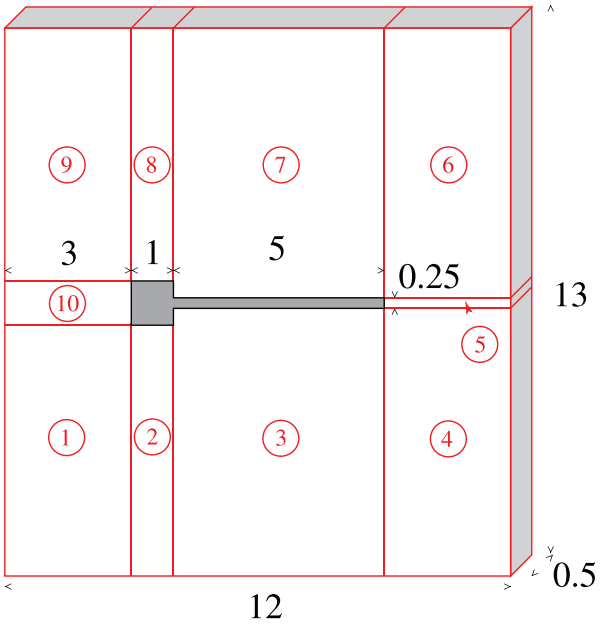


Figure 11. Sketch of the topology of the grid structure with the splitting into ten blocks.

In this study the cylinder is exposed to a rotation of 30° around the center. Furthermore, the splitter plate is deformed by a trigonometric function according to $0.2\sin(2\pi x)$, where x is the distance from the point of attachment. Hence, the splitter plate is subjected to two different kinds of deformations. The grid possesses 8 cells in the spanwise direction. Each of the spanwise planes is further sub-divided into square cells of dimension $1/16$ units leading to 363,033 nodes (= $193 \times 209 \times 9$ without subtracting the nodes inside the structure) in the entire domain under consideration.

4.2. Results of the elementary and hybrid methods

The results obtained by the elementary and hybrid methods are depicted in Fig. 12. For the grid deformed by TFI-TFI a C^1-discontinuity is observed at the interfaces between the blocks (see Fig. 12(a)). This grid discontinuity is not taken into account by the *skew quality metric*. A jump in the f_{skew} level appears but its value remains high. Therefore, the harmonic average $\langle f_{\text{skew}} \rangle$ reported in Table IV is found to be high. That shows that $\langle f_{\text{skew}} \rangle$ is not sufficient for the evaluation of the overall grid quality. This sudden change at the block interface is attributed to the differences in the interpolating polynomials used in distinct sub-domains. Thus as noted for *problem 1*, the pure TFI algorithm is unable to maintain the orthogonality and is inappropriate for FSI-LES predictions.

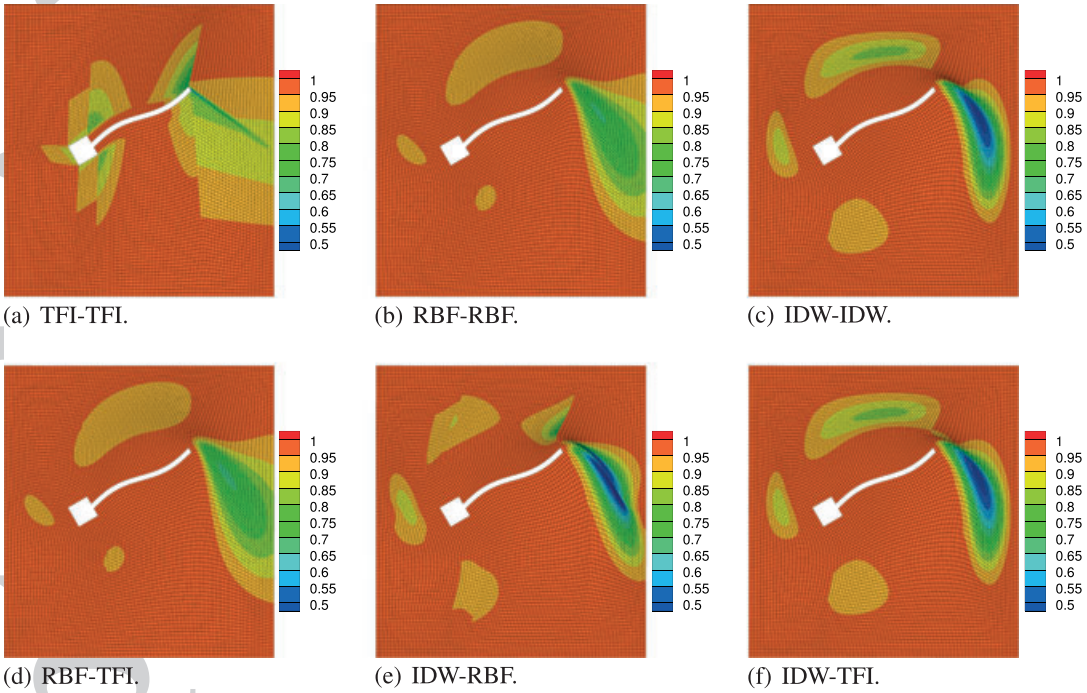
As visible in Fig. 12(b) the RBF-RBF deformation strategy works quite well. The orthogonality is preserved in the deformed grid. Although at the far field, especially in the downstream direction, some nodes are skewed, their *skew quality metric* is still high and a reasonable deformation of the grid leading to an overall high-quality grid can be observed. As usual IDW generates a mesh, which preserves the orthogonality in the vicinity of the deformed splitter plate to a great extent. However, in the region downstream of the splitter plate some cells with a relatively high skewness are observed in Fig. 12(c). This is consistent with earlier observations that the IDW interpolation pushes the skewness away from the deforming body.

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Table IV. Comparison of relative computational costs and global measure of the mesh quality $\langle f_{\text{skew}} \rangle$ for elementary and hybrid mesh deformation strategies applied to a block-structured grid.

Method	Rel. comp. costs	Remark	$\langle f_{\text{skew}} \rangle$
TFI-TFI	1	Block interfaces strongly non-smooth	0.96
RBF-RBF	2013.71	Overall high grid quality	0.96
IDW-IDW	19.71	High grid quality near moving boundaries	0.96
RBF-TFI	2000.78	Overall high grid quality	0.96
IDW-RBF	418.00	Overall good grid quality	0.95
IDW-TFI	**5.46**	Best compromise of quality & CPU-time	0.95



(a) TFI-TFI. (b) RBF-RBF. (c) IDW-IDW.

(d) RBF-TFI. (e) IDW-RBF. (f) IDW-TFI.

Figure 12. Deformed mesh and *skew quality metric* contours computed by the three elementary and the three hybrid grid deformation methods.

The two hybrid versions involving IDW for the block interfaces (see Fig. 12(e) and 12(f)) lead to similar grid qualities as obtained by the pure IDW method. Similarly the hybrid RBF-TFI method (see Fig. 12(d)) delivers a grid of almost the same quality as the pure RBF technique. Thus, the lesson which could be learned from the present results is that the method applied for the deformation of the block interfaces is of major importance for the grid quality achieved, whereas the method used for the displacements of the inner grid points plays a minor role.

The corresponding relative computational efforts related to all six computations are given in Table IV. Similar to the previous cases huge differences are already found between the pure elementary methods. As expected, pure TFI produces deformed meshes with the lowest computational effort. TFI is about 2000 times faster than pure RBF and still about 20 times faster than pure IDW. As visible in Table IV the major disadvantage of any RBF implementation is the large computational effort required consumed. As anticipated a combination of RBF and TFI only slightly reduces the computational costs in comparison with the pure RBF algorithm, since

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Figure 13. Air inflated structure in Gentofte (Denmark).

the time-consuming determination of the coefficients of the basis functions are required in both variants. Hence, this hybrid technique is also not suitable. Combining IDW with RBF reduces the computational costs of the pure RBF method significantly. However, it is quite slow compared to the IDW-TFI method. The hybrid IDW-TFI method is very promising since it only takes about one quarter of the computational effort of the pure IDW technique while a grid of comparably high quality is achieved.

5. APPLICATION OF THE HYBRID METHOD TO A REAL-LIFE FSI PROBLEM

In the previous section the hybrid IDW-TFI method proves its ability to deform the mesh conserving a high quality with reasonable computational effort. This hybrid method is now evaluated based on a real-life FSI problem (see Fig. 13): A flexible membranous hemisphere (diameter $D = 1.5 \times 10^{-1}$ m) is attached on a flat plate and placed in an approaching thick turbulent boundary layer ($\mathrm{Re} = \rho_{\mathrm{air}}\, D\, U_\infty \,/\, \mu_{\mathrm{air}} = 50{,}000$) (see Fig. 14). This FSI-LES case is derived from the turbulent flow past a rigid hemisphere [46] with two modifications: First, the grid is coarsened and consists of about 4.5×10^6 control volumes. Second, the deformation of the hemisphere is imposed based on the following movement of the surface in spherical coordinates $(r_{\mathrm{FSI}}, \theta, \phi)$:

$$r_{\mathrm{FSI}} = A_{\mathrm{def}} \frac{z}{D/2} \sin(\omega_t\, t)\, \cos(4\,\phi) \quad \text{with} \quad A_{\mathrm{def}} = 1.1 \times 10^{-2}\ \mathrm{m} \quad \text{and} \quad \omega_t = 2\pi\, 10^3\ \mathrm{rad\ s^{-1}}.$$

Hence, it presently describes only a one-way coupled simulation which, however, is sufficient here for the evaluation of the hybrid technique. Note that this imposed movement is not physical. However, the chosen parameters leads to severe deformations in critical parts of the geometry: On the top of the hemisphere, on the sides and at the junction between the membrane and the bottom wall. The amplitude of the deformation A_{def} is set to a value much larger than what is expected in a real physical case in order to stress the robustness of the proposed hybrid method. A snapshot

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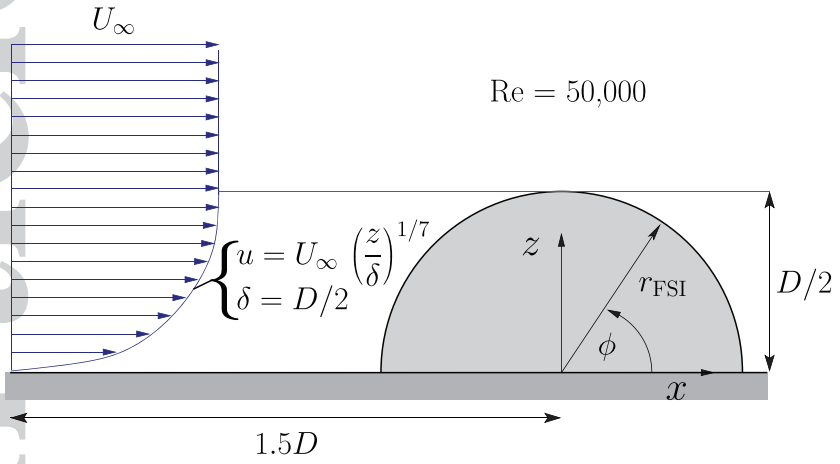


Figure 14. Surface-mounted hemisphere within a turbulent boundary layer.

of the geometry and the flow is shown in Fig. 15 at the time instant of the maximum deformation ($t = 2.5 \times 10^{-4}$ s). The solver (FASTEST-3D) and the techniques used to compute this FSI-LES problem are described in Breuer et al. [1] and were validated by complementary experimental-numerical FSI investigations [2, 3].

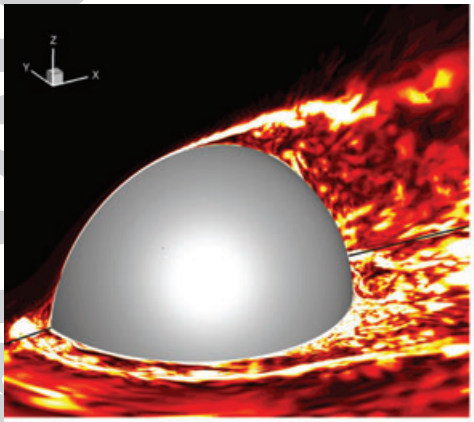
Figures 15(a) and 15(b) show the vorticity magnitude of the turbulent flow around the rigid and the flexible hemisphere in two slices ($y/D = 0$, $z/D = 0.014$). Non-physical numerical oscillations are not detected, which is a first rough indicator for a grid of high quality. To evaluate the grid quality, the *skew quality metric* is plotted in the slice $y/D = 0$ in Figs. 15(c) and 15(d). The undeformed grid possesses a high quality. The first cells are all perfectly orthogonal to the wall and the *skew quality metric* is close to unity in the whole mesh. After the deformation the quality of the grid is slightly altered in several regions around the hemisphere (see Fig. 15(d)). However, the *skew quality metric* remains high in these areas (> 0.9). Furthermore, it is important to mention that the high grid quality is maintained near the moving membrane (see Fig. 15(e)) and near the bottom wall (see Fig. 15(f)). As explained in Section 3.2.4 this is one of the main advantages of a method based on IDW: The deformation is pushed away from the FSI interface and the node distribution remains unchanged in the wall vicinity. This particularity ensures that the boundary layer can be correctly resolved by the LES solver.

The junction between the bottom wall and the flexible hemisphere is a critical region during the mesh deformation based on IDW. In case of a wall-resolved LES the mesh is very fine here. When the structure inflates, the area is reduced and degenerated cells can appear. In case of the hybrid IDW-TFI method, IDW is only applied on the block boundaries, which are far away from the FSI interface. Therefore, this is not an issue for the present flexible hemisphere. The mesh does not contain any bad cells and the orthogonality is more or less conserved also in this critical region.

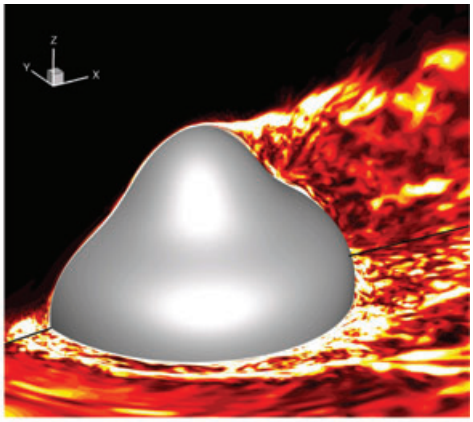
Concerning the CPU-time consumption the hybrid IDW-TFI method is about 6.7 times faster than the pure IDW method for exactly the same grid quality (see Table V). Note that the corresponding ratio of floating-point operations is about 7.3. Thus the pure IDW methods needs more floating-point operations but runs slightly more efficiently than the hybrid IDW-TFI method. No comparison with the elementary TFI and RBF methods is possible. The former leads to cross-overs, whereas for

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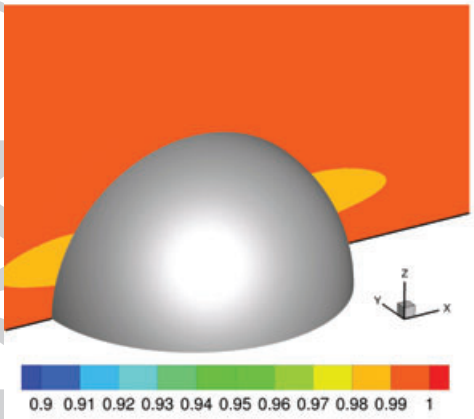
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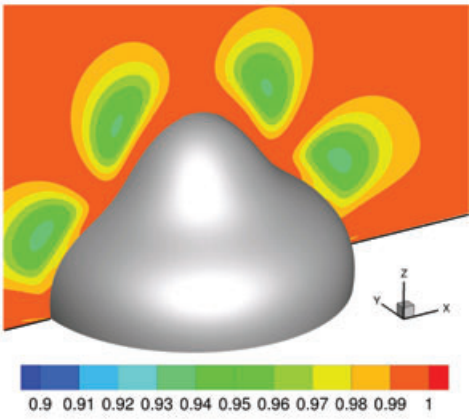
(a) Full view of the rigid hemisphere: Slices ($y/D = 0$, $z/D = 0.014$) with the vorticity magnitude.



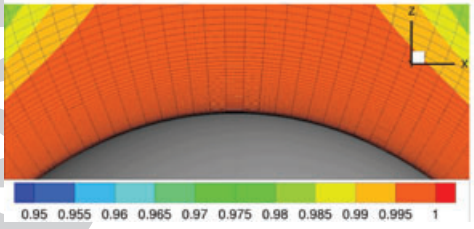
(b) Full view of the deformed hemisphere: Slices ($y/D = 0$, $z/D = 0.014$) with the vorticity magnitude.



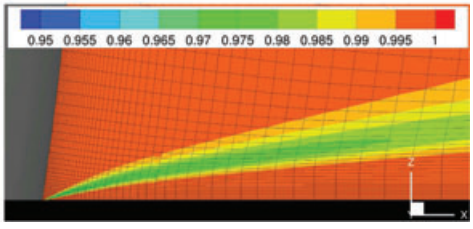
(c) Full view of the rigid hemisphere: Slice ($y/D = 0$) with the grid *skew quality metric*.



(d) Full view of the deformed hemisphere: Slice ($y/D = 0$) with the grid *skew quality metric*.



(e) Close view of the deformed grid at the top: Slice ($y/D = 0$) with the grid *skew quality metric*.



(f) Close view of the deformed grid at the bottom: Slice ($y/D = 0$) with the grid *skew quality metric*.

Figure 15. Snapshot of the flexible hemisphere in a turbulent flow predicted by LES.

the latter the system of equations can not be solved in a reasonable time due to the large number of boundary points.

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Table V. Comparison of the floating-point operations (GFLOP for one time step) and of the relative computational costs for elementary and hybrid mesh deformation strategies applied to a real-life FSI problem.

Method	GFLOP	Rel. comp. costs	Remark
TFI-TFI	-	-	Cross-overs
RBF-RBF	-	-	System too large to be solved
IDW-IDW	9675	6.68	High grid quality near moving boundaries
IDW-TFI	1328	1	Best compromise of quality & CPU-time

6. CONCLUSIONS

Three elementary mesh deformation methods are thoroughly investigated regarding their responses to situations involving substantial deformations. As expected, TFI is able to deform the grid quite fast, but is incapable of dealing with substantial rotation. Moreover, in case of a block-structured grid the deformed mesh is not smooth at the block interfaces. Nevertheless, in convex domains with prescribed deformations on the boundaries, its performance is reliable, although some loss of orthogonality is observed. RBF on the other hand produces a mesh of high quality in almost all situations, but is computationally expensive. Direct solvers for RBF are faster than iterative ones but limited to small problems. Thus, the latter is preferred for general utilization. At a glance of the test cases studied, the TPS basis function with global support and Wendland's C^2 function with local support are the best choices. W C^2 is able to preserve the orthogonality, but is time-consuming. TPS represents a good compromise between quality and speed. A well-thought-out use of IDW is also an attractive alternative guaranteeing a reasonable computational effort. An IDW method split into a rotational and translational deformation maintains a perfect orthogonality in the near-wall region, which is mandatory for LES. However, it requires to set many problem-dependent parameters, where unfavorable choices can lead to a degenerated mesh with cross-overs. The IDW grid deformation relying on a rotation and a translation step introduces a coordinate dependence. To solve this problem the time-consuming *transformation division steps* can be applied. However, for FSI-LES simulations involving small time steps and thus small relative displacements between subsequent time steps, IDW can renounce this additional procedure. For this purpose is has to be implemented in such a manner that it relies on the last deformed grid mimicking the *transformation division steps* and thus eliminating the coordinate dependence and the high computational costs.

Hybrid methods represent an appropriate compromise between quality requirements and computational effort. Since for block-structured grids the deformation consists of two steps (deformation of the block interfaces [$\sim N_v^{2/3}$] and displacement of the inner points [$\sim N_v$]), in principle six combinations of TFI, RBF and IDW are possible. Since the application of TFI for the block interfaces in not appropriate and the hybrid method RBF-IDW is canceled owing to the expected high effort, three possible hybrid variants remain: IDW-TFI, IDW-RBF and RBF-TFI. These are analyzed based on a realistic FSI scenario.

It is found that the deformation of the block boundaries plays the major role for the quality of block-structured grids. The block boundaries of a typical FSI-LES prediction are best adapted by IDW since it correctly captures the rotation of a deforming body. Once these block boundaries are deformed in an efficient manner, the deformation of the inner mesh is less critical and can

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be performed either by TFI or RBF. In case of substantial deformations RBF is favored for the inner deformation, but even in conjunction with the *greedy* algorithm the computational effort is still extremely high, which renders it unfeasible for a regular usage within each time step or even each FSI sub-iteration step of a coupled FSI-LES prediction. In many cases the application of TFI within the second step of the hybrid algorithm leads to deformed grids of high quality, while the CPU-time requirements stay at a reasonable level. Based on these arguments the hybrid IDW-TFI methodology is clearly favored and additionally tested based on a real-life FSI problem of a hemispherical membranous dome with an imposed time-dependent movement. It turns out that the proposed hybrid IDW-TFI method appropriately deforms the grid maintaining the original distribution and the orthogonality of the cells in the near-wall region. Thus, it generates high-quality grids in a reasonable time as mandatory for FSI-LES.

ACKNOWLEDGMENTS

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A. DECOMPOSITION OF THE DISPLACEMENTS INTO ROTATION (MODELED WITH QUATERNIONS) AND TRANSLATION

The displacement at each boundary node can be decomposed into a rotation (quaternion Q) and a translation T. Q and T can be determined by using the steps outlined by Maruyama et al. [32]:

1. Determine unit normal vectors $\hat{\boldsymbol{n}}_u$ and $\hat{\boldsymbol{n}}_d$ at each surface grid point corresponding to the undeformed and deformed mesh, respectively. This can be accomplished in two steps. The first step is to represent the coordinates of the neighboring nodes with respect to the node under consideration (green node in Fig. 16). These vertices are now denoted $R'_u = [0, \boldsymbol{r}'_u]$ or $R'_d = [0, \boldsymbol{r}'_d]$ for the undeformed and deformed mesh, respectively. A schematic representation for one point can be found in Fig. 16(a) and 16b, respectively. The second step is to predict the unit normals to the face corner position vectors and to take the average leading to the computation of $\hat{\boldsymbol{n}}_u$ and $\hat{\boldsymbol{n}}_d$, respectively.
2. The first quaternion Q_1 is then defined as

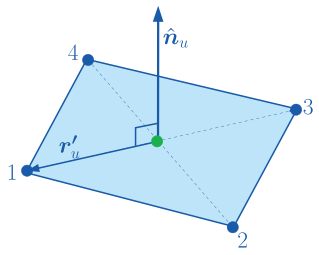
$$Q_1 = \left[\cos\frac{\theta_1}{2}, \hat{\boldsymbol{a}}\sin\frac{\theta_1}{2}\right], \quad \text{with} \quad \theta_1 = \cos^{-1}(\hat{\boldsymbol{n}}_u.\hat{\boldsymbol{n}}_d), \quad \hat{\boldsymbol{a}} = \frac{\hat{\boldsymbol{n}}_u \times \hat{\boldsymbol{n}}_d}{\| \hat{\boldsymbol{n}}_u \times \hat{\boldsymbol{n}}_d \|}. \tag{21}$$

The quaternion Q_1 represents the change of the surface orientation during deformation and aligns $\hat{\boldsymbol{n}}_u$ to $\hat{\boldsymbol{n}}_d$ as depicted in Fig. 16(c).

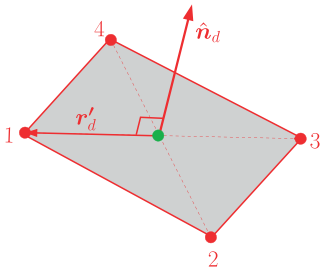
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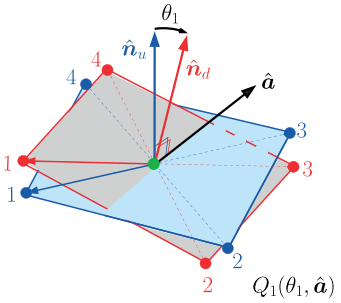
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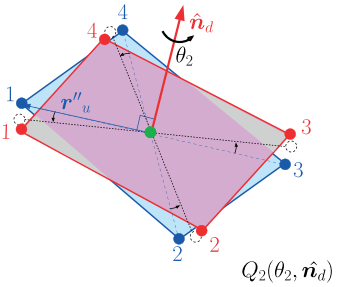
(a) Unit normal of the undeformed mesh.



(b) Unit normal of the deformed mesh.



(c) Computation of the first quaternion Q_1.



(d) Computation of the second quaternion Q_2.

Figure 16. Representation of the procedure to obtain quaternions at a surface node having four neighboring nodes.

3. Now the position vectors of the undeformed face are updated to $R''_u = [0, \boldsymbol{r}''_u]$ using the relation

$$R''_u = Q_1 R'_u Q_1^*. \tag{22}$$

4. Next, it is necessary to rotate the updated undeformed surface in order to align it as good as possible with the corners of the deformed face. The corresponding rotation axis is $\hat{\boldsymbol{n}}_d$ and the corresponding angle θ_2 can be calculated as follows:

$$\theta_2 = \frac{1}{N_{vrtx}} \sum_{n=1}^{N_{vrtx}} \pm \cos^{-1} \left[\frac{\boldsymbol{r}''_u . \boldsymbol{r}'_d}{\| \boldsymbol{r}''_u \| \| \boldsymbol{r}'_d \|} \right], \tag{23}$$

where N_{vrtx} is the number of adjacent vertices. This process can be seen in Fig. 16(d). Thus, a second quaternion Q_2 which represents the rotation due to the cell shape modification during the mesh deformation procedure can be defined as

$$Q_2 = \left[\cos \frac{\theta_2}{2}, \hat{\boldsymbol{n}}_d \sin \frac{\theta_2}{2} \right]. \tag{24}$$

5. Finally, the quaternion Q is attained by using the composition of Q_1 and Q_2,

$$Q = Q_2 \, Q_1. \tag{25}$$

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Once the rotation quaternion Q is determined at a boundary node, the translation quaternion $T = R_d - QR_uQ^*$ can be calculated following Eq. (20). However, this translation vector is coordinate dependent since the rotation has been calculated with respect to a local coordinate system. To eliminate this coordinate dependence, a strategy called *transformation division* algorithm has been proposed by Maruyama et al. [32]. This algorithm consists of repeating the identical elementary process m times. Each iteration of this elementary transformation is composed of a rotation about the same axis but with an incremental angle θ/m, where θ is the angle of rotation prescribed by the quaternion Q. This is followed by the corresponding calculation of the translation vector. The entire procedure can be summarized as

6. Using $Q = [\cos(\frac{\theta}{2}), \hat{\boldsymbol{b}}\sin(\frac{\theta}{2})]$ one computes

$$Q^{1/m} = \left[\cos\left(\frac{\theta}{2m}\right), \hat{\boldsymbol{b}}\sin\left(\frac{\theta}{2m}\right)\right]. \tag{26}$$

The corresponding calculation of the translation vector is based on the relation

$$T_m = \mathcal{M}^{-1}(R_d - \mathcal{N}R_u), \tag{27}$$

where $\mathcal{N}$ is the 3×3 element matrix corresponding to the quaternion Q and

$$\mathcal{M} = \sum_{i=0}^{m-1} \mathcal{N}_{1/m}^{i}. \tag{28}$$

B. COORDINATE DEPENDENCY OF IDW AND TRANSFORMATION DIVISION STEPS

In order to illustrate the coordinate dependency of the interpolated quaternions used for IDW leading to different grid deformations for an identical structural movement, the simple test case of a bending beam in a rectangular region is considered. Furthermore, this setup allows to discuss the subsequent strategy developed to avoid this phenomenon.

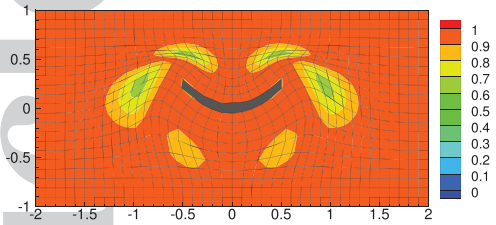
Assuming a two-dimensional problem, a beam of unit length is immersed in a rectangle of dimension 4×2. Computations are carried out using two different grids, a coarse mesh of size 41×22 and a finer one of size 401×202. Since the beam is immaterial, its thickness has been set to one grid spacing in y-direction. The beam is deformed by prescribing a parabolic displacement $y = x^2$, where x is the distance from the center of the beam.

In IDW all the displacements are determined in terms of a global coordinate system, but the quaternions are calculated locally [32]. This leads to a situation where the translation vector, computed at a surface point, depends on its global position, whereas the rotation, expressed by quaternions, does not. Hence, the displacements of the interpolated volume grid points loses symmetry.

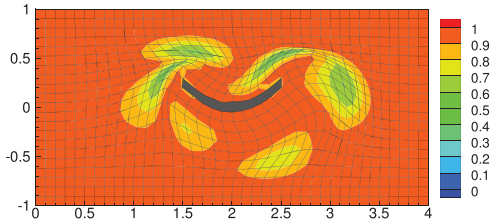
The so-called *transformation division* procedure introduced by Maruyama et al. [32] intends to make this interpolation independent of the coordinate system by dividing the deformation into small steps. Thus, it is a finite implementation of an essentially infinite procedure. In case of an unsteady computation it is found that with a small time step the deformations are most often small in each

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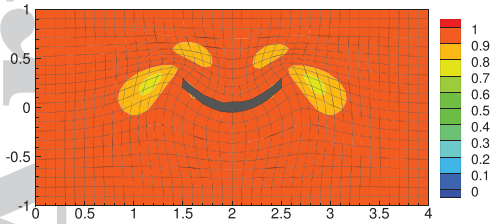
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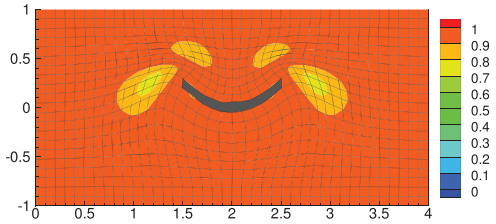
(a) IDW with origin located at the center of the mesh.



(b) IDW with origin located away from the center of the mesh.



(c) IDW with origin located away from the center of the mesh but based on an unsteady computation.



(d) IDW with origin located away from the center of the mesh but using the *transformation division* procedure.

Figure 17. Deformed mesh of the beam and *skew quality metric* computed by IDW with or without the *transformation division* procedure.

step. Hence the *transformation division* technique may not be necessary. To validate this procedure, the problem is tackled as a steady and an unsteady process, where the latter is completed in unit non-dimensional time.

In Fig. 17 the deformed mesh and its *skew quality metric* predicted by various strategies are presented. Figures 17(a), 17(b) and 17(c) are obtained without using the *transformation division* technique. In Fig. 17(a) a symmetric deformation is visible because the origin of the coordinate system is located at the center of the geometry. However, a non-symmetric grid is predicted if this origin is shifted away from the center (see Fig. 17(b)). In Fig. 17(c) a time-marching strategy is used dividing the entire deformation into 25 time steps. Obviously, this leads to a symmetric mesh. The *transformation division* procedure is applied in the fourth case shown in Fig. 17(d). Again 25 steps similar to the time-marching procedure are used. Obviously the results obtained by the time-marching strategy and the *transformation division* algorithm are indistinguishable from each other. A particularly interesting observation that requires attention is that neither Fig. 17(c) nor Fig. 17(d) matches with Fig. 17(a). Both procedures relying on a splitting of the entire deformation into a finite number of sub-steps yield a higher grid quality than the original procedure. Thus, it can be concluded that making the node displacement procedure independent of the coordinate system is fundamental.

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ON GENERALIZED WEAVING FRAMES IN HILBERT SPACES

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DEEPSHIKHA AND P.K. DAS

ABSTRACT. Generalized frames (in short, g-frames) are a natural generalization of standard frames in separable Hilbert spaces. Motivated by the concept of weaving frames in separable Hilbert spaces by [**1**] in the context of distributed signal processing, we study weaving properties of g-frames. Firstly, we present necessary and sufficient conditions for weaving g-frames in Hilbert spaces. We extend some results of [**1, 6**] regarding conversion of standard weaving frames to g-weaving frames. Some Paley-Wiener type perturbation results for weaving g-frames are obtained. Finally, we give necessary and sufficient conditions for weaving g-Riesz bases.

1. Introduction. Frames in Hilbert spaces were originally introduced by Duffin and Schaeffer [**13**] in 1952 in the context of nonharmonic Fourier series and popularized in 1986 by Daubechies, Grossmann and Meyer [**9**]. Frames are basis-like building blocks that span a vector space but allow for linear dependency, which is useful for reducing noise and finding sparse representations, spherical codes, compressed sensing, signal processing, wavelet analysis, etc., see [**5**]. Motivated by a problem regarding distributed signal processing where redundant building blocks, e.g., frames, play an important role, Bemrose, et al., [**1**] introduced weaving frames in separable Hilbert spaces. Weaving frames have potential applications in wireless sensor networks that require distributed processing under different frames, as well as

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preprocessing of signals using Gabor frames. Sun introduced the notion of *generalized frames* or *g-frames* in [**17**]. It is well known that g-frames include standard frames and bounded invertible linear operators, as well as many recent generalizations of frames, e.g., bounded quasi-projectors and frames of subspaces. It is of interest to find the weaving properties of g-frames in separable Hilbert spaces.

1.1. Outline of the paper. The paper is organized as follows. Section 2 contains basic definitions and results regarding frames, weaving frames and g-frames in Hilbert spaces. In Section 3, we study weaving g-frames. Necessary and sufficient conditions for weaving g-frames in Hilbert spaces are given. We present sufficient conditions in terms of lower g-frame bounds for a sequence of operators not to be weaving g-frames. Some Paley-Wiener type perturbation results for weaving g-frames are obtained. In Section 4, we discuss weaving properties of g-Riesz bases.

2. Preliminaries. In this section, we review the concepts of frames, g-frames and weaving frames. We begin with some notation. The set of all positive integers is denoted by $\mathbb{N}$, and $\mathbb{J}$ denotes a subset of $\mathbb{N}$. As is standard, $\ell^2(\mathbb{N})$ is the space of all square summable complex-valued sequences indexed by $\mathbb{N}$.

2.1. Frames in Hilbert spaces. A sequence $\{x_k\}_{k\in\mathbb{N}}$ in a separable Hilbert space H is called a *frame* (or *Hilbert frame*) for H if there exist positive numbers $A_0 \leq B_0 < \infty$ such that

$$A_0\|x\|^2 \leq \sum_{k\in\mathbb{N}} |\langle x, x_k\rangle|^2 \leq B_0\|x\|^2 \quad \text{for all } x \in H. \tag{2.1}$$

The numbers A_0 and B_0 are called *lower* and *upper frame bounds*, respectively. If the upper inequality in (2.1) is satisfied, then we say that $\{x_k\}_{k\in\mathbb{N}}$ is a *Bessel sequence* (or *Hilbert Bessel sequence*) with *Bessel bound* B_0. The frame $\{x_k\}_{k\in\mathbb{N}}$ is *tight* if it is possible to choose $A_0 = B_0$. The frame operator $S : H \to H$ for the frame $\{x_k\}_{k\in\mathbb{N}}$ is a bounded, linear, invertible and positive operator, given by

$$Sx = \sum_{k\in\mathbb{N}} \langle x, x_k\rangle x_k.$$

This gives the *reconstruction formula* for all $x \in H$,

$$x = SS^{-1}x = \sum_{k\in\mathbb{N}} \langle S^{-1}x, x_k\rangle x_k = \sum_{k\in\mathbb{N}} \langle x, S^{-1}x_k\rangle x_k.$$

The basic theory of frames may be found in Han, et al., [**14**], Christensen [**7, 8**], Casazza and Kutyniok [**5**], Casazza [**2, 3**] and Han and Larson [**15**].

2.2. Weaving frames. We recall some elementary facts about weaving frames. Let $m \in \mathbb{N}$ be fixed, and let

$$[m] = \{1, 2, \ldots, m\} \quad \text{and} \quad [m]^c = \mathbb{N} \setminus [m] = \{m+1, m+2, \ldots\}.$$

Definition 2.1 ([**1**])**.** A family of frames $\{\phi_{ij}\}_{i\in\mathbb{N}, j\in[m]}$ for a Hilbert space H is said to be *woven* if there are universal constants A and B so that, for every partition $\{\sigma_j\}_{j\in[m]}$ of $\mathbb{N}$, the family $\{\phi_{ij}\}_{i\in\sigma_j, j\in[m]}$ is a frame for H with lower and upper frame bounds A and B, respectively.

Definition 2.2 ([**1**])**.** A family of frames $\{\phi_{ij}\}_{i\in\mathbb{N}, j\in[m]}$ for a Hilbert space H is *weakly woven* if, for every partition $\{\sigma_j\}_{j\in[m]}$ of $\mathbb{N}$, the family $\{\phi_{ij}\}_{i\in\sigma_j, j\in[m]}$ is a frame for H.

It may be observed that weakly woven frames do not require universal frame bounds for each weaving.

It is proven in [**1**] that this weaker form of weaving, given in Definition 2.2, is equivalent to weaving. Bemrose, et al., in [**1**] proved necessary and sufficient conditions for weaving frames (which depend on frame bounds). They classified when Riesz bases and Riesz basic sequences can be woven and provided a characterization in terms of distances between subspaces. Furthermore, they proved that, if two Riesz bases are woven, then every weaving is, in fact, a Riesz basis and not just a frame. A geometric characterization of woven Riesz bases in terms of distance between subspaces of a Hilbert space H is given in [**1**]. Casazza and Lynch [**6**] reviewed fundamental properties of weaving frames. They considered a relation of frames to projections and gave a better understanding of what it really means for two frames to be woven. Finally, they discussed a weaving equivalent of an unconditional basis.

Casazza, Freeman and Lynch [**4**] extended the concept of weaving Hilbert space frames to the Banach space setting. They introduced and studied *weaving Schauder frames* in Banach spaces. Deepshikha and Vashisht [**10**] studied weaving properties of an infinite family of frames in separable Hilbert spaces. They also studied vector-valued weaving frames [**11**] and weaving frames with respect to measure spaces in [**19**]. Deepshikha and Vashisht [**12**] studied weaving properties of K-frames in separable Hilbert spaces.

2.3. g-frames in Hilbert spaces. Sun [**17**] introduced g-frames which are generalized frames and include ordinary frames and many recent generalizations of frames, e.g., bounded quasi-projectors and frames of subspaces. For stability of the g-frame, see [**18**]. Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces, and let $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$ be a sequence of closed subspaces of $\mathcal{K}$. By $B(\mathcal{H},\mathcal{H}_n)$ we denote the space of bounded linear operators from $\mathcal{H}$ into $\mathcal{H}_n$.

Definition 2.3. A sequence $\Lambda \equiv \{\Lambda_n\}_{n\in\mathbb{N}}$, where $\Lambda_n \in B(\mathcal{H},\mathcal{H}_n)$ for each $n \in \mathbb{N}$, is a *generalized frame* (in short, *g-frame*) for $\mathcal{H}$ with respect to $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$ if there exist positive constants $A \le B$ such that

$$A\|x\|^2 \le \sum_{n\in\mathbb{N}} \|\Lambda_n x\|^2 \le B\|x\|^2 \quad \text{for all } x \in \mathcal{H}. \tag{2.2}$$

As in the case of standard frames, the constants A and B are called *lower* and *upper g-frame bounds*, respectively. If the right-hand inequality of (2.2) holds, then Λ is said to be a g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$. Associated with a g-Bessel sequence Λ, we shall denote the representation space as follows:

$$\left(\sum_{n\in\mathbb{N}}\bigoplus \mathcal{H}_n\right)_{\ell^2} = \left\{\{z_n\}_{n\in\mathbb{N}} : z_n \in \mathcal{H}_n \ (n\in\mathbb{N}), \sum_{n\in\mathbb{N}} \|z_n\|^2 < +\infty\right\}.$$

The operator

$$T_\Lambda : \left(\sum_{n\in\mathbb{N}}\bigoplus \mathcal{H}_n\right)_{\ell^2} \longrightarrow \mathcal{H}$$

defined by

$$T_\Lambda(\{z_n\}_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} \Lambda_n^* z_n,$$

is called the *pre-frame operator* or *synthesis operator*, and the adjoint of T_Λ, given by

$$T_\Lambda^* : \mathcal{H} \longrightarrow \left(\sum_{i\in\mathbb{N}} \bigoplus \mathcal{H}_i \right)_{\ell^2}$$
$$T_\Lambda^* : x \longrightarrow \{\Lambda_n x\}_{n\in\mathbb{N}}, \quad x \in \mathcal{H},$$

is called the *analysis operator* of Λ. The *frame operator* S_Λ associated with Λ is defined as

$$S_\Lambda = T_\Lambda T_\Lambda^* : \mathcal{H} \longrightarrow \mathcal{H}$$
$$S_\Lambda : x \longrightarrow \sum_{n\in\mathbb{N}} \Lambda_n^* \Lambda_n x, \quad x \in \mathcal{H}.$$

If Λ is a g-frame for $\mathcal{H}$, then S_Λ is a linear, bounded, positive and invertible operator.

Definition 2.4 ([**17**])**.** A sequence $\Lambda \equiv \{\Lambda_n\}_{n\in\mathbb{N}}$, where $\Lambda_n \in B(\mathcal{H}, \mathcal{H}_n)$ for each $n \in \mathbb{N}$, is called a *generalized Riesz basis* (abbreviated g-Riesz basis) for $\mathcal{H}$ with respect to $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$, if

(i) Λ is complete in $\mathcal{H}$, i.e.,

$$\{x : \Lambda_n x = 0,\ n \in \mathbb{N}\} = \{0\},$$

and

(ii) there are positive constants A_Λ and B_Λ such that, for any finite subset $J \subset \mathbb{N}$,

$$A_\Lambda \sum_{j\in J} \|x_j\|^2 \le \left\| \sum_{j\in J} \Lambda_j^* x_j \right\|^2 \le B_\Lambda \sum_{j\in J} \|x_j\|^2, \quad x_j \in H_j,\ j \in J.$$

The reader is referred to [**16, 17, 18**] for basic properties about g-frames and g-Riesz bases.

3. Weaving g-frames. We begin with the definition of weaving g-frames for separable Hilbert spaces.

Definition 3.1. A family of g-frames

$$\left\{ \{\Lambda_{ni}\}_{n\in\mathbb{N}} : i \in [m] \right\}$$

for a separable Hilbert space $\mathcal{H}$ with respect to $\{\mathcal{H}_n : n \in \mathbb{N}\}$ is said to be *g-woven* if there are universal constants A and B so that, for every partition $\{\sigma_i\}_{i\in[m]}$ of $\mathbb{N}$, the family $\{\Lambda_{ni}\}_{n\in\sigma_i, i\in[m]}$ is a g-frame for $\mathcal{H}$ with lower and upper g-frame bounds A and B, respectively.

Sun [**17**] obtained a characterization of g-frames in terms of ordinary frames in separable Hilbert spaces.

Theorem 3.2 ([**17**])**.** *Let $\Lambda_n \in B(\mathcal{H}, \mathcal{H}_n)$ and $\{e_{n,m}\}_{m\in\mathbb{J}_n}$ be an orthonormal basis for $\mathcal{H}_n$, where $\mathbb{J}_n \subset \mathbb{N}$, $n \in \mathbb{N}$. Then, $\{\Lambda_n\}_{n\in\mathbb{N}}$ is a g-frame for $\mathcal{H}$ if and only if $\{\Lambda_n^* e_{n,m}\}_{m\in\mathbb{J}_n, n\in\mathbb{N}}$ is a frame for $\mathcal{H}$.*

As an immediate consequence, we have the following result for weaving g-frames.

Corollary 3.3. *Let $\Lambda \equiv \{\Lambda_n\}_{n\in\mathbb{N}}$ and $\Omega \equiv \{\Omega_n\}_{n\in\mathbb{N}}$ be g-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_n : n \in \mathbb{N}\}$ and, for every $n \in \mathbb{N}$, let $\{e_{n,m}\}_{m\in\mathbb{J}_n}$ be an orthonormal basis for $\mathcal{H}_n$. Then, Λ and Ω are g-woven if and only if $\{\Lambda_n^* e_{n,m}\}_{m\in\mathbb{J}_n, n\in\mathbb{N}}$ and $\{\Omega_n^* e_{n,m}\}_{m\in\mathbb{J}_n, n\in\mathbb{N}}$ are woven frames for $\mathcal{H}$.*

Proof. Since $\Lambda_n, \Omega_n \in B(\mathcal{H}, \mathcal{H}_n)$ for all $n \in \mathbb{N}$, the mappings

$$x \longmapsto \langle \Lambda_n x, e_{n,m}\rangle \quad \text{and} \quad x \longmapsto \langle \Omega_n x, e_{n,m}\rangle$$

define bounded linear functionals on $\mathcal{H}$ for every $m \in \mathbb{J}_n$, $n \in \mathbb{N}$. Consequently, we can find some $v_{n,m} \in \mathcal{H}$ and $w_{n,m} \in \mathcal{H}$ such that, for all $x \in \mathcal{H}$,

$$\langle x, v_{n,m}\rangle = \langle \Lambda_n x, e_{n,m}\rangle \quad \text{and} \quad \langle x, w_{n,m}\rangle = \langle \Omega_n x, e_{n,m}\rangle.$$

Hence, for all $x \in \mathcal{H}$, we have

$$\Lambda_n x = \sum_{m\in\mathbb{J}_n} \langle x, v_{n,m}\rangle e_{n,m} \quad \text{and} \quad \Omega_n x = \sum_{m\in\mathbb{J}_n} \langle x, w_{n,m}\rangle e_{n,m}.$$

Let $\{\sigma, \sigma^c\}$ be any partition of $\mathbb{N}$, and write $\{\Gamma_n\}_{n\in\mathbb{N}} = \{\Lambda_n\}_{n\in\sigma} \cup \{\Omega_n\}_{n\in\sigma^c}$. Then,

$$\Gamma_n x = \begin{cases} \Lambda_n x & n \in \sigma, \\ \Omega_n x & n \in \sigma^c \end{cases} = \begin{cases} \sum_{m\in\mathbb{J}_n} \langle x, v_{n,m}\rangle e_{n,m} & n \in \sigma, \\ \sum_{m\in\mathbb{J}_n} \langle x, w_{n,m}\rangle e_{n,m} & n \in \sigma^c. \end{cases}$$

This gives

$$\sum_{n\in\mathbb{N}} \|\Gamma_n x\|^2 = \sum_{n\in\sigma}\sum_{m\in\mathbb{J}_n} |\langle x, v_{n,m}\rangle|^2 + \sum_{n\in\sigma^c}\sum_{m\in\mathbb{J}_n} |\langle x, w_{n,m}\rangle|^2 \quad \text{for all } x\in\mathcal{H}.$$

Hence, $\{\Lambda_n\}_{n\in\sigma}\cup\{\Omega_n\}_{n\in\sigma^c}$ is a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_n : n\in\mathbb{N}\}$ if and only if

$$\{u_{n,m} : m\in\mathbb{J}_n, n\in\mathbb{N}\} = \{v_{n,m} : m\in\mathbb{J}_n, n\in\sigma\} \cup \{w_{n,m} : m\in\mathbb{J}_n, n\in\sigma^c\}$$

is a frame for $\mathcal{H}$. Furthermore, for any $x\in\mathcal{H}$ and for any $y_n\in\mathcal{H}_n$, we have

$$\langle x, \Lambda_n^* y_n\rangle = \langle \Lambda_n x, y_n\rangle = \sum_{m\in\mathbb{J}_n} \langle x, v_{n,m}\rangle\langle e_{n,m}, y_n\rangle = \left\langle x, \sum_{m\in\mathbb{J}_n} \langle y_n, e_{n,m}\rangle v_{n,m}\right\rangle,$$

and

$$\langle x, \Omega_n^* y_n\rangle = \langle \Omega_n x, y_n\rangle = \sum_{m\in\mathbb{J}_n} \langle x, w_{n,m}\rangle\langle e_{n,m}, y_n\rangle = \left\langle x, \sum_{m\in\mathbb{J}_n} \langle y_n, e_{n,m}\rangle w_{n,m}\right\rangle.$$

This gives

$$\Lambda_n^* y_n = \sum_{m\in\mathbb{J}_n} \langle y_n, e_{n,m}\rangle v_{n,m}$$

and

$$\Omega_n^* y_n = \sum_{m\in\mathbb{J}_n} \langle y_n, e_{n,m}\rangle w_{n,m} \text{ for all } y_n\in\mathcal{H}_n,\ n\in\mathbb{N}.$$

In particular,

$$v_{n,m} = \Lambda_n^* e_{n,m}$$

and

$$w_{n,m} = \Omega_n^* e_{n,m} \text{ for any } m\in\mathbb{J}_n,\ n\in\mathbb{N}.$$

This completes the proof. □

3.1. Application of Corollary 3.3. Let $\mathcal{H} = \ell^2(\mathbb{N})$ and $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. Choose $\mathcal{H}_n = \overline{\text{span}}\{e_k\}_{k=n}^{\infty}$ for $n \in \mathbb{N}$. Then, $\{e_{n,m}\}_{m=1}^{\infty} = \{e_{n+m-1}\}_{m=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}_n$, $n \in \mathbb{N}$.

(i) Let $\Lambda \equiv \{\Lambda_n\}_{n=1}^{\infty}$ and $\Omega \equiv \{\Omega_n\}_{n=1}^{\infty}$, where $\Lambda_n \in B(\mathcal{H}, \mathcal{H}_n)$ is the orthogonal projection of $\mathcal{H}$ onto $\overline{\text{span}}\{e_n\}$ and $\Omega_n \in B(\mathcal{H}, \mathcal{H}_n)$ is the orthogonal projection of $\mathcal{H}$ onto $\overline{\text{span}}\{e_n, e_{n+1}\}$. Clearly,

$$\Lambda_n^* e_{n,m} = \begin{cases} e_n & m = 1, \\ 0 & m > 1, \end{cases}$$

and

$$\Omega_n^* e_{n,m} = \begin{cases} e_n & m = 1, \\ e_{n+1} & m = 2, \\ 0 & m > 2. \end{cases}$$

Note that $\{\Lambda_n^* e_{n,m}\}_{n,m=1}^{\infty}$ and $\{\Omega_n^* e_{n,m}\}_{n,m=1}^{\infty}$ are frames for $\mathcal{H}$.

Next, we show that $\{\Lambda_n^* e_{n,m}\}_{n,m=1}^{\infty}$ and $\{\Omega_n^* e_{n,m}\}_{n,m=1}^{\infty}$ are woven. Let $\sigma \subset \mathbb{N}$ be any arbitrary subset. We compute

$$\begin{aligned} \|x\|^2 &\le \sum_{n\in\sigma}\sum_{m\in\mathbb{N}} |\langle x, \Lambda_n^* e_{n,m}\rangle|^2 + \sum_{n\in\sigma^c}\sum_{m\in\mathbb{N}} |\langle x, \Omega_n^* e_{n,m}\rangle|^2 \\ &= \sum_{n\in\sigma} |\langle x, \Lambda_n^* e_{n,1}\rangle|^2 + \sum_{n\in\sigma^c} |\langle x, \Omega_n^* e_{n,1}\rangle|^2 \\ &\quad + \sum_{n\in\sigma^c} |\langle x, \Omega_n^* e_{n,2}\rangle|^2 \\ &= \sum_{n\in\sigma} |\langle x, e_n\rangle|^2 + \sum_{n\in\sigma^c} |\langle x, e_n\rangle|^2 \\ &\quad + \sum_{n\in\sigma^c} |\langle x, e_{n+1}\rangle|^2 \le 2\sum_{n\in\mathbb{N}} |\langle x, e_n\rangle|^2 \\ &= 2\|x\|^2 \quad \text{for all } x \in \mathcal{H}. \end{aligned}$$

Thus,

$$\{\Lambda_n^* e_{n,m}\}_{\substack{n\in\sigma \\ m\in\mathbb{N}}} \cup \{\Omega_n^* e_{n,m}\}_{\substack{n\in\sigma^c \\ m\in\mathbb{N}}}$$

is a frame for $\mathcal{H}$ for any $\sigma \subset \mathbb{N}$. Hence, by Corollary 3.3, Λ and Ω are g-woven.

(ii) Let $\Lambda \equiv \{\Lambda_n\}_{n=1}^{\infty}$ and $\Omega \equiv \{\Omega_n\}_{n=2}^{\infty}$ be the same as in part (i) except for Ω_1 which is the zero mapping. Then, $\{\Lambda_n^* e_{n,m}\}_{n,m=1}^{\infty}$ and $\{\Omega_n^* e_{n,m}\}_{n,m=1}^{\infty}$ are not woven. Indeed, let $\{\Lambda_n^* e_{n,m}\}_{m,n=1}^{\infty}$ and $\{\Omega_n^* e_{n,m}\}_{n,m=1}^{\infty}$ be woven with universal frame bounds A and B. Choose $\sigma = \mathbb{N} \setminus \{1\}$. Then, compute

$$\begin{aligned}
&\sum_{n\in\sigma}\sum_{m\in\mathbb{N}} |\langle e_1, \Lambda_n^* e_{n,m}\rangle|^2 + \sum_{n\in\sigma^c}\sum_{m\in\mathbb{N}} |\langle e_1, \Omega_n^* e_{n,m}\rangle|^2 \\
&\quad = \sum_{n\in\mathbb{N}\setminus\{1\}} |\langle e_1, \Lambda_n^* e_{n,1}\rangle|^2 + |\langle e_1, 0\rangle|^2 \\
&\quad = \sum_{n\in\mathbb{N}\setminus\{1\}} |\langle e_1, e_n\rangle|^2 + |\langle e_1, 0\rangle|^2 \\
&\quad = 0 < A\|e_1\|^2.
\end{aligned}$$

This is a contradiction. Hence, by Corollary 3.3, Λ and Ω are not g-woven.

Inspired by [**1,** Lemma 4.3], the next theorem provides sufficient conditions for a sequence of operators not to be woven g-frames for $\mathcal{H}$.

Theorem 3.4. *Suppose that $\Lambda \equiv \{\Lambda_n\}_{n\in\mathbb{N}}$ and $\Omega \equiv \{\Omega_n\}_{n\in\mathbb{N}}$ are g-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_n : n \in \mathbb{N}\}$. Assume that, for every two disjoint finite sets $I, J \subset \mathbb{N}$ and every $\epsilon > 0$, there are subsets $\sigma, \delta \subset \mathbb{N} \setminus (I \cup J)$ with $\delta = \mathbb{N} \setminus (I \cup J \cup \sigma)$ such that the lower g-frame bound of*

$$\{\Lambda_n\}_{n\in I\cup\sigma} \cup \{\Omega_n\}_{n\in J\cup\delta}$$

is less than ϵ. Then, there exists a subset $\mathcal{M} \subset \mathbb{N}$ so that

$$\{\Lambda_n\}_{n\in\mathcal{M}} \cup \{\Omega_n\}_{n\in\mathcal{M}^c}$$

is not a g-frame. Hence, Λ and Ω are not g-woven.

Proof. Let $\epsilon > 0$ be arbitrary. By hypothesis, for $I_0 = J_0 = \varnothing$, we can choose $\sigma_1 \subset \mathbb{N}$ such that, if $\delta_1 = \sigma_1^c$, then a lower g-frame bound of $\{\Lambda_n\}_{n\in\sigma_1} \cup \{\Omega_n\}_{n\in\delta_1}$ is less than ϵ. Thus, there exists an $x_1 \in \mathcal{H}$ with $\|x_1\| = 1$ such that

$$\sum_{n\in\sigma_1} \|\Lambda_n x_1\|^2 + \sum_{n\in\delta_1} \|\Omega_n x_1\|^2 < \epsilon.$$

Since

$$\sum_{n=1}^{\infty} \|\Lambda_n x_1\|^2 + \sum_{n=1}^{\infty} \|\Omega_n x_1\|^2 < \infty,$$

there is a positive integer k_1 such that

$$\sum_{n=k_1+1}^{\infty} \|\Lambda_n x_1\|^2 + \sum_{n=k_1+1}^{\infty} \|\Omega_n x_1\|^2 < \epsilon.$$

Let $I_1 = \sigma_1 \cap [k_1]$ and $J_1 = \delta_1 \cap [k_1]$. Then, $I_1 \cap J_1 = \varnothing$ and $I_1 \cup J_1 = [k_1]$.

By assumption, there are subsets $\sigma_2, \delta_2 \subset [k_1]^c$ with $\delta_2 = [k_1]^c \setminus \sigma_2$ such that a lower g-frame bound of

$$\{\Lambda_n\}_{n \in I_1 \cup \sigma_2} \cup \{\Omega_n\}_{n \in J_1 \cup \delta_2}$$

is less than $\epsilon/2$, that is, there exists a vector $x_2 \in \mathcal{H}$ with $\|x_2\| = 1$ such that

$$\sum_{n \in I_1 \cup \sigma_2} \|\Lambda_n x_2\|^2 + \sum_{n \in J_1 \cup \delta_2} \|\Omega_n x_2\|^2 < \frac{\epsilon}{2}.$$

Similar to the above, there is a $k_2 > k_1$ such that

$$\sum_{n=k_2+1}^{\infty} \|\Lambda_n x_2\|^2 + \sum_{n=k_2+1}^{\infty} \|\Omega_n x_2\|^2 < \frac{\epsilon}{2}.$$

Set $I_2 = I_1 \cup (\sigma_2 \cap [k_2])$ and $J_2 = J_1 \cup (\delta_2 \cap [k_2])$. Note that $I_2 \cap J_2 = \varnothing$ and $I_2 \cup J_2 = [k_2]$. Thus, by the induction method, we obtain:

(i) a sequence of positive integers $\{k_n\}_{n \in \mathbb{N}} \subset \mathbb{N}$ with $k_n < k_{n+1}$ for all $n \in \mathbb{N}$;
(ii) a sequence of vectors $\{x_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ with $\|x_n\| = 1$ for all $n \in \mathbb{N}$;
(iii) subsets $\sigma_n \subset [k_{n-1}]^c$, $\delta_n = [k_{n-1}]^c \setminus \sigma_n$, $n \in \mathbb{N}$; and
(iv) $I_n = I_{n-1} \cup (\sigma_n \cap [k_n])$, $J_n = J_{n-1} \cup (\delta_n \cap [k_n])$, $n \in \mathbb{N}$,

which satisfy both

$$\sum_{i \in I_{n-1} \cup \sigma_n} \|\Lambda_i x_n\|^2 + \sum_{i \in J_{n-1} \cup \delta_n} \|\Omega_i x_n\|^2 < \frac{\epsilon}{n}, \tag{3.1}$$

and

$$\sum_{i=k_n+1}^{\infty} \|\Lambda_i x_n\|^2 + \sum_{i=k_n+1}^{\infty} \|\Omega_i x_n\|^2 < \frac{\epsilon}{n}. \tag{3.2}$$

By construction, $I_n \cap J_n = \varnothing$ and $I_n \cup J_n = [k_n]$ for all $n \in \mathbb{N}$ such that

$$\left(\bigcup_{i=1}^{\infty} I_i\right) \bigsqcup \left(\bigcup_{j=1}^{\infty} J_j\right) = \mathbb{N},$$

where $\sqcup$ represents disjoint union. Choose $\mathcal{M} = \cup_{i=1}^{\infty} I_i$. Note that

$$\mathcal{M}^c = \bigcup_{j=1}^{\infty} J_j.$$

We compute

$$\begin{aligned}
&\sum_{i\in\mathcal{M}} \|\Lambda_i x_n\|^2 + \sum_{i\in\mathcal{M}^c} \|\Omega_i x_n\|^2 \\
&\quad= \left(\sum_{i\in I_n} \|\Lambda_i x_n\|^2 + \sum_{i\in J_n} \|\Omega_i x_n\|^2\right) \\
&\qquad+ \left(\sum_{i\in A\cap[k_n]^c} \|\Lambda_i x_n\|^2 + \sum_{i\in A^c\cap[k_n]^c} \|\Omega_i x_n\|^2\right) \\
&\quad\leqslant \left(\sum_{i\in I_{n-1}\cup\sigma_n} \|\Lambda_i x_n\|^2 + \sum_{i\in J_{n-1}\cup\delta_n} \|\Omega_i x_n\|^2\right) \\
&\qquad+ \left(\sum_{i=k_n+1}^{\infty} \|\Lambda_i x_n\|^2 + \sum_{i=k_n+1}^{\infty} \|\Omega_i x_n\|^2\right) \\
&\quad< \frac{\epsilon}{n} + \frac{\epsilon}{n} = \frac{2\epsilon}{n}.
\end{aligned}$$

This shows that a lower g-frame bound of $\{\Lambda_n\}_{n\in\mathcal{M}} \cup \{\Omega_n\}_{n\in\mathcal{M}^c}$ is zero, a contradiction. Hence, the g-frames Λ and Ω are not g-woven. □

Theorem 3.4 gives a necessary condition for weaving g-frames in terms of lower frame bounds.

Proposition 3.5. *Suppose that the family of g-frames*

$$\{\{\Lambda_{ni}\}_{n\in\mathbb{N}} : i \in [m]\}$$

for $\mathcal{H}$ with respect to $\{\mathcal{H}_n : n \in \mathbb{N}\}$ is g-woven. Then, there exists a partition $\{\tau_i\}_{i\in[m]}$ of some finite subset of $\mathbb{N}$ and $A > 0$ such that, for

any partition $\{\sigma_i\}_{i\in[m]}$ *of* $\mathbb{N}\setminus\{\tau_i\}_{i\in[m]}$, *the family*

$$\bigcup_{i\in[m]}\{\Lambda_{in}\}_{n\in\sigma_i\cup\tau_i}$$

has a lower g-frame bound A.

The next proposition gives a universal g-Bessel bound for a family of g-Bessel sequences. This is an adaptation of [**1,** Proposition 3.1].

Proposition 3.6. *For each* $i\in[m]$, *let* $\{\Lambda_{ni}\}_{n\in\mathbb{N}}$ *be a g-Bessel sequence for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_n : n\in\mathbb{N}\}$ *and with g-Bessel bounds* B_i. *Then, every weaving is a g-Bessel sequence with*

$$\sum_{i=1}^{m} B_i$$

as a g-Bessel bound.

Proof. Let $\{\Lambda_{ni}\}_{n\in\sigma_i,i\in[m]}$ be a weaving for any partition $\{\sigma_i\}_{i\in[m]}$ of $\mathbb{N}$. Then,

$$\begin{aligned}\sum_{i=1}^{m}\sum_{n\in\sigma_i}\|\Lambda_{ni}x\|^2 &\leqslant \sum_{i=1}^{m}\sum_{n\in\mathbb{N}}\|\Lambda_{ni}x\|^2\\ &\leqslant \left(\sum_{i=1}^{m}B_i\right)\|x\|^2 \quad \text{for all } x\in\mathcal{H}.\end{aligned}$$

The proof is complete. □

As in the case of standard weaving frames [**6,** Proposition 15], it is enough to check g-weaving on smaller sets than the original.

Proposition 3.7. *Let* $\Lambda\equiv\{\Lambda_n\}_{n\in\mathbb{N}}$ *and* $\Omega\equiv\{\Omega_n\}_{n\in\mathbb{N}}$ *be g-Bessel sequences in* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_n : n\in\mathbb{N}\}$ *with g-Bessel bounds* B_1 *and* B_2, *respectively. If* $J\subset\mathbb{N}$, *and* $\Lambda_J\equiv\{\Lambda_i\}_{i\in J}$ *and* $\Omega_J\equiv\{\Omega_i\}_{i\in J}$ *are g-woven frames, then* Λ *and* Ω *are g-woven frames for* $\mathcal{H}$.

Proof. Let A be a lower universal g-frame bound for Λ_J and Ω_J, and let $\sigma \subset \mathbb{N}$ be an arbitrary subset. Then,

$$\begin{aligned} A\|x\|^2 &\leqslant \sum_{i\in\sigma\cap J} \|\Lambda_i x\|^2 + \sum_{i\in\sigma^c\cap J} \|\Omega_i x\|^2 \\ &\leqslant \sum_{i\in\sigma} \|\Lambda_i x\|^2 + \sum_{i\in\sigma^c} \|\Omega_i x\|^2 \\ &\leqslant (B_1 + B_2)\|x\|^2 \quad \text{for all } x \in \mathcal{H} \end{aligned}$$

(by Proposition 3.6). Hence, Λ and Ω are g-woven frames for $\mathcal{H}$. □

Recall that, after removal of a vector from a discrete frame, the resultant family is either a frame or an incomplete set, see **[8,** Theorem 5.4.7]. Casazza and Lynch **[6]** proved that removal of vectors from woven frames leaves them woven. In the direction of g-frames we have following result.

Proposition 3.8. *Let $\Lambda \equiv \{\Lambda_n\}_{n\in\mathbb{N}}$ and $\Omega \equiv \{\Omega_n\}_{n\in\mathbb{N}}$ be g-woven frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_n : n \in \mathbb{N}\}$ with universal g-frame bounds A and B. If $J \subset \mathbb{N}$ and*

$$\sum_{i\in J} \|\Lambda_i x\|^2 \leq D_0\|x\|^2$$

for all $x \in \mathcal{H}$ and for some $0 < D_0 < A$, then $\Lambda_0 \equiv \{\Lambda_i\}_{i\in\mathbb{N}\setminus J}$ and $\Omega_0 \equiv \{\Omega_i\}_{i\in\mathbb{N}\setminus J}$ are g-woven frames for $\mathcal{H}$ with universal g-frame bounds $A - D_0$ and B.

Proof. Let $\sigma \subset \mathbb{N} \setminus J$ be arbitrary. We compute

$$\begin{aligned} &\sum_{i\in\sigma} \|\Lambda_i x\|^2 + \sum_{i\in(\mathbb{N}\setminus J)\setminus\sigma} \|\Omega_i x\|^2 \\ &\quad= \Big(\sum_{i\in\sigma\cup J} \|\Lambda_i x\|^2 - \sum_{i\in J} \|\Lambda_i x\|^2 \Big) + \sum_{i\in(\mathbb{N}\setminus J)\setminus\sigma} \|\Omega_i x\|^2 \\ &\quad= \Big(\sum_{i\in\sigma\cup J} \|\Lambda_i x\|^2 + \sum_{i\in(\mathbb{N}\setminus J)\setminus\sigma} \|\Omega_i x\|^2 \Big) - \sum_{i\in J} \|\Lambda_i x\|^2 \\ &\quad\geqslant (A - D_0)\|x\|^2 \quad \text{for all } x \in \mathcal{H}. \end{aligned}$$

On the other hand, for all $x \in \mathcal{H}$, we have

$$\sum_{i\in\sigma} \|\Lambda_i x\|^2 + \sum_{i\in(\mathbb{N}\setminus J)\setminus\sigma} \|\Omega_i x\|^2 \leqslant \sum_{i\in\sigma\cup J} \|\Lambda_i x\|^2 + \sum_{i\in(\mathbb{N}\setminus J)\setminus\sigma} \|\Omega_i x\|^2 \leqslant B\|x\|^2.$$

Hence, Λ_0 and Ω_0 are g-woven frames for $\mathcal{H}$ with the required universal g-frame bounds. □

4. Perturbation of weaving g-frames. It is well known that perturbation theory is an important area in applied mathematics. For applications of perturbation theory for frames in various directions, the reader is referred to **[2, 5, 7, 8]** and the references therein. Bemrose, et al., **[1]** proved sufficient conditions for weaving frames by means of perturbation theory and diagonal dominance. We begin this section with the following Paley-Wiener type perturbation of weaving g-frames.

Theorem 4.1. *Let $\Lambda \equiv \{\Lambda_i\}_{i\in\mathbb{N}}$ and $\Omega \equiv \{\Omega_i\}_{i\in\mathbb{N}}$ be g-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{N}\}$ with g-frame bounds A_1, B_1 and A_2, B_2, respectively. Assume that there are constants $0 < \lambda_1$, λ_2, $\mu < 1$ such that*

$$\lambda_1\sqrt{B_1} + \lambda_2\sqrt{B_2} + \mu \leqslant \frac{A_1}{2(\sqrt{B_1} + \sqrt{B_2})}$$

and

(4.1)
$$\left\| \sum_{i\in\mathbb{N}} (\Lambda_i^* x_i - \Omega_i^* x_i) \right\| \leqslant \lambda_1 \left\| \sum_{i\in\mathbb{N}} \Lambda_i^* x_i \right\| + \lambda_2 \left\| \sum_{i\in\mathbb{N}} \Omega_i^* x_i \right\| + \mu \|\{x_i\}_{i\in\mathbb{N}}\|,$$

for all

$$\{x_i\}_{i\in\mathbb{N}} \in \left(\sum_{i\in\mathbb{N}} \bigoplus \mathcal{H}_i \right)_{\ell^2}.$$

Then, Λ and Ω are g-woven with universal g-frame bounds $A_1/2$, $B_1 + B_2$.

Proof. Let T and R be the synthesis operators for the frames $\{\Lambda_i\}_{i\in\mathbb{N}}$ and $\{\Omega_i\}_{i\in\mathbb{N}}$, respectively. For each $\sigma \subset \mathbb{N}$, define bounded operators

$$T_\sigma, R_\sigma : \left(\sum_{i\in\mathbb{N}} \bigoplus \mathcal{H}_i \right)_{\ell^2} \longrightarrow \mathcal{H},$$

$$T_\sigma(\{x_i\}_{i\in\mathbb{N}}) = \sum_{i\in\sigma} \Lambda_i^*(x_i),$$

and

$$R_\sigma(\{x_i\}_{i\in\mathbb{N}}) = \sum_{i\in\sigma} \Omega_i^*(x_i).$$

Note that $\|T_\sigma\| \leqslant \|T\|$, $\|R_\sigma\| \leqslant \|R\|$ and $\|T_\sigma - R_\sigma\| \leqslant \|T - R\|$.

By using (4.1), we have

$$\begin{aligned}
&\lambda_1\|T(\{x_i\}_{i\in\mathbb{N}})\| + \lambda_2\|R(\{x_i\}_{i\in\mathbb{N}})\| + \mu\|\{x_i\}_{i\in\mathbb{N}}\| \\
&\quad \geqslant \left\| \sum_{i\in\mathbb{N}} (\Lambda_i^* - \Omega_i^*)(x_i) \right\| \\
&\quad = \|(T-R)(\{x_i\}_{i\in\mathbb{N}})\|, \quad \{x_i\}_{i\in\mathbb{N}} \in \left(\sum_{i\in\mathbb{N}} \bigoplus \mathcal{H}_i \right)_{\ell^2}.
\end{aligned}$$

This gives $\|T - R\| \leq \lambda_1\|T\| + \lambda_2\|R\| + \mu$. Using this, for any $\sigma \subset \mathbb{N}$, we compute

(4.2)

$$\begin{aligned}
\left\| \sum_{i\in\sigma} \Lambda_i^*\Lambda_i x - \sum_{i\in\sigma} \Omega_i^*\Omega_i x \right\| &= \|T_\sigma(\{\Lambda_i x\}_{i\in\sigma}) - R_\sigma(\{\Omega_i x\}_{i\in\sigma})\| \\
&= \|T_\sigma T_\sigma^* x - R_\sigma R_\sigma^* x\| \\
&\leqslant \|(T_\sigma T_\sigma^* - T_\sigma R_\sigma^*)(x)\| + \|(T_\sigma R_\sigma^* - R_\sigma R_\sigma^*)(x)\| \\
&\leqslant \|T_\sigma\|\|T_\sigma^* - R_\sigma^*\|\|x\| + \|T_\sigma - R_\sigma\|\|R_\sigma^*\|\|x\| \\
&\leqslant \|T\|\|T - R\|\|x\| + \|T - R\|\|R\|\|x\| \\
&\leqslant (\lambda_1\|T\| + \lambda_2\|R\| + \mu)(\|T\| + \|R\|)\|x\| \\
&\leqslant (\lambda_1\sqrt{B_1} + \lambda_2\sqrt{B_2} + \mu)(\sqrt{B_1} + \sqrt{B_2})\|x\| \\
&< \left(\frac{A_1}{2(\sqrt{B_1} + \sqrt{B_2})} \right)(\sqrt{B_1} + \sqrt{B_2})\|x\| \\
&= \frac{A_1}{2}\|x\| \quad \text{for all } x \in \mathcal{H}.
\end{aligned}$$

By using (4.2), it follows that

$$\begin{aligned}
&\Big\| \sum_{i\in\sigma^c} \Lambda_i^*\Lambda_i x + \sum_{i\in\sigma} \Omega_i^*\Omega_i x \Big\| \\
&\quad = \Big\| \sum_{i\in\sigma^c} \Lambda_i^*\Lambda_i x + \sum_{i\in\sigma} \Lambda_i^*\Lambda_i x - \sum_{i\in\sigma} \Lambda_i^*\Lambda_i x + \sum_{i\in\sigma} \Omega_i^*\Omega_i x \Big\| \\
&\quad = \Big\| \sum_{i\in\mathbb{N}} \Lambda_i^*\Lambda_i x + \sum_{i\in\sigma} \Omega_i^*\Omega_i x - \sum_{i\in\sigma} \Lambda_i^*\Lambda_i x \Big\| \\
&\quad \geqslant \Big\| \sum_{i\in\mathbb{N}} \Lambda_i^*\Lambda_i x \Big\| - \Big\| \sum_{i\in\sigma} \Omega_i^*\Omega_i x - \sum_{i\in\sigma} \Lambda_i^*\Lambda_i x \Big\| \\
&\quad \geqslant A_1\|x\| - \Big\| \sum_{i\in\sigma} \Lambda_i^*\Lambda_i x - \sum_{i\in\sigma} \Omega_i^*\Omega_i x \Big\| \\
&\quad \geqslant A_1\|x\| - \frac{A_1}{2}\|x\| \\
&\quad = \frac{A_1}{2}\|x\| \quad \text{for all } x \in \mathcal{H}.
\end{aligned}$$

This gives a universal lower g-frame bound. The upper universal g-frame bound can be obtained from Proposition 3.6. Hence, Λ and Ω are g-woven. □

The next theorem gives another variant of Paley-Wiener type perturbation of weaving g-frames in terms of frame operators associated with Λ and Ω.

Theorem 4.2. *Let $\Lambda \equiv \{\Lambda_i\}_{i\in\mathbb{N}}$ and $\Omega \equiv \{\Omega_i\}_{i\in\mathbb{N}}$ be g-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{N}\}$ with frame bounds A_1, B_1 and A_2, B_2, respectively. Assume that there are constants $0 < \lambda, \mu, \gamma < 1$ such that*

$$\lambda B_1 + \mu B_2 + \gamma\sqrt{B_1} < A_1$$

and

(4.3)

$$\begin{aligned}
\Big\| \sum_{i\in\sigma} (\Lambda_i^*\Lambda_i x - \Omega_i^*\Omega_i x) \Big\| \leqslant \lambda \Big\| \sum_{i\in\sigma} \Lambda_i^*\Lambda_i x \Big\| \\
+ \mu \Big\| \sum_{i\in\sigma} \Omega_i^*\Omega_i x \Big\| + \gamma \Big(\sum_{i\in\sigma} \|\Lambda_i x\|^2 \Big)^{1/2},
\end{aligned}$$

for all $x \in \mathcal{H}$ and for every $\sigma \subset \mathbb{N}$. Then, Λ and Ω are g-woven with universal g-frame bounds $(A_1 - \lambda\sqrt{B_1} - \mu B_2 - \gamma)$ and $(B_1 + \lambda\sqrt{B_1} + \mu B_2 + \gamma)$.

Proof. By using the fact that

$$\left\| \sum_{i\in\sigma} \Lambda_i^* \Lambda_i x \right\| \leqslant B_1 \|x\| \quad \text{and} \quad \left\| \sum_{i\in\sigma} \Omega_i^* \Omega_i x \right\| \leqslant B_2 \|x\|$$

for any $\sigma \subset \mathbb{N}$ and $x \in \mathcal{H}$, we compute

$$\begin{aligned}
\left\| \sum_{i\in\sigma^c} \Lambda_i^* \Lambda_i x + \sum_{i\in\sigma} \Omega_i^* \Omega_i x \right\| &= \left\| \sum_{i\in\mathbb{N}} \Lambda_i^* \Lambda_i x + \sum_{i\in\sigma} \Omega_i^* \Omega_i x - \sum_{i\in\sigma} \Lambda_i^* \Lambda_i x \right\| \\
&\geqslant \left\| \sum_{i\in\mathbb{N}} \Lambda_i^* \Lambda_i x \right\| - \left\| \sum_{i\in\sigma} \Omega_i^* \Omega_i x - \sum_{i\in\sigma} \Lambda_i^* \Lambda_i x \right\| \\
&\geqslant A_1 \|x\| - \lambda \left\| \sum_{i\in\sigma} \Lambda_i^* \Lambda_i x \right\| - \mu \left\| \sum_{i\in\sigma} \Omega_i^* \Omega_i x \right\| \\
&\quad - \gamma \left(\sum_{i\in\sigma} \|\Lambda_i x\|^2 \right)^{1/2} \\
&\geqslant (A_1 - \lambda B_1 - \mu B_2 - \gamma\sqrt{B_1}) \|x\|, \qquad (4.4)
\end{aligned}$$

and

$$\begin{aligned}
\left\| \sum_{i\in\sigma^c} \Lambda_i^* \Lambda_i x + \sum_{i\in\sigma} \Omega_i^* \Omega_i x \right\| &= \left\| \sum_{i\in\mathbb{N}} \Lambda_i^* \Lambda_i x + \sum_{i\in\sigma} \Omega_i^* \Omega_i x - \sum_{i\in\sigma} \Lambda_i^* \Lambda_i x \right\| \\
&\leqslant \left\| \sum_{i\in\mathbb{N}} \Lambda_i^* \Lambda_i x \right\| + \left\| \sum_{i\in\sigma} \Omega_i^* \Omega_i x - \sum_{i\in\sigma} \Lambda_i^* \Lambda_i x \right\| \\
&\leqslant B_1 \|x\| + \lambda \left\| \sum_{i\in\sigma} \Lambda_i^* \Lambda_i x \right\| + \mu \left\| \sum_{i\in\sigma} \Omega_i^* \Omega_i x \right\| \\
&\quad + \gamma \left(\sum_{i\in\sigma} \|\Lambda_i x\|^2 \right)^{1/2} \\
&\leqslant (B_1 + \lambda B_1 + \mu B_2 + \gamma\sqrt{B_1}) \|x\|. \qquad (4.5)
\end{aligned}$$

Therefore, by (4.4) and (4.5), the g-frames Λ and Ω are g-woven with the required universal g-frame bounds. □

We end this section with perturbation of weaving g-frames in terms of certain closeness between the vectors in $\mathcal{H}_i$.

Theorem 4.3. *Let $\Lambda \equiv \{\Lambda_i\}_{i\in\mathbb{N}}$ and $\Omega \equiv \{\Omega_i\}_{i\in\mathbb{N}}$ be g-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{N}\}$ and with g-frame bounds A_1, B_1 and A_2, B_2, respectively. Assume that there is a constant $M > 0$ such that, for every $J \subset \mathbb{N}$,*

$$\sum_{i\in J} \|\Lambda_i x - \Omega_i x\|^2 \leqslant M \min\left\{ \sum_{i\in J} \|\Lambda_i x\|^2, \sum_{i\in J} \|\Omega_i x\|^2 \right\}, \quad x \in \mathcal{H}. \tag{4.6}$$

Then, Λ and Ω are g-woven with universal g-frame bounds $(A_1 + A_2)/(2M + 3)$ and $B_1 + B_2$.

Proof. Let $\sigma \subset \mathbb{N}$ be arbitrary. Then, by using (4.6), we compute

$$\begin{aligned}
(A_1 + A_2)\|x\|^2 &\leqslant \sum_{i\in\mathbb{N}} \|\Lambda_i x\|^2 + \sum_{i\in\mathbb{N}} \|\Omega_i x\|^2 \\
&= \sum_{i\in\sigma} \|\Lambda_i x\|^2 + \sum_{i\in\sigma^c} \|\Lambda_i x\|^2 + \sum_{i\in\sigma} \|\Omega_i x\|^2 + \sum_{i\in\sigma^c} \|\Omega_i x\|^2 \\
&\leqslant \sum_{i\in\sigma} \|\Lambda_i x\|^2 + 2\bigg(\sum_{i\in\sigma^c} \|(\Lambda_i - \Omega_i)(x)\|^2 + \sum_{i\in\sigma^c} \|\Omega_i x\|^2 \bigg) \\
&\quad + 2\bigg(\sum_{i\in\sigma} \|(\Lambda_i - \Omega_i)(x)\|^2 + \sum_{i\in\sigma} \|\Lambda_i x\|^2 \bigg) + \sum_{i\in\sigma^c} \|\Omega_i x\|^2 \\
&\leqslant \sum_{i\in\sigma} \|\Lambda_i x\|^2 + 2\bigg(M \sum_{i\in\sigma^c} \|\Omega_i x\|^2 + \sum_{i\in\sigma^c} \|\Omega_i x\|^2 \bigg) \\
&\quad + 2\bigg(M \sum_{i\in\sigma} \|\Lambda_i x\|^2 + \sum_{i\in\sigma} \|\Lambda_i x\|^2 \bigg) + \sum_{i\in\sigma^c} \|\Omega_i x\|^2 \\
&= (2M+3)\bigg(\sum_{i\in\sigma} \|\Lambda_i x\|^2 + \sum_{i\in\sigma^c} \|\Omega_i x\|^2 \bigg) \quad \text{for all } x \in \mathcal{H}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{A_1 + A_2}{2M + 3}\|x\|^2 &\leq \sum_{i\in\sigma} \|\Lambda_i x\|^2 + \sum_{i\in\sigma^c} \|\Omega_i x\|^2 \\
&\leq (B_1 + B_2)\|x\|^2, \quad x \in \mathcal{H}.
\end{aligned}$$

Hence, Λ and Ω are g-woven with the desired universal g-frame bounds. □

5. Weaving g-Riesz bases. Bemrose, et al., [**1**] classified when Riesz bases and Riesz basic sequences can be woven and proved a characterization in terms of distances between subspaces. We present a necessary and sufficient condition for weaving g-Riesz bases in terms of standard woven Riesz bases. The proof is based upon the technique developed by Sun [**17**], which may be found in the following theorem.

Theorem 5.1 ([**17**]). *Let $\Lambda_n \in B(\mathcal{H}, \mathcal{H}_n)$ and $\{e_{n,m}\}_{m\in\mathbb{J}_n}$ be an orthonormal basis for $\mathcal{H}_n$, where $\mathbb{J}_n \subset \mathbb{N}$, $n \in \mathbb{N}$. Then, $\{\Lambda_n\}_{n\in\mathbb{N}}$ is a g-Riesz basis for $\mathcal{H}$ if and only if $\{\Lambda_n^* e_{n,m}\}_{m\in\mathbb{J}_n, n\in\mathbb{N}}$ is a Riesz basis for $\mathcal{H}$.*

As a corollary, we have the next result for weaving g-Riesz bases.

Corollary 5.2. *Let $\Lambda \equiv \{\Lambda_n\}_{n\in\mathbb{N}}$, and $\Omega \equiv \{\Omega_n\}_{n\in\mathbb{N}}$ be g-Riesz bases for $\mathcal{H}$ with respect to $\{\mathcal{H}_n : n \in \mathbb{N}\}$, and let $\{e_{n,m}\}_{m\in\mathbb{J}_n}$ be an orthonormal basis for $\mathcal{H}_n$, for each $n \in \mathbb{N}$. Then, Λ and Ω are g-woven Riesz bases for $\mathcal{H}$ if and only if $\{\Lambda_n^* e_{n,m}\}_{n\in\mathbb{N}, m\in\mathbb{J}_n}$ and $\{\Omega_n^* e_{n,m}\}_{n\in\mathbb{N}, m\in\mathbb{J}_n}$ are woven Riesz bases for $\mathcal{H}$.*

Proof. For each $n \in \mathbb{N}$, since $\{e_{n,m}\}_{m\in\mathbb{J}_n}$ is an orthonormal basis for $\mathcal{H}_n$, every $y_n \in \mathcal{H}_n$ has an expansion of the form

$$y_n = \sum_{m\in\mathbb{J}_n} c_{n,m} e_{n,m},$$
$$\text{where } \{c_{n,m}\}_{\substack{n\in\mathbb{N}\\ m\in\mathbb{J}_n}} \in \ell^2(\mathbb{N}).$$

Let $J \subset \mathbb{N}$ be any arbitrary finite subset and $\{\sigma, \sigma^c\}$ any partition of $\mathbb{N}$. We write $\{\Gamma_n\}_{n\in\mathbb{N}} = \{\Lambda_n\}_{n\in\sigma} \cup \{\Omega_n\}_{n\in\sigma^c}$ and $v_{n,m}, w_{n,m} \in \mathcal{H}$ for vectors defined as in the proof of Corollary 3.3. Compute

$$\begin{aligned}\left\| \sum_{n\in J} \Gamma_n^* y_n \right\|^2 &= \left\| \sum_{n\in J\cap\sigma} \Lambda_n^* y_n + \sum_{n\in J\cap\sigma^c} \Omega_n^* y_n \right\|^2 \\ &= \left\| \sum_{n\in J\cap\sigma} \sum_{m\in\mathbb{J}_n} \langle y_n, e_{n,m}\rangle v_{n,m} \right. \\ &\qquad \left. + \sum_{n\in J\cap\sigma^c} \sum_{m\in\mathbb{J}_n} \langle y_n, e_{n,m}\rangle w_{n,m} \right\|^2\end{aligned}$$

$$= \left\| \sum_{n\in J\cap\sigma} \sum_{m\in\mathbb{J}_n} c_{n,m} v_{n,m} + \sum_{n\in J\cap\sigma^c} \sum_{m\in\mathbb{J}_n} c_{n,m} w_{n,m} \right\|^2,$$

and

$$\sum_{n\in J} \|y_n\|^2 = \sum_{n\in J} \left\| \sum_{m\in\mathbb{J}_n} c_{n,m} e_{n,m} \right\|^2 = \sum_{n\in J} \sum_{m\in\mathbb{J}_n} |c_{n,m}|^2.$$

Hence, it follows that

$$A \sum_{n\in J} \|y_n\|^2 \leqslant \left\| \sum_{n\in J} \Gamma_n^* y_n \right\|^2 \leqslant B \sum_{n\in J} \|y_n\|^2$$

is equivalent to

$$\begin{aligned} A \sum_{n\in J} \sum_{m\in\mathbb{J}_n} |c_{n,m}|^2 &\leqslant \left\| \sum_{n\in J\cap\sigma} \sum_{m\in\mathbb{J}_n} c_{n,m} v_{n,m} + \sum_{n\in J\cap\sigma^c} \sum_{m\in\mathbb{J}_n} c_{n,m} w_{n,m} \right\|^2 \\ &\leqslant B \sum_{n\in J} \sum_{m\in\mathbb{J}_n} |c_{n,m}|^2, \end{aligned}$$

that is, $\{\Lambda_n\}_{n\in\sigma} \cup \{\Omega_n\}_{n\in\sigma^c}$ is a g-Riesz sequence if and only if

$$\{\Lambda_n^* e_{n,m}\}_{\substack{n\in\sigma \\ m\in\mathbb{J}_n}} \cup \{\Omega_n^* e_{n,m}\}_{\substack{n\in\sigma^c \\ m\in\mathbb{J}_n}}$$

is a Riesz sequence.

Next, we show that $\{\Gamma_n\}_{n\in\mathbb{N}}$ is g-complete if and only if

$$\{\Lambda_n^* e_{n,m}\}_{\substack{n\in\sigma \\ m\in\mathbb{J}_n}} \cup \{\Omega_n^* e_{n,m}\}_{\substack{n\in\sigma^c \\ m\in\mathbb{J}_n}}$$

is complete.

$$\begin{aligned} \{x : \Gamma_n x = 0, n \in \mathbb{N}\} &= \{x : \Lambda_n x = 0, n \in \sigma\} \cup \{x : \Omega_n x = 0, n \in \sigma^c\} \\ &= \left\{ x : \sum_{m\in\mathbb{J}_n} \langle x, v_{n,m}\rangle e_{n,m} = 0, n \in \sigma \right\} \\ &\cup \left\{ x : \sum_{m\in\mathbb{J}_n} \langle x, w_{n,m}\rangle e_{n,m} = 0, n \in \sigma^c \right\} \\ &= \{x : \langle x, v_{n,m}\rangle = 0, n \in \sigma, m \in \mathbb{J}_n\} \\ &\cup \{x : \langle x, w_{n,m}\rangle = 0, n \in \sigma^c, m \in \mathbb{J}_n\}. \end{aligned}$$

This completes the proof. □

Example 5.3. Let $\mathcal{H} = \mathbb{C}^N$, where $N > 1$ is any odd natural number, and let $\{e_n\}_{n=1}^N$ be the canonical orthonormal basis for $\mathcal{H}$, i.e.,

$$e_n = \Big(0, \dots, 0, \underbrace{1}_{n\text{th-place}}, 0, \dots, 0\Big).$$

Suppose that $\mathcal{H}_n = \text{span}\{e_n + e_{n+1}\}$ for $n \in [N-1]$ and $\mathcal{H}_N = \text{span}\{e_1 + e_N\}$. Then,

$$\{e_{n,m}\}_{m=1} = \left\{ \frac{1}{\sqrt{2}} \Big(0, \dots, 0, \underbrace{1}_{n\text{th-place}}, 1, 0, \dots, 0\Big) \right\}$$

is an orthonormal basis of $\mathcal{H}_n$ $(n \in [n-1])$ and

$$\{e_{N,m}\}_{m=1} = \left\{ \frac{1}{\sqrt{2}} (1, 0, \dots, 0, 1) \right\}$$

is an orthonormal basis of $\mathcal{H}_N$.

Let $\Lambda \equiv \{\Lambda_n\}_{n=1}^N$ and $\Omega \equiv \{\Omega_n\}_{n=1}^N$, where Λ_n is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_n$, and Ω_n is the orthogonal projection of $\mathcal{H}$ onto $\text{span}\{e_n\}$ for each n, $1 \leq n \leq N$. Clearly,

$$\Lambda_n^* e_{n,1} = e_{n,1} \quad \text{and} \quad \Omega_n^* e_{n,1} = \frac{1}{\sqrt{2}} e_n.$$

It is easy to verify that $\{\Lambda_n^* e_{n,m}\}_{n\in[N],m=1}$ and $\{\Omega_n^* e_{n,m}\}_{n\in[N],m=1}$ are Riesz bases for $\mathcal{H}$. Furthermore, for any $\sigma \subset \mathbb{N}$,

$$\{\Lambda_n^* e_{n,m}\}_{\substack{n\in\sigma \\ m=1}} \bigcup \{\Omega_n^* e_{n,m}\}_{\substack{n\in\sigma^c \\ m=1}}$$

is a Riesz basis for $\mathcal{H}$. Hence, by Corollary 5.2, Λ and Ω are g-woven.

The next theorem provides sufficient conditions for weaving g-Riesz bases in terms of g-Riesz sequences. This generalizes [**1,** Theorem 5.2].

Theorem 5.4. *Let $\Lambda \equiv \{\Lambda_i\}_{i\in\mathbb{N}}$ and $\Omega \equiv \{\Omega_i\}_{i\in\mathbb{N}}$ be g-Riesz bases for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{N}\}$, for which there are uniform constants $0 < A \leqslant B < \infty$ so that, for every $\sigma \subset \mathbb{N}$, the family*

$$\{\Lambda_i\}_{i\in\sigma} \cup \{\Omega_i\}_{i\in\sigma^c}$$

is a g-Riesz sequence with g-Riesz bounds A and B. Then, for every $\sigma \subset \mathbb{N}$, the family $\{\Lambda_i\}_{i\in\sigma} \cup \{\Omega_i\}_{i\in\sigma^c}$ is a g-Riesz basis.

Proof. We prove the result in the following steps.

Step 1. First, we discuss the case $|\sigma| < \infty$. We prove the result by induction on the cardinality of σ. The case $|\sigma| = 0$ is trivial. Suppose that the result is true for every σ with $|\sigma| = n$.

Now, let $\sigma \subset \mathbb{N}$ with $|\sigma| = n+1$, and choose $i_0 \in \sigma$. Let $\sigma_1 = \sigma \backslash \{i_0\}$. Then,

$$\{\Lambda_i\}_{i\in\sigma_1} \cup \{\Omega_i\}_{i\in\sigma_1^c}$$

is a g-Riesz basis by induction hypothesis. Assume that

$$\{\Lambda_i\}_{i\in\sigma} \cup \{\Omega_i\}_{i\in\sigma^c}$$

is not a g-Riesz basis, that is,

$$\{\Lambda_i^* e_{i,k}\}_{\substack{i\in\sigma\\k\in\mathbb{N}}} \cup \{\Omega_i^* e_{i,k}\}_{\substack{i\in\sigma^c\\k\in\mathbb{N}}}$$

is not complete in $\mathcal{H}$. Then,

$$\Omega_{i_0}^* e_{i_0,k} \notin \operatorname{span}\left(\{\Lambda_i^* e_{i,k}\}_{\substack{i\in\sigma\\k\in\mathbb{N}}} \cup \{\Omega_i^* e_{i,k}\}_{\substack{i\in\sigma^c\\k\in\mathbb{N}}}\right).$$

Indeed, if

$$\Omega_{i_0}^* e_{i_0,k} \in \operatorname{span}\left(\{\Lambda_i^* e_{i,k}\}_{\substack{i\in\sigma\\k\in\mathbb{N}}} \cup \{\Omega_i^* e_{i,k}\}_{\substack{i\in\sigma^c\\k\in\mathbb{N}}}\right),$$

then

$$\overline{\operatorname{span}}\left(\{\Lambda_i^* e_{i,k}\}_{\substack{i\in\sigma\\k\in\mathbb{N}}} \cup \{\Omega_i^* e_{i,k}\}_{\substack{i\in\sigma^c\\k\in\mathbb{N}}}\right) \supset \overline{\operatorname{span}}\left(\{\Lambda_i^* e_{i,k}\}_{\substack{i\in\sigma_1\\k\in\mathbb{N}}} \cup \{\Omega_i^* e_{i,k}\}_{\substack{i\in\sigma_1^c\\k\in\mathbb{N}}}\right) = \mathcal{H},$$

that is,

$$\{\Lambda_i^* e_{i,k}\}_{\substack{i\in\sigma\\k\in\mathbb{N}}} \cup \{\Omega_i^* e_{i,k}\}_{\substack{i\in\sigma^c\\k\in\mathbb{N}}}$$

is complete in $\mathcal{H}$, which is a contradiction. Hence,

$$\{\Gamma_i\}_{i\in\mathbb{N}} \equiv \{\Lambda_i^* e_{i,k}\}_{\substack{i\in\sigma\\k\in\mathbb{N}}} \cup \{\Omega_i^* e_{i,k}\}_{\substack{i\in\sigma^c\\k\in\mathbb{N}}} \cup \{\Omega_{i_0}^* e_{i_0,k}\}$$

is a Riesz sequence in $\mathcal{H}$.

Now, $\sigma_1^c = \sigma^c \cup \{i_0\}$. We obtained $\{\Lambda_i^* e_{i,k}\}_{i\in\sigma_1,k\in\mathbb{N}} \cup \{\Omega_i^* e_{i,k}\}_{i\in\sigma_1^c,k\in\mathbb{N}}$ by deleting the element $\Lambda_{i_0}^* e_{i_0,k}$ from the Riesz sequence $\{\Gamma_i\}_{i\in\mathbb{N}}$.

Therefore, $\{\Lambda_i^* e_{i,k}\}_{i\in\sigma_1,k\in\mathbb{N}} \cup \{\Omega_i^* e_{i,k}\}_{i\in\sigma_1^c,k\in\mathbb{N}}$ cannot be a Riesz basis for $\mathcal{H}$, i.e., $\{\Lambda_i\}_{i\in\sigma_1} \cup \{\Omega_i\}_{i\in\sigma_1^c}$ cannot be a g-Riesz basis, which is a contradiction. Hence,

$$\{\Lambda_i\}_{i\in\sigma} \cup \{\Omega_i\}_{i\in\sigma^c}$$

is a g-Riesz basis.

Step 2. Consider $|\sigma| = \infty$. Suppose that there exists a $\sigma \in \mathbb{N}$ with both σ and σ^c infinite, such that $\{\Lambda_i\}_{i\in\sigma} \cup \{\Omega_i\}_{i\in\sigma^c}$ is not g-complete, i.e., $\{\Lambda_i^* e_{i,k}\}_{i\in\sigma,k\in\mathbb{N}} \cup \{\Omega_i^* e_{i,k}\}_{i\in\sigma^c,k\in\mathbb{N}}$ is not complete in $\mathcal{H}$. Then,

$$M = \overline{\operatorname{span}}\left(\{\Lambda_i^* e_{i,k}\}_{\substack{i\in\sigma\\k\in\mathbb{N}}} \cup \{\Omega_i^* e_{i,k}\}_{\substack{i\in\sigma^c\\k\in\mathbb{N}}}\right) \neq \mathcal{H}.$$

Thus, there exists a non-zero vector $x_0 \in \mathcal{H}$ such that $x_0 \perp M$. Since $\{\Omega_i^* e_{i,k}\}_{i,k\in\mathbb{N}}$ is a Bessel sequence, we can find $\sigma_1 \subset \sigma$ with $|\sigma| < \infty$ such that

$$\sum_{i\in\sigma\setminus\sigma_1} \sum_{k\in\mathbb{N}} |\langle x_0, \Omega_i^* e_{i,k}\rangle|^2 < \frac{A}{2}\|x_0\|^2.$$

From Step 1, the family

$$\{\Lambda_i^* e_{i,k}\}_{\substack{i\in\sigma_1\\k\in\mathbb{N}}} \cup \{\Omega_i^* e_{i,k}\}_{\substack{i\in\sigma\setminus\sigma_1\\k\in\mathbb{N}}} \cup \{\Omega_i^* e_{i,k}\}_{\substack{i\in\sigma^c\\k\in\mathbb{N}}}$$

is a Riesz basis with Riesz bounds A and B. Using $x_0 \perp M$, we compute

$$\begin{aligned} A\|x_0\|^2 &\leqslant \sum_{i\in\sigma_1} \sum_{k\in\mathbb{N}} |\langle x_0, \Lambda_i^* e_{i,k}\rangle|^2 \\ &\quad + \sum_{i\in\sigma\setminus\sigma_1} \sum_{k\in\mathbb{N}} |\langle x_0, \Omega_i^* e_{i,k}\rangle|^2 \\ &\quad + \sum_{i\in\sigma^c} \sum_{k\in\mathbb{N}} |\langle x_0, \Omega_i^* e_{i,k}\rangle|^2 \\ &= \sum_{i\in\sigma\setminus\sigma_1} \sum_{k\in\mathbb{N}} |\langle x_0, \Omega_i^* e_{i,k}\rangle|^2 < \frac{A}{2}\|x_0\|^2, \end{aligned}$$

which is absurd. Thus, $\{\Lambda_i\}_{i\in\sigma} \cup \{\Omega_i\}_{i\in\sigma^c}$ is g-complete, and hence, a g-Riesz basis. □

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Existence of Solutions to a New Class of Abstract Non-Instantaneous Impulsive Fractional Integro-Differential Equations

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Abstract: In this paper we prove the sufficient conditions for the existence and uniqueness of piecewise continuous mild solutions to fractional integro-differential equations in a Banach space with non instantaneous impulses. The results are established by using the theory of sectorial operators and the fixed point theorem. We discuss an example to illustrate the analytical results obtained.

Keywords: *sectorial operator; solution operator; non-instantaneous impulses; Krasnoselskii's fixed point theorem.*

Mathematics Subject Classification (2010): 34G20, 34K30, 34K40, 47N20.

1 Introduction

Let $(X, \|\cdot\|)$ be a complex Banach space. The objective of this paper is to study the solutions to a new class of abstract integro-differential equations of fractional order with non-instantaneous impulses in X :

$$\left.\begin{aligned} {}^cD_t^\alpha[u(t)+\varphi(t,u(t))] &= Au(t)+J_t^{1-\alpha}f(t,u(t)), \\ &\qquad t\in(s_i,t_{i+1}],\ i=0,1,\cdots,N,\ 0<\alpha<1, \\ u(t) &= g_i(t,u(t)),\quad t\in(t_i,s_i],\quad i=1,2,\cdots,N, \\ u(0) &= u_0, \end{aligned}\right\} \tag{1}$$

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Research Article

Various spectra and energies of commuting graphs of finite rings

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Abstract

The commuting graph of a non-commutative ring R with center $Z(R)$ is a simple undirected graph whose vertex set is $R \setminus Z(R)$ and two vertices x, y are adjacent if and only if $xy = yx$. In this paper, we compute various spectra and energies of commuting graphs of some classes of finite rings and study their consequences.

Mathematics Subject Classification (2010). 16P10, 05C50, 15A18, 05C25

Keywords. commuting graph, spectrum, energy, finite ring

1. Introduction

Let R be a non-commutative ring with center $Z(R)$. The commuting graph of R, denoted by Γ_R, is a simple undirected graph whose vertex set is $R \setminus Z(R)$ and two vertices x, y are adjacent if and only if $xy = yx$. In recent years, many mathematicians have considered commuting graph of different rings and studied various graph theoretic aspects (see [1, 3, 12, 13, 17, 19, 20, 23]). Some generalizations of Γ_R are also considered in [2, 9].

In Section 2, we compute spectrum, Laplacian spectrum and Signless Laplacian spectrum of commuting graphs of some classes of finite rings. Recall that the spectrum of a graph $\mathcal{G}$ denoted by $\mathrm{Spec}(\mathcal{G})$ is the set

$$\left\{\lambda_1^{k_1}, \lambda_2^{k_2}, \ldots, \lambda_n^{k_n}\right\},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the adjacency matrix of $\mathcal{G}$ with multiplicities $k_1, k_2, \ldots, k_n$ respectively. Let $A(\mathcal{G})$ and $D(\mathcal{G})$ denote the adjacency matrix and degree matrix of a graph $\mathcal{G}$ respectively. Then the Laplacian matrix and Signless Laplacian matrix of $\mathcal{G}$ are given by $L(\mathcal{G}) = D(\mathcal{G}) - A(\mathcal{G})$ and $Q(\mathcal{G}) = D(\mathcal{G}) + A(\mathcal{G})$ respectively. Let L-spec($\mathcal{G}$) and Q-spec($\mathcal{G}$) be the Laplacian spectrum and Signless Laplacian spectrum of $\mathcal{G}$ respectively. Then L-spec($\mathcal{G}$) $= \left\{\beta_1^{b_1}, \beta_2^{b_2}, \ldots, \beta_m^{b_m}\right\}$ and Q-spec($\mathcal{G}$) $= \{\gamma_1^{c_1}, \gamma_2^{c_2}, \ldots, \gamma_n^{c_n}\}$, where $\beta_1, \beta_2, \ldots, \beta_m$ are the eigenvalues of $L(\mathcal{G})$ with multiplicities $b_1, b_2, \ldots, b_m$ and $\gamma_1, \gamma_2, \ldots, \gamma_n$ are the eigenvalues of $Q(\mathcal{G})$ with multiplicities $c_1, c_2, \ldots, c_n$ respectively. The energy, Laplacian energy and Signless Laplacian energy of a graph $\mathcal{G}$ are given by

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$$E(\mathcal{G}) = \sum_{\lambda \in \mathrm{Spec}(\mathcal{G})} |\lambda|,$$
$$LE(\mathcal{G}) = \sum_{\mu \in \text{L-spec}(\mathcal{G})} \left|\mu - \frac{2|e(\mathcal{G})|}{|v(\mathcal{G})|}\right| \text{ and} \tag{1.1}$$
$$LE^{+}(\mathcal{G}) = \sum_{\nu \in \text{Q-spec}(\mathcal{G})} \left|\nu - \frac{2|e(\mathcal{G})|}{|v(\mathcal{G})|}\right|,$$

where $v(\mathcal{G})$ and $e(\mathcal{G})$ are the set of vertices and edges of $\mathcal{G}$, respectively.

Throughout the paper R denotes a non-commutative finite ring and p, q denote distinct primes. $\frac{R}{Z(R)}$ denotes the additive quotient group. Also, K_n denotes a complete graph on n vertices and lK_n denotes the disjoint union of l copies of K_n.

2. Various spectra

In [19], various spectra of commuting graphs of some small order finite non-commutative rings have been computed. In this section we consider more classes of finite non-commutative rings. The following theorem is useful in computing various spectra of commuting graphs of finite rings.

Theorem 2.1 ([18, Theorem 2.1]). *If $\mathcal{G} = l_1K_{n_1} \sqcup l_2K_{n_2} \sqcup \cdots \sqcup l_mK_{n_m}$, then*

(a) $\mathrm{Spec}(\mathcal{G}) = \left\{(-1)^{\sum_{i=1}^{m} l_i(n_i-1)}, (n_1-1)^{l_1}, (n_2-1)^{l_2}, \ldots, (n_m-1)^{l_m}\right\}$.

(b) $\text{L-spec}(\mathcal{G}) = \left\{0^{\sum_{i=1}^{m} l_i}, n_1^{l_1(n_1-1)}, n_2^{l_2(n_2-1)}, n_m^{l_m(n_m-1)}\right\}$.

(c) $\text{Q-spec}(\mathcal{G}) = \Big\{(2n_1-2)^{l_1}, (n_1-2)^{l_1(n_1-1)}, (2n_2-2)^{l_2}, (n_2-2)^{l_2(n_2-1)}, \ldots,$
$(2n_m-2)^{l_m}, (n_m-2)^{l_m(n_m-1)}\Big\}$.

Theorem 2.2. *Let $|R| = p^4$ and R has unity.*

(a) *If $|Z(R)| = p$ then* $\mathrm{Spec}(\Gamma_R) = \left\{(-1)^{(p^2+p+1)(p^2-p-1)}, (p^2-p-1)^{p^2+p+1}\right\}$,
$\text{L-spec}(\Gamma_R) = \left\{0^{p^2+p+1}, (p^2-p)^{(p^2+p+1)(p^2-p-1)}\right\}$ *and*
$\text{Q-spec}(\Gamma_R) = \left\{(2p^2-2p-2)^{p^2+p+1}, (p^2-p-2)^{(p^2+p+1)(p^2-p-1)}\right\}$; *or*
$\mathrm{Spec}(\Gamma_R) = \left\{(-1)^{l_1(p^2-p-1)+l_2(p^3-p-1)}, (p^2-p-1)^{l_1}, (p^3-p-1)^{l_2}\right\}$,
$\text{L-spec}(\Gamma_R) = \left\{0^{l_1+l_2}, (p^2-p)^{l_1(p^2-p-1)}, (p^3-p)^{l_2(p^3-p-1)}\right\}$ *and* $\text{Q-spec}(\Gamma_R) =$
$\left\{(2p^2-2p-2)^{l_1}, (p^2-p-2)^{l_1(p^2-p-1)}, (2p^3-2p-2)^{l_2}, (p^3-p-2)^{l_2(p^3-p-1)}\right\}$,
where $l_1 + l_2(p+1) = p^2+p+1$.

(b) *If $|Z(R)| = p^2$ then* $\mathrm{Spec}(\Gamma_R) = \left\{(-1)^{(p+1)(p^3-p^2-1)}, (p^3-p^2-1)^{p+1}\right\}$,
$\text{L-spec}(\Gamma_R) = \left\{0^{p+1}, (p^3-p^2)^{(p+1)(p^3-p^2-1)}\right\}$ *and*
$\text{Q-spec}(\Gamma_R) = \left\{(2p^3-2p^2-2)^{p+1}, (p^3-p^2-2)^{(p+1)(p^3-p^2-1)}\right\}$.

Proof. (a) If $|Z(R)| = p$ then, by Theorem 2.5 of [22], we have $\Gamma_R = (p^2+p+1)K_{(p^2-p)}$ or $l_1K_{(p^2-p)} \sqcup l_2K_{(p^3-p)}$, where $l_1 + l_2(p+1) = p^2+p+1$. Hence, the result follows from Theorem 2.1.

(b) If $|Z(R)| = p^2$ then, by Theorem 2.5 of [22], we have $\Gamma_R = (p+1)K_{(p^3-p^2)}$. Hence, the result follows from Theorem 2.1. □

Theorem 2.3. *Let $|R| = p^5$ with unity and $Z(R)$ is not a field.*

(a) *If $|Z(R)| = p^2$ then* $\text{Spec}(\Gamma_R) = \left\{(-1)^{(p^2+p+1)(p^3-p^2-1)}, (p^3-p^2-1)^{p^2+p+1}\right\}$,
$\text{L-spec}(\Gamma_R) = \left\{0^{p^2+p+1}, (p^3-p^2)^{(p^2+p+1)(p^3-p^2-1)}\right\}$ *and*
$\text{Q-spec}(\Gamma_R) = \left\{(2p^3-2p^2-2)^{p^2+p+1}, (p^3-p^2-2)^{(p^2+p+1)(p^3-p^2-1)}\right\}$; *or*
$\text{Spec}(\Gamma_R) = \left\{(-1)^{l_1(p^3-p^2-1)+l_2(p^3-p-1)}, (p^3-p^2-1)^{l_1}, (p^3-p-1)^{l_2}\right\}$,
$\text{L-spec}(\Gamma_R) = \left\{0^{l_1+l_2}, \ (p^3-p^2)^{l_1(p^3-p^2-1)}, \ (p^3-p)^{l_2(p^3-p-1)}\right\}$ *and*
$\text{Q-spec}(\Gamma_R) = \left\{(2p^3-2p^2-2)^{l_1}, (p^3-p^2-2)^{l_1(p^3-p^2-1)}, (2p^3-2p-2)^{l_2}, (p^3-p-2)^{l_2(p^3-p-1)}\right\}$, *where* $l_1 + l_2(p+1) = p^2+p+1$.

(b) *If $|Z(R)| = p^3$ then* $\text{Spec}(\Gamma_R) = \left\{(-1)^{(p+1)(p^4-p^3-1)}, (p^4-p^3-1)^{p+1}\right\}$,
$\text{L-spec}(\Gamma_R) = \left\{0^{p+1}, (p^4-p^3)^{(p+1)(p^4-p^3-1)}\right\}$ *and*
$\text{Q-spec}(\Gamma_R) = \left\{(2p^4-2p^3-2)^{p+1}, (p^4-p^3-2)^{(p+1)(p^4-p^3-1)}\right\}$.

Proof. (a) If $|Z(R)| = p^2$ then, by Theorem 2.7 of [22], we have $\Gamma_R = (p^2+p+1)K_{(p^3-p^2)}$ or $l_1K_{(p^3-p^2)} \sqcup l_2K_{(p^3-p)}$, where $l_1 + l_2(p+1) = p^2+p+1$. Hence, the result follows from Theorem 2.1.

(b) If $|Z(R)| = p^3$ then, by Theorem 2.7 of [22], we have $\Gamma_R = (p+1)K_{(p^4-p^3)}$. Hence, the result follows from Theorem 2.1. □

Theorem 2.4. *Let $|R| = pq$ and $Z(R) = \{0\}$.*

(a) *If $(p-1) \mid (pq-1)$ then* $\text{Spec}(\Gamma_R) = \left\{(-1)^{\frac{(pq-1)(p-2)}{p-1}}, (p-2)^{\frac{pq-1}{p-1}}\right\}$, $\text{L-spec}(\Gamma_R) = \left\{0^{\frac{pq-1}{p-1}}, (p-1)^{\frac{(pq-1)(p-2)}{p-1}}\right\}$ *and* $\text{Q-spec}(\Gamma_R) = \left\{(2p-4)^{\frac{pq-1}{p-1}}, (p-3)^{\frac{(pq-1)(p-2)}{p-1}}\right\}$.

(b) *If $(q-1) \mid (pq-1)$ then* $\text{Spec}(\Gamma_R) = \left\{(-1)^{\frac{(pq-1)(q-2)}{q-1}}, (q-2)^{\frac{pq-1}{q-1}}\right\}$, $\text{L-spec}(\Gamma_R) = \left\{0^{\frac{pq-1}{q-1}}, (q-1)^{\frac{(pq-1)(q-2)}{q-1}}\right\}$ *and* $\text{Q-spec}(\Gamma_R) = \left\{(2q-4)^{\frac{pq-1}{q-1}}, (q-3)^{\frac{(pq-1)(q-2)}{q-1}}\right\}$.

(c) *If $l_1(p-1) + l_2(q-1) = pq-1$ then* $\text{Spec}(\Gamma_R) = \left\{(-1)^{l_1(p-2)+l_2(q-2)}, (p-2)^{l_1}, (q-2)^{l_2}\right\}$, $\text{L-spec}(\Gamma_R) = \left\{0^{l_1+l_2}, (p-1)^{l_1(p-2)}, (q-1)^{l_2(q-2)}\right\}$ *and*
$\text{Q-spec}(\Gamma_R) = \left\{(2p-4)^{l_1}, (p-3)^{l_1(p-2)}, (2q-4)^{l_2}, (q-3)^{l_2(q-2)}\right\}$.

Proof. It was shown in [23, Theorem 2.8] that

$$\Gamma_R = \begin{cases} \frac{pq-1}{p-1}K_{p-1}, & \text{if } (p-1) \mid (pq-1) \\ \frac{pq-1}{q-1}K_{q-1}, & \text{if } (q-1) \mid (pq-1) \\ l_1K_{p-1} \sqcup l_2K_{q-1}, & \text{if } l_1(p-1) + l_2(q-1) = pq-1. \end{cases}$$

Hence, the result follows from Theorem 2.1. □

Theorem 2.5. *Let $|R| = p^2q$ and $Z(R) = \{0\}$.*

(a) *If $t \in \{p, q, p^2, pq\}$ and $(t-1) \mid (p^2q-1)$ then* $\text{Spec}(\Gamma_R) = \left\{(-1)^{\frac{(p^2q-1)(t-2)}{t-1}}, (t-2)^{\frac{p^2q-1}{t-1}}\right\}$, $\text{L-spec}(\Gamma_R) = \left\{0^{\frac{p^2q-1}{t-1}}, (t-1)^{\frac{(p^2q-1)(t-2)}{t-1}}\right\}$ *and*
$\text{Q-spec}(\Gamma_R) = \left\{(2t-4)^{\frac{p^2q-1}{t-1}}, (t-3)^{\frac{(p^2q-1)(t-2)}{t-1}}\right\}$.

(b) *If $l_1(p-1) + l_2(q-1) + l_3(p^2-1) + l_4(pq-1) = p^2q-1$ then* $\text{Spec}(\Gamma_R) = \left\{(-1)^{l_1(p-2)+l_2(q-2)+l_3(p^2-2)+l_4(pq-2)}, (p-2)^{l_1}, (q-2)^{l_2}, (p^2-2)^{l_3}, (pq-2)^{l_4}\right\}$,

$$\text{L-spec}(\Gamma_R) = \Big\{0^{l_1+l_2+l_3+l_4}, (p-1)^{l_1(p-2)}, (q-1)^{l_2(q-2)}, (p^2-1)^{l_3(p^2-2)}, (pq-1)^{l_4(pq-2)}\Big\} \quad \textit{and} \quad \text{Q-spec}(\Gamma_R) = \Big\{(2p-4)^{l_1}, (p-3)^{l_1(p-2)}, (2q-4)^{l_2}, (q-3)^{l_2(q-2)}, (2p^2-4)^{l_3}, (p^2-3)^{l_3(p^2-2)}, (2pq-4)^{l_4}, (pq-3)^{l_4(pq-2)}\Big\}.$$

Proof. (a) By [23, Theorem 2.9], we have $\Gamma_R = \frac{p^2q-1}{t-1}K_{t-1}$ if $t \in \{p, q, p^2, pq\}$ and $(t-1) \mid (p^2q-1)$. Hence, the result follows from Theorem 2.1.

(b) By [23, Theorem 2.9], we also have $\Gamma_R = l_1K_{p-1} \sqcup l_2K_{q-1} \sqcup l_3K_{p^2-1} \sqcup l_4K_{pq-1}$ if $l_1(p-1) + l_2(q-1) + l_3(p^2-1) + l_4(pq-1) = p^2q-1$. Hence, the result follows from Theorem 2.1. □

Theorem 2.6. *Let $|R| = p^3q$ and R has unity. If $|Z(R)| = pq$ then*

$$\text{Spec}(\Gamma_R) = \Big\{(-1)^{(p+1)(p^2q-pq-1)}, (p^2q-pq-1)^{p+1}\Big\},$$
$$\text{L-spec}(\Gamma_R) = \Big\{0^{p+1}, (p^2q-pq)^{(p+1)(p^2q-pq-1)}\Big\} \quad \textit{and}$$
$$\text{Q-spec}(\Gamma_R) = \Big\{(2p^2q-2pq-2)^{p+1}, (p^2q-pq-2)^{(p+1)(p^2q-pq-1)}\Big\}.$$

Proof. If $|Z(R)| = pq$ then, by [23, Theorem 2.12], we have $\Gamma_R = (p+1)K_{p^2q-pq}$. Hence, the result follows from Theorem 2.1. □

We conclude this section with the following result.

Theorem 2.7. *Let $|R| = p^3q$, R has unity and $|Z(R)| = p^2$.*

(a) *If $(p-1) \mid (pq-1)$ then* $\text{Spec}(\Gamma_R) = \Big\{(-1)^{\frac{(pq-1)(p^3-p^2-1)}{p-1}}, (p^3-p^2-1)^{\frac{pq-1}{p-1}}\Big\}$,
$\text{L-spec}(\Gamma_R) = \Big\{0^{\frac{pq-1}{p-1}}, (p^3-p^2)^{\frac{(pq-1)(p^3-p^2-1)}{p-1}}\Big\}$ *and*
$\text{Q-spec}(\Gamma_R) = \Big\{(2p^3-2p^2-2)^{\frac{pq-1}{p-1}}, (p^3-p^2-2)^{\frac{(pq-1)(p^3-p^2-1)}{p-1}}\Big\}$.

(b) *If $(q-1) \mid (pq-1)$ then* $\text{Spec}(\Gamma_R) = \Big\{(-1)^{\frac{(pq-1)(p^2q-p^2-1)}{q-1}}, (p^2q-p^2-1)^{\frac{pq-1}{q-1}}\Big\}$,
$\text{L-spec}(\Gamma_R) = \Big\{0^{\frac{pq-1}{q-1}}, (p^2q-p^2)^{\frac{(pq-1)(p^2q-p^2-1)}{q-1}}\Big\}$ *and*
$\text{Q-spec}(\Gamma_R) = \Big\{(2p^2q-2p^2-2)^{\frac{pq-1}{q-1}}, (p^2q-p^2-2)^{\frac{(pq-1)(p^2q-p^2-1)}{q-1}}\Big\}$.

(c) *If $l_1(p-1) + l_2(q-1) = pq-1$ then*
$\text{Spec}(\Gamma_R) = \Big\{(-1)^{l_1(p^3-p^2-1)+l_2(p^2q-p^2-1)}, (p^3-p^2-1)^{l_1}, (p^2q-p^2-1)^{l_2}\Big\}$,
$\text{L-spec}(\Gamma_R) = \Big\{0^{l_1+l_2}, (p^3-p^2)^{l_1(p^3-p^2-1)}, (p^2q-p^2)^{l_2(p^2q-p^2-1)}\Big\}$ *and*
$\text{Q-spec}(\Gamma_R) = \Big\{(2p^3-2p^2-2)^{l_1}, (p^3-p^2-2)^{l_1(p^3-p^2-1)}, (2p^2q-2p^2-2)^{l_2}, (p^2q-p^2-2)^{l_2(p^2q-p^2-1)}\Big\}$.

Proof. If $|Z(R)| = p^2$ then, by [23, Theorem 2.12], we have

$$\Gamma_R = \begin{cases} \frac{pq-1}{p-1}K_{p^3-p^2}, & \text{if } (p-1) \mid (pq-1) \\ \frac{pq-1}{q-1}K_{p^2q-p^2}, & \text{if } (q-1) \mid (pq-1) \\ l_1K_{p^3-p^2} \sqcup l_2K_{p^2q-p^2}, & \text{if } l_1(p-1) + l_2(q-1) = pq-1. \end{cases}$$

Hence, the result follows from Theorem 2.1. □

3. Various energies

If $\frac{R}{Z(R)}$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, then it was shown in Theorem 3.1 of [19] that

$$E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = 2(p^2-1)|Z(R)| - 2(p+1). \tag{3.1}$$

As a consequence of (3.1), in the following results, we compute various energies of commuting graphs of several well-known classes of finite rings.

Theorem 3.1. *If* $|R| = p^2$, *then* $E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = 2(p^2-p-2)$.

Proof. If R is a non-commutative ring of order p^2, then $Z(R)$ has only one element. Therefore, the additive group $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence the result follows from (3.1). □

Theorem 3.2. *If* $|R| = p^3$ *and* R *has unity, then*

$$E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = 2(p^3-2p-1).$$

Proof. If R is a non-commutative ring with unity of order p^3, then $Z(R)$ has p elements. Therefore, the additive group $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence the result follows from (3.1). □

A ring R is called an n-centralizer ring if $|\operatorname{Cent}(R)| = n$, where $\operatorname{Cent}(R) = \{C_R(x) : x \in R\}$. Various properties of n-centralizer rings can be found in [5, 10, 11]. In the following results we compute various energies of some finite n-centralizer rings.

Theorem 3.3. *If* $|\operatorname{Cent}(R)| = 4$, *then* $E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = 6|Z(R)| - 6$.

Proof. It was shown in [11, Theorem 3.2] that the additive quotient group $\frac{R}{Z(R)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ if R is a finite 4-centralizer ring. Hence, the result follows from (3.1) putting $p = 2$. □

Theorem 3.4. *If* $|\operatorname{Cent}(R)| = 5$, *then* $E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = 16|Z(R)| - 8$.

Proof. It was shown in [11, Theorem 4.3] that the additive quotient group $\frac{R}{Z(R)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ if R is a finite 5-centralizer ring. Hence, the result follows from (3.1). □

Theorem 3.5. *If* R *is a finite* p*-ring and* $|\operatorname{Cent}(R)| = (p+2)$, *then*

$$E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = 2(p^2-1)|Z(R)| - 2(p+1).$$

Proof. It was shown in [11, Theorem 2.12] that the additive quotient group $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ if R is a finite $(p+2)$-centralizer p-ring. Hence, the result follows from (3.1). □

In 1976, MacHale [16] initiated the study of commutativity degree of a finite ring R denoted by $\Pr(R)$. Recall that the commutativity degree of R is the probability that a randomly chosen pair of elements of R commute. Recent results on $\Pr(R)$ can be found in [4, 6–8]. In the following theorem we compute various energies of Γ_R for some given values of $\Pr(R)$.

Theorem 3.6. *Let* p *be the smallest prime dividing* $|R|$. *If* $\Pr(R) = \frac{p^2+p-1}{p^3}$ *then*

$$E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = 2(p^2-1)|Z(R)| - 2(p+1).$$

Proof. By Theorem 2 of [16] we have $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence, the result follows from (3.1). □

We have the following corollary to the above theorem.

Corollary 3.7. *If* $\Pr(R) = \frac{5}{8}$ *then*

$$E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = 6|Z(R)| - 6.$$

Now we compute various energies of Γ_R for the rings considered in Section 2. Note that one can do this using Theorems 2.1 - 2.7 and (1.1). However, using the following theorem one can also compute various energies.

Theorem 3.8. *If* $\mathcal{G} = l_1K_{n_1} \sqcup l_2K_{n_2}$, *then* $E(\mathcal{G}) = 2l_1(n_1-1) + 2l_2(n_2-1)$. *Further, if* $n_1 = n_2 = n$ *then*

$$E(lK_n) = LE(lK_n) = LE^+(lK_n) = 2l(n-1),$$

where $l = l_1 + l_2$.

Proof. By Theorem 2.1(a) we have

$$\mathrm{Spec}(\mathcal{G}) = \left\{(-1)^{\sum_{i=1}^{2} l_i(n_i-1)},\ (n_1-1)^{l_1},\ (n_2-1)^{l_2}\right\}.$$

Therefore, (1.1) gives

$$\begin{aligned} E(\mathcal{G}) &= |-1|\sum_{i=1}^{2} l_i(n_i-1) + l_1|n_1-1| + l_2|n_2-1| \\ &= l_1(n_1-1) + l_2(n_2-1) + l_1(n_1-1) + l_2(n_2-1) \\ &= 2l_1(n_1-1) + 2l_2(n_2-1). \end{aligned}$$

If $n_1 = n_2 = n$ and $l = l_1 + l_2$ then $\mathcal{G} = lK_n$ and so $E(lK_n) = 2l(n-1)$. In this case, we also have $|v(lK_n)| = ln$, $|e(lK_n)| = \frac{ln(n-1)}{2}$ and so $\frac{2|e(lK_n)|}{|v(lK_n)|} = n-1$.

By Theorem 2.1(b) we have L-spec$(lK_n) = \left\{0^l, n^{l(n-1)}\right\}$. Therefore

$$\left|0 - \frac{2|e(lK_n)|}{|v(lK_n)|}\right| = n-1 \text{ and } \left|n - \frac{2|e(lK_n)|}{|v(lK_n)|}\right| = 1.$$

Hence, (1.1) gives

$$LE(lK_n) = (n-1)l + l(n-1) = 2l(n-1).$$

Again, by Theorem 2.1(c) we also have Q-spec$(lK_n) = \left\{(2n-2)^l, (n-2)^{l(n-1)}\right\}$. Therefore

$$\left|2n-2 - \frac{2|e(lK_n)|}{|v(lK_n)|}\right| = n-1 \text{ and } \left|n-2 - \frac{2|e(lK_n)|}{|v(lK_n)|}\right| = 1.$$

Hence, (1.1) gives

$$LE^+(lK_n) = (n-1)l + l(n-1) = 2l(n-1).$$

This completes the proof. □

Theorem 3.9. *Let* R *have unity and* $|R| = p^4$.

(a) *If* $|Z(R)| = p$ *then* $E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = 2(p^2+p+1)(p^2-p-1)$ *or* $E(\Gamma_R) = 2l_1(p^2-p-1) + 2l_2(p^3-p-1)$, *where* $l_1 + l_2(p+1) = p^2+p+1$.

(b) *If* $|Z(R)| = p^2$ *then* $E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = 2(p+1)(p^3-p^2-1)$.

Proof. By Theorem 2.5 of [22], we have $\Gamma_R = (p^2+p+1)K_{(p^2-p)}$ or $l_1K_{(p^2-p)} \sqcup l_2K_{(p^3-p)}$ (where $l_1 + l_2(p+1) = p^2+p+1$) if $|Z(R)| = p$ and $(p+1)K_{(p^3-p^2)}$ if $|Z(R)| = p^2$. Hence, the result follows from Theorem 3.8. □

Theorem 3.10. *Let* R *have unity,* $|R| = p^5$ *and* $Z(R)$ *is not a field.*

(a) *If* $|Z(R)| = p^2$ *then* $E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = 2(p^2+p+1)(p^3-p^2-1)$ *or* $E(\Gamma_R) = 2l_1(p^3-p^2-1) + 2l_2(p^3-p-1)$, *where* $l_1 + l_2(p+1) = p^2+p+1$.

(b) *If* $|Z(R)| = p^3$ *then* $E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = 2(p+1)(p^4-p^3-1)$.

Proof. By Theorem 2.7 of [22], we have $\Gamma_R = (p^2+p+1)K_{(p^3-p^2)}$ or $l_1K_{(p^3-p^2)} \sqcup l_2K_{(p^3-p)}$ (where $l_1 + l_2(p+1) = p^2+p+1$) if $|Z(R)| = p^2$ and $(p+1)K_{(p^4-p^3)}$ if $|Z(R)| = p^3$. Hence, the result follows from Theorem 3.8. □

Theorem 3.11. *Let* $|R| = pq$ *and* $Z(R) = \{0\}$.

(a) *If* $(p-1) \mid (pq-1)$ *then* $E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = \frac{2(pq-1)(p-2)}{p-1}$.

(b) *If* $(q-1) \mid (pq-1)$ *then* $E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = \frac{2(pq-1)(q-2)}{q-1}$.

(c) *If* $l_1(p-1) + l_2(q-1) = pq-1$ *then* $E(\Gamma_R) = 2l_1(p-2) + 2l_2(q-2)$.

Proof. It was shown in [23, Theorem 2.8] that

$$\Gamma_R = \begin{cases} \frac{pq-1}{p-1}K_{p-1}, & \text{if } (p-1) \mid (pq-1) \\ \frac{pq-1}{q-1}K_{q-1}, & \text{if } (q-1) \mid (pq-1) \\ l_1K_{p-1} \sqcup l_2K_{q-1}, & \text{if } l_1(p-1) + l_2(q-1) = pq-1. \end{cases}$$

Hence, the result follows from Theorem 3.8. □

Theorem 3.12. *Let* $|R| = p^2q$ *and* $Z(R) = \{0\}$.

(a) *If* $t \in \{p, q, p^2, pq\}$ *and* $(t-1) \mid (p^2q-1)$ *then*

$$E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = \frac{2(p^2q-1)(t-2)}{t-1}.$$

(b) *If* $l_1(p-1) + l_2(q-1) + l_3(p^2-1) + l_4(pq-1) = p^2q-1$ *then*

$$E(\Gamma_R) = 2\left(p^2q - 1 - (l_1 + l_2 + l_3 + l_4\right).$$

Proof. (a) By [23, Theorem 2.9], we have $\Gamma_R = \frac{p^2q-1}{t-1}K_{t-1}$ if $t \in \{p, q, p^2, pq\}$ and $(t-1) \mid (p^2q-1)$. Hence, the result follows from Theorem 3.8.

(b) By [23, Theorem 2.9], we also have $\Gamma_R = l_1K_{p-1} \sqcup l_2K_{q-1} \sqcup l_3K_{p^2-1} \sqcup l_4K_{pq-1}$ if $l_1(p-1) + l_2(q-1) + l_3(p^2-1) + l_4(pq-1) = p^2q-1$. Hence, the result follows from Theorem 2.5 and (1.1). □

Theorem 3.13. *Let* R *have unity and* $|R| = p^3q$. *If* $|Z(R)| = pq$ *then*

$$E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = 2(p+1)(p^2q - pq - 1).$$

Proof. If $|Z(R)| = pq$ then, by [23, Theorem 2.12], we have $\Gamma_R = (p+1)K_{p^2q-pq}$. Hence the result follows from Theorem 3.8. □

Theorem 3.14. *Let* R *have unity,* $|R| = p^3q$ *and* $|Z(R)| = p^2$.

(a) *If* $(p-1) \mid (pq-1)$ *then* $E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = \frac{2(pq-1)(p^3-p^2-1)}{p-1}$.

(b) *If* $(q-1) \mid (pq-1)$ *then*

$$E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = \frac{2(pq-1)(p^2q-p^2-1)}{q-1}.$$

(c) *If* $l_1(p-1) + l_2(q-1) = pq-1$ *then*

$$E(\Gamma_R) = 2l_1(p^3 - p^2 - 1) + 2l_2(p^2q - p^2 - 1).$$

Proof. If $|Z(R)| = p^2$ then, by [23, Theorem 2.12], we have

$$\Gamma_R = \begin{cases} \frac{pq-1}{p-1}K_{p^3-p^2}, & \text{if } (p-1) \mid (pq-1) \\ \frac{pq-1}{q-1}K_{p^2q-p^2}, & \text{if } (q-1) \mid (pq-1) \\ l_1K_{p^3-p^2} \sqcup l_2K_{p^2q-p^2}, & \text{if } l_1(p-1) + l_2(q-1) = pq-1. \end{cases}$$

Hence, the result follows from Theorem 3.8. □

Note that the rings considered above are CC-rings. Recall that a non-commutative ring R is called a CC-ring if all the centralizers of its non-central elements are commutative. In other words, $C_R(x)$ for all $x \in R \backslash Z(R)$ is commutative, where $C_R(x) := \{y \in R : xy = yx\}$ is the centralizer of x. The study of CC-rings was initiated by Erfanian et al. in [13]. In the following theorem we compute energy of a CC-ring.

Theorem 3.15. *Let R be a finite CC-ring with distinct centralizers $S_1, S_2, \ldots, S_n$ of non-central elements of R. Then $E(\Gamma_R) = 2(|R| - |Z(R)| - n)$.*

Proof. By Theorem 2.1 of [12] we have

$$\mathrm{Spec}(\Gamma_R) = \{(-1)^{\sum_{i=1}^{n}|S_i|-n(|Z(R)|+1)}, (|S_1| - |Z(R)| - 1)^1, \ldots, (|S_n| - |Z(R)| - 1)^1\}.$$

Therefore

$$\begin{aligned} E(\Gamma_R) &= \sum_{i=1}^{n}|S_i| - n(|Z(R)| + 1) + (|S_1| - |Z(R)| - 1) + \cdots \\ &\qquad + (|S_n| - |Z(R)| - 1) \\ &= 2\sum_{i=1}^{n}|S_i| - 2n|Z(R)| - 2n. \end{aligned}$$

Since $\sum_{i=1}^{n}|S_i| = |R| + (n-1)|Z(R)|$, we get the required expression for $E(\Gamma_R)$. □

Corollary 3.16. *Let R be a finite CC-ring and A be any finite commutative ring. Then $E(\Gamma_{R\times A}) = 2(|R||A| - |Z(R)||A| - n)$, where $n = |\mathrm{Cent}(R)| - 1$.*

Proof. Follows from Theorem 3.15 noting that $R\times A$ is a CC-ring, $|\mathrm{Cent}(R)| = |\mathrm{Cent}(R\times A)|$ and $Z(R \times A) = Z(R) \times A$. □

4. Some consequences

A finite non-commutative ring R is called super integral if spectrum, Laplacian spectrum and Signless Laplacian spectrum of Γ_R contain only integers. The notion of super integral ring was introduced in [19]. It can be seen that all the rings considered in Section 2 are super integral.

A finite graph $\mathcal{G}$ is called hyperenergetic and borderenergetic if $E(\mathcal{G}) > E(K_{|v(\mathcal{G})|})$ and $E(\mathcal{G}) = E(K_{|v(\mathcal{G})|})$ respectively. Similarly, $\mathcal{G}$ is called L-hyperenergetic and L-borderenergetic if $LE(\mathcal{G}) > LE(K_{|v(\mathcal{G})|})$ and $LE(\mathcal{G}) = LE(K_{|v(\mathcal{G})|})$ respectively; $\mathcal{G}$ is called Q-hyperenergetic and Q-borderenergetic if $LE^+(\mathcal{G}) > LE^+(K_{|v(\mathcal{G})|})$ and $LE^+(\mathcal{G}) = LE^+(K_{|v(\mathcal{G})|})$ respectively. The study of hyperenergetic graph was initiated by Walikar et al. [24] and Gutman [15] in 1999. The concepts of borderenergetic and L-borderenergetic graphs were introduced by Gong et al. [14] and Tura [21] in the years 2015 and 2017 respectively.

A finite graph $\mathcal{G}$ is called super hyperenergetic if it is hyperenergetic, L-hyperenergetic and Q-hyperenergetic. Similarly, we define super borderenergetic graph. In this section, we show that the commuting graphs of the rings considered in Section 3 are neither super hyperenergetic nor super borderenergetic.

Theorem 4.1. *If $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ then Γ_R is neither super hyperenergetic nor super borderenergetic.*

Proof. We have $|v(\Gamma_R)| = |Z(R)|(p^2-1)$, since $|R| = p^2|Z(R)|$ and $|v(\Gamma_R)| = |R|-|Z(R)|$. Therefore

$$E(K_{|v(\Gamma_R)|}) = LE(K_{|v(\Gamma_R)|}) = LE^+(K_{|v(\Gamma_R)|}) = 2(|Z(R)|(p^2-1) - 1).$$

Since $2(|Z(R)|(p^2-1)) - (p+1) < 2(|Z(R)|(p^2-1) - 1)$, by (3.1) the result follows. □

Corollary 4.2. *Γ_R is neither super hyperenergetic nor super borderenergetic if*

(a) *R is of order p^2.*
(b) *R is of order p^3 with unity.*
(c) *R is a 4-centralizer ring.*

(d) *R is a 5-centralizer ring.*
(e) *R is a $(p+2)$-centralizer p-ring.*
(f) *p is the smallest prime dividing $|R|$ and* $\Pr(R) = \frac{p^2+p-1}{p^3}$.
(g) $\Pr(R) = \frac{5}{8}$.

Proof. In any of the above cases, the additive quotient group $\frac{R}{Z(R)}$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p. Hence, the result follows from Theorem 4.1. □

Theorem 4.3. *If R is a non-commutative ring with unity of order p^4 then Γ_R is neither super hyperenergetic nor super borderenergetic.*

Proof. If $|Z(G)| = p$ then $|v(\Gamma_R)| = p^4 - p$. Therefore

$$E(K_{|v(\Gamma_R)|}) = LE(K_{|v(\Gamma_R)|}) = LE^+(K_{|v(\Gamma_R)|}) = 2(p^4 - p - 1)$$
$$= 2((p^2 + p + 1)(p^2 - p) - 1).$$

We have $2(p^2 + p + 1)(p^2 - p - 1) < 2(p^4 - p - 1)$. Also

$$2l_1(p^2 - p - 1) + 2l_2(p^3 - p - 1) < 2((p^2 + p + 1)(p^2 - p) - 1)$$

if l_1, l_2 are positive integers such that $l_1 + l_2(p+1) = p^2 + p + 1$. Hence, the result follows Theorem 3.9.

If $|Z(G)| = p^2$ then $|v(\Gamma_R)| = p^4 - p^2$. Therefore

$$E(K_{|v(\Gamma_R)|}) = LE(K_{|v(\Gamma_R)|}) = LE^+(K_{|v(\Gamma_R)|}) = 2(p^4 - p^2 - 1).$$

We have

$$2(p + 1)(p^3 - p^2 - 1) < 2(p^4 - p^2 - 1).$$

Hence, the result follows Theorem 3.9. □

Theorem 4.4. *If R is a non-commutative ring with unity of order p^5 such that $Z(R)$ is not a field then Γ_R is neither super hyperenergetic nor super borderenergetic.*

Proof. If $|Z(R)| = p^2$ then $|v(\Gamma_R)| = p^5 - p^2$. Therefore

$$E(K_{|v(\Gamma_R)|}) = LE(K_{|v(\Gamma_R)|}) = LE^+(K_{|v(\Gamma_R)|}) = 2(p^5 - p^2 - 1)$$
$$= 2((p^3 - p^2)(p^2 + p + 1) - 1).$$

We have $2(p^2 + p + 1)(p^3 - p^2 - 1) < 2(p^5 - p^2 - 1)$. Also

$$2l_1(p^3 - p^2 - 1) + 2l_2(p^3 - p - 1) < 2((p^3 - p^2)(p^2 + p + 1) - 1)$$

if l_1, l_2 are positive integers such that $l_1 + l_2(p+1) = p^2 + p + 1$. Hence, the result follows from Theorem 3.10.

If $|Z(R)| = p^3$ then $|v(\Gamma_R)| = p^5 - p^3$. Therefore

$$E(K_{|v(\Gamma_R)|}) = LE(K_{|v(\Gamma_R)|}) = LE^+(K_{|v(\Gamma_R)|}) = 2(p^5 - p^3 - 1).$$

We have $2(p + 1)(p^4 - p^3 - 1) < 2(p^5 - p^3 - 1)$. Hence, the result follows from Theorem 3.10. □

Theorem 4.5. *Let R be a non-commutative ring of order pq such that $Z(R) = \{0\}$. Then Γ_R is neither super hyperenergetic nor super borderenergetic.*

Proof. We have $|v(\Gamma_R)| = pq - 1$. Therefore

$$E(K_{|v(\Gamma_R)|}) = LE(K_{|v(\Gamma_R)|}) = LE^+(K_{|v(\Gamma_R)|}) = 2(pq - 2).$$

If $t \in \{p, q\}$ and $(t-1) \mid (pq-1)$ then $\frac{2(pq-1)(t-2)}{t-1} < 2(pq-2)$. Also $2l_1(p-2)+2l_2(q-2) < 2(pq-2)$ if l_1, l_2 are positive integers and $l_1(p-1) + l_2(q-1) = pq - 1$. Hence, the result follows from Theorem 3.11. □

Theorem 4.6. *Let R be a non-commutative ring of order p^2q such that $Z(R) = \{0\}$. Then Γ_R is neither super hyperenergetic nor super borderenergetic.*

Proof. We have $|v(\Gamma_R)| = p^2q - 1$. Therefore

$$E(K_{|v(\Gamma_R)|}) = LE(K_{|v(\Gamma_R)|}) = LE^+(K_{|v(\Gamma_R)|}) = 2(p^2q - 2).$$

If $t \in \{p, q, p^2, pq\}$ and $(t-1) \mid (p^2q-1)$ then we have $\frac{2(p^2q-1)(t-2)}{t-1} < 2(p^2q-2)$. Also, $2\left(p^2q - 1 - (l_1 + l_2 + l_3 + l_4\right) < 2(p^2q-2)$ if l_1, l_2 are positive integers such that $l_1(p-1) + l_2(q-1) + l_3(p^2-1) + l_4(pq-1) = p^2q - 1$. Hence, the result follows from Theorem 3.12. □

Theorem 4.7. *Let R be a non-commutative ring with unity having order p^3q. If $|Z(R)|$ is not a prime then Γ_R is neither super hyperenergetic nor super borderenergetic.*

Proof. If $|Z(R)| = pq$ then $|v(\Gamma_R)| = p^3q - pq$. Therefore

$$E(K_{|v(\Gamma_R)|}) = LE(K_{|v(\Gamma_R)|}) = LE^+(K_{|v(\Gamma_R)|}) = 2(p^3q - pq - 1).$$

We have $2(p+1)(p^2q - pq - 1) < 2(p^3q - pq - 1)$. Hence, the result follows from Theorem 3.13.

If $|Z(R)| = p^2$ then $|v(\Gamma_R)| = p^3q - p^2$. Therefore

$$E(K_{|v(\Gamma_R)|}) = LE(K_{|v(\Gamma_R)|}) = LE^+(K_{|v(\Gamma_R)|}) = 2(p^3q - p^2 - 1).$$

We have $\frac{2(pq-1)(p^3-p^2-1)}{p-1} < 2(p^3q - p^2 - 1)$, $\frac{2(pq-1)(p^2q-p^2-1)}{q-1} < 2(p^3q - p^2 - 1)$ and $2l_1(p^3 - p^2 - 1) + 2l_2(p^2q - p^2 - 1) < 2(p^3q - p^2 - 1)$ if $(p-1) \mid (pq-1)$, $(q-1) \mid (pq-1)$ and $l_1(p-1) + l_2(q-1) = pq - 1$ respectively. Hence, the result follows from Theorem 3.14. □

We conclude this paper with the following general result.

Theorem 4.8. *If R is finite CC-ring then Γ_R is neither super hyperenergetic nor super borderenergetic.*

Proof. We have $|v(\Gamma_R)| = |R| - |Z(R)|$. Therefore

$$E(K_{|v(\Gamma_R)|}) = LE(K_{|v(\Gamma_R)|}) = LE^+(K_{|v(\Gamma_R)|}) = 2(|R| - |Z(R)| - 1).$$

Also, $2(|R| - |Z(R)| - n) < 2(|R| - |Z(R)| - 1)$, where n is the number of distinct centralizers of non-central elements of R. Hence, the results follows from Theorem 3.15. □

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Research Article

Solvable graphs of finite groups

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Abstract

Let G be a finite non-solvable group with solvable radical $\mathrm{Sol}(G)$. The solvable graph $\Gamma_s(G)$ of G is a graph with vertex set $G \setminus \mathrm{Sol}(G)$ and two distinct vertices u and v are adjacent if and only if $\langle u, v\rangle$ is solvable. We show that $\Gamma_s(G)$ is not a star graph, a tree, an n-partite graph for any positive integer $n \geq 2$ and not a regular graph for any non-solvable finite group G. We compute the girth of $\Gamma_s(G)$ and derive a lower bound of the clique number of $\Gamma_s(G)$. We prove the non-existence of finite non-solvable groups whose solvable graphs are planar, toroidal, double-toroidal, triple-toroidal or projective. We conclude the paper by obtaining a relation between $\Gamma_s(G)$ and the solvability degree of G.

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Keywords. solvable graph, genus, solvability degree, finite group

1. Introduction

Let G be a finite group and $u \in G$. The solvabilizer of u, denoted by $\mathrm{Sol}_G(u)$, is the set given by $\{v \in G : \langle u, v\rangle \text{ is solvable}\}$. Note that the centralizer $C_G(u) := \{v \in G : uv = vu\}$ is a subset of $\mathrm{Sol}_G(u)$ and hence the center $Z(G) \subseteq \mathrm{Sol}_G(u)$ for all $u \in G$. By [21, Proposition 2.13], $|C_G(u)|$ divides $|\mathrm{Sol}_G(u)|$ for all $u \in G$ though $\mathrm{Sol}_G(u)$ is not a subgroup of G in general. A group G is called a S-group if $\mathrm{Sol}_G(u)$ is a subgroup of G for all $u \in G$. A finite group G is a S-group if and only if it is solvable (see [21, Proposition 2.22]). Many other properties of $\mathrm{Sol}_G(u)$ can be found in [21]. We write $\mathrm{Sol}(G) = \{u \in G : \langle u, v\rangle \text{ is solvable for all } v \in G\}$. It is easy to see that $\mathrm{Sol}(G) = \bigcap_{u \in G} \mathrm{Sol}_G(u)$. Also, $\mathrm{Sol}(G)$ is the solvable radical of G (see [18]). The solvable graph of a finite non-solvable group G is a simple undirected graph whose vertex set is $G \setminus \mathrm{Sol}(G)$, and two vertices u and v are adjacent if $\langle u, v\rangle$ is a solvable. We write $\Gamma_s(G)$ to denote this graph. It is worth mentioning that $\Gamma_s(G)$ is the complement of the non-solvable graph of G considered in [4, 21] and extension of commuting and nilpotent graphs of finite groups that are studied extensively in [1–3, 5, 6, 9–11, 13–16, 25, 26]. It is worth mentioning that the study of commuting graphs of finite groups is originated from a question posed by Erdös [23].

In this paper, we show that $\Gamma_s(G)$ is not a star graph, a tree, an n-partite graph for any positive integer $n \geq 2$ and not a regular graph for any non-solvable finite group G. In Section 2, we also show that the girth of $\Gamma_s(G)$ is 3 and the clique number of $\Gamma_s(G)$ is

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greater than or equal to 4. In Section 3, we first show that for a given non-negative integer k, there are at the most finitely many finite non-solvable groups whose solvable graph have genus k. We also show that there is no finite non-solvable group, whose solvable graph is planar, toroidal, double-toroidal, triple-toroidal or projective. We conclude the paper by obtaining a relation between $\Gamma_s(G)$ and $P_s(G)$ in Section 4, where $P_s(G)$ is the probability that a randomly chosen pair of elements of G generate a solvable group (see [20]).

The reader may refer to [27] and [28] for various standard graph theoretic terminologies. For any subset X of the vertex set of a graph Γ, we write $\Gamma[X]$ to denote the induced subgraph of Γ on X. The girth of Γ is the minimum of the lengths of all cycles in Γ, and is denoted by girth(Γ). We write $\omega(\Gamma)$ to denote the clique number of Γ which is the least upper bound of the sizes of all the cliques of Γ. The smallest non-negative integer k is called the genus of a graph Γ if Γ can be embedded on the surface obtained by attaching k handles to a sphere. Let $\gamma(\Gamma)$ be the genus of Γ. Then, it is clear that $\gamma(\Gamma) \geq \gamma(\Gamma_0)$ for any subgraph Γ_0 of Γ. Let K_n be the complete graph on n vertices and mK_n the disjoint union of m copies of K_n. It was proved in [7, Corollary 1] that $\gamma(\Gamma) \geq \gamma(K_m) + \gamma(K_n)$ if Γ has two disjoint subgraphs isomorphic to K_m and K_n. Also, by [28, Theorem 6-38] we have

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \text{ if } n \geq 3. \tag{1.1}$$

A graph Γ is called planar, toroidal, double-toroidal and triple-toroidal if $\gamma(\Gamma) = 0, 1, 2$ and 3 respectively.

Let N_k be the connected sum of k projective planes. A simple graph which can be embedded in N_k but not in N_{k-1}, is called a graph of crosscap k. The notation $\bar{\gamma}(\Gamma)$ stand for the crosscap of a graph Γ. It is easy to see that $\bar{\gamma}(\Gamma) \geq \bar{\gamma}(\Gamma_0)$ for any subgraph Γ_0 of Γ. It was shown in [8] that

$$\bar{\gamma}(K_n) = \begin{cases} \lceil \frac{1}{6}(n-3)(n-4) \rceil & \text{if } n \geq 3 \text{ and } n \neq 7, \\ 3 & \text{if } n = 7. \end{cases} \tag{1.2}$$

A graph Γ is called a projective graph if $\bar{\gamma}(\Gamma) = 1$. It is worth mentioning that $2K_5$ is not projective graph (see [17]).

2. Graph realization

We begin with the following lemma.

Lemma 2.1. *For every $u \in G \setminus \mathrm{Sol}(G)$ we have*

$$\deg(u) = |\mathrm{Sol}_G(u)| - |\mathrm{Sol}(G)| - 1.$$

Proof. Note that $\deg(u)$ represents the number of vertices from $G \setminus \mathrm{Sol}(G)$ which are adjacent to u. Since $u \in \mathrm{Sol}_G(u)$, therefore $|\mathrm{Sol}_G(u)| - 1$ represents the number of vertices which are adjacent to u. Since we are excluding $\mathrm{Sol}(G)$ from the vertex set therefore $\deg(u) = |\mathrm{Sol}_G(u)| - |\mathrm{Sol}(G)| - 1$. □

Proposition 2.2. *$\Gamma_s(G)$ is not a star.*

Proof. Suppose for a contradiction $\Gamma_s(G)$ is a star. Let $|G| - |\mathrm{Sol}(G)| = n$. Then there exists $u \in G \setminus \mathrm{Sol}(G)$ such that $\deg(u) = n - 1$. Therefore, by Lemma 2.1, $|\mathrm{Sol}_G(u)| = |G|$. This gives $u \in \mathrm{Sol}(G)$, a contradiction. Hence, the result follows. □

Proposition 2.3. *$\Gamma_s(G)$ is not complete bipartite.*

Proof. Let $\Gamma_s(G)$ be complete bipartite. Suppose that A_1 and A_2 are parts of the bipartition. Then, by Proposition 2.2, $|A_1| \geq 2$ and $|A_2| \geq 2$. Let $u \in A_1, v \in A_2$. If $|\langle u, v \rangle \mathrm{Sol}(G) \setminus \mathrm{Sol}(G)| > 2$, then there exists $y \in \langle u, v \rangle \mathrm{Sol}(G) \setminus \mathrm{Sol}(G)$ with $u \neq y \neq v$ such that $\langle u, y \rangle$ and $\langle v, y \rangle$ are both solvable. But then $y \notin A_1$ and $y \notin A_2$, a contradiction.

It follows that $|\langle u, v\rangle \operatorname{Sol}(G) \setminus \operatorname{Sol}(G)| = 2$. In particular, $\operatorname{Sol}(G) = 1$ and $\langle u, v\rangle$ is cyclic of order 3 or $|\operatorname{Sol}(G)| = 2$ and $v = uz$ for z an involution in $\operatorname{Sol}(G)$. Now the neighbours of $u \in A_1$ is just $u^2 \in A_2$ or uz in the respective cases. Hence $|A_2| = |A_1| = 1$, a contradiction. Hence, the result follows. □

Following similar arguments as in the proof of Proposition 2.3 we get the following result.

Proposition 2.4. *$\Gamma_s(G)$ is not complete n-partite.*

Proposition 2.5. *For any finite non-solvable group G, $\Gamma_s(G)$ has no isolated vertex.*

Proof. Suppose x is an isolated vertex of $\Gamma_s(G)$. Then $|\operatorname{Sol}(G)| = 1$; otherwise x is adjacent to xz for any $z \in \operatorname{Sol}(G) \setminus \{1\}$. Thus it follows that $o(x) = 2$; otherwise x is adjacent to x^2. Let $y \in G$. Then $\langle x, x^y\rangle$ is dihedral and so $x = x^y$ as x is isolated. Hence $x \in Z(G)$ and so $x \in Z(G) \leq \operatorname{Sol}(G)$, a contradiction. Hence, $\Gamma_s(G)$ has no isolated vertex. □

The following lemma is useful in proving the next two results as well as some results in subsequent sections.

Lemma 2.6. *Let G be a finite non-solvable group. Then there exist $x \in G$ such that $x, x^2 \notin \operatorname{Sol}(G)$.*

Proof. Suppose that for all $x \in G$, we have $x^2 \in \operatorname{Sol}(G)$. Therefore, $G/\operatorname{Sol}(G)$ is elementary abelian and hence solvable. Also, $\operatorname{Sol}(G)$ is solvable. It follows that G is solvable, a contradiction. Hence, the result follows. □

Theorem 2.7. *Let G be a finite non-solvable group. Then* $\operatorname{girth}(\Gamma_s(G)) = 3$.

Proof. Suppose for a contradiction that $\Gamma_s(G)$ has no 3-cycle. Let $x \in G$ such that $x, x^2 \notin \operatorname{Sol}(G)$ (by Lemma 2.6). Suppose $|\operatorname{Sol}(G)| \geq 2$. Let $z \in \operatorname{Sol}(G), z \neq 1$, then x, x^2 and xz form a 3-cycle, which is a contradiction. Thus $|\operatorname{Sol}(G)| = 1$. In this case, every element of G has order 2 or 3; otherwise, $\{x, x^2, x^3\}$ forms a 3-cycle in $\Gamma_s(G)$ for all $x \in G$ with $o(x) > 3$. Therefore, $|G| = 2^m 3^n$ for some non-negative integers m and n. By Burnside's Theorem, it follows that G is solvable; a contradiction. Hence, $\operatorname{girth}(\Gamma_s(G)) = 3$. □

Theorem 2.8. *Let G be a finite non-solvable group. Then $\omega(\Gamma_s(G)) \geq 4$.*

Proof. Suppose for a contradiction that G is a finite non-solvable group with $\omega(\Gamma_s(G)) \leq 3$. Let $x \in G \setminus \operatorname{Sol}(G)$ such that $x^2 \notin \operatorname{Sol}(G)$ according to Lemma 2.6. Suppose $|\operatorname{Sol}(G)| \geq 2$. Let $z \in \operatorname{Sol}(G), z \neq 1$, then $\{x, x^2, xz, x^2z\}$ is a clique which is a contradiction. Thus $|\operatorname{Sol}(G)| = 1$. In this case every element of $G \setminus \operatorname{Sol}(G)$ has order $2, 3$ or 4 otherwise $\{x, x^2, x^3, x^4\}$ is a clique with $o(x) > 4$, which is a contradiction. Therefore $|G| = 2^m 3^n$ where m, n are non-negative integers. Again, by Burnside's Theorem, it follows that G is solvable; a contradiction. This completes the proof. □

As a consequence of Theorem 2.7 and Theorem 2.8 we have the following corollary.

Corollary 2.9. *The solvable graph of a finite non-solvable group is not a tree.*

We conclude this section with the following result.

Proposition 2.10. *$\Gamma_s(G)$ is not regular.*

Proof. Follows from [21, Corollary 3.17], noting the fact that a graph is regular if and only if its complement is regular. □

3. Genus and diameter

We begin this section with the following useful lemma.

Lemma 3.1. *Let G be a finite group and H a solvable subgroup of G. Then $\langle H, \operatorname{Sol}(G)\rangle$ is a solvable subgroup of G.*

Proposition 3.2. *Let G be a finite non-solvable group such that $\gamma(\Gamma_s(G)) = m$.*

(a) *If S is a nonempty subset of $G \setminus \operatorname{Sol}(G)$ such that $\langle x, y\rangle$ is solvable for all $x, y \in S$, then $|S| \leq \left\lfloor \frac{7+\sqrt{1+48m}}{2} \right\rfloor$.*

(b) $|\operatorname{Sol}(G)| \leq \frac{1}{t-1}\left\lfloor \frac{7+\sqrt{1+48m}}{2} \right\rfloor$, *where* $t = \max\{o(x\operatorname{Sol}(G)) \mid x\operatorname{Sol}(G) \in G/\operatorname{Sol}(G)\}$.

(c) *If H is a solvable subgroup of G, then* $|H| \leq \left\lfloor \frac{7+\sqrt{1+48m}}{2} \right\rfloor + |H \cap \operatorname{Sol}(G)|$.

Proof. We have $\Gamma_s(G)[S] \cong K_{|S|}$ and $\gamma(K_{|S|}) = \gamma(\Gamma_s(G)[S]) \leq \gamma(\Gamma_s(G))$. Therefore, if $m = 0$ then $\gamma(K_{|S|}) = 0$. This gives $|S| \leq 4$, otherwise $K_{|S|}$ will have a subgraph K_5 having genus 1. If $m > 0$ then, by Heawood's formula [27, Theorem 6.3.25], we have

$$|S| = \omega(\Gamma_s(G)[S]) \leq \omega(\Gamma_s(G)) \leq \chi(\Gamma_s(G)) \leq \left\lfloor \frac{7+\sqrt{1+48m}}{2} \right\rfloor$$

where $\chi(\Gamma_s(G))$ is the chromatic number of $\Gamma_s(G)$. Hence part (a) follows.

Part (b) follows from Lemma 3.1 and part (a) considering $S = \bigsqcup_{i=1}^{t-1} y^i \operatorname{Sol}(G)$, where $y \in G \setminus \operatorname{Sol}(G)$ such that $o(y\operatorname{Sol}(G)) = t$.

Part (c) follows from part (a) noting that $H = (H \setminus \operatorname{Sol}(G)) \cup (H \cap \operatorname{Sol}(G))$. □

Theorem 3.3. *Let G be a finite non-solvable group. Then $|G|$ is bounded above by a function of $\gamma(\Gamma_s(G))$.*

Proof. Let $\gamma(\Gamma_s(G)) = m$ and $h_m = \left\lfloor \frac{7+\sqrt{1+48m}}{2} \right\rfloor$. By Lemma 3.1, we have $\Gamma_s(G)[x\operatorname{Sol}(G)] \cong K_{|\operatorname{Sol}(G)|}$, where $x \in G \setminus \operatorname{Sol}(G)$. Therefore by Proposition 3.2(a), $|\operatorname{Sol}(G)| \leq h_m$.

Let P be a Sylow p-subgroup of G for any prime p dividing $|G|$ having order p^n for some positive integer n. Then P is a solvable. Therefore, by Proposition 3.2(c), we have $|P| \leq h_m + |\operatorname{Sol}(G)| \leq 2h_m$. Hence, $|G| < (2h_m)^{h_m}$ noting that the number of primes less than $2h_m$ is at most h_m. This completes the proof. □

As an immediate consequence of Theorem 3.3 we have the following corollary.

Corollary 3.4. *Let n be a non-negative integer. Then there are at the most finitely many finite non-solvable groups G such that $\gamma(\Gamma_s(G)) = n$.*

The following two lemmas are essential in proving the main results of this section.

Lemma 3.5. [24, Lemma 3.4] *Let G be a finite group.*

(a) *If $|G| = 7m$ and the Sylow 7-subgroup is normal in G, then G has an abelian subgroup of order at least 14 or $|G| \leq 42$.*

(b) *If $|G| = 9m$, where $3 \nmid m$ and the Sylow 3-subgroup is normal in G, then G has an abelian subgroup of order at least 18 or $|G| \leq 72$.*

Lemma 3.6. *If G is a non-solvable group of order not exceeding 120 then $\Gamma_s(G)$ has a subgraph isomorphic to K_{11} and $\gamma(\Gamma_s(G)) \geq 5$.*

Proof. If G is a non-solvable group and $|G| \leq 120$ then G is isomorphic to A_5, $A_5 \times \mathbb{Z}_2$, S_5 or $SL(2,5)$. Note that $|\operatorname{Sol}(A_5)| = |\operatorname{Sol}(S_5)| = 1$ and $|\operatorname{Sol}(A_5 \times \mathbb{Z}_2)| = |\operatorname{Sol}(SL(2,5))| = 2$. Also, A_5 has a solvable subgroup of order 12 and S_5, $A_5 \times \mathbb{Z}_2$, $SL(2,5)$ have solvable subgroups of order 24. It follows that $\Gamma_s(G)$ has a subgraph isomorphic to K_{11}. Therefore, by (1.1), $\gamma(\Gamma_s(G)) \geq \gamma(K_{11}) = 5$. □

Theorem 3.7. *The solvable graph of a finite non-solvable group is neither planar, toroidal, double-toroidal nor triple-toroidal.*

Proof. Let G be a finite non-solvable group. Note that it is enough to show $\gamma(\Gamma_s(G)) \geq 4$ to complete the proof. Suppose that $\gamma(\Gamma_s(G)) \leq 3$. Let $x \in G \setminus \mathrm{Sol}(G)$ such that $x^2 \notin \mathrm{Sol}(G)$. Such element exists by Lemma 2.6. Since any two elements of the set $A = x\,\mathrm{Sol}(G) \cup x^2\,\mathrm{Sol}(G)$ generate a solvable group, by Proposition 3.2(a), we have $2|\,\mathrm{Sol}(G)| = |A| \leq \left\lfloor \frac{7+\sqrt{1+48\cdot 3}}{2} \right\rfloor = 9$. Thus $|\,\mathrm{Sol}(G)| \leq 4$. Let p be a prime divisor of $|G|$ and P is a Sylow p-subgroup of G. Since P is solvable, by Proposition 3.2(c), we get $|P| \leq 9 + |P \cap \mathrm{Sol}(G)| \leq 13$. If $|P| = 11$ or 13 then $|P \cap \mathrm{Sol}(G)| = 1$. Therefore, $\Gamma_s(G)[P \setminus \mathrm{Sol}(G)] \cong K_{10}$ or K_{12}. Using (1.1), we get $\gamma(\Gamma_s(G)[P \setminus \mathrm{Sol}(G)]) = 4$ or 6. Therefore, $\gamma(\Gamma_s(G)) \geq \gamma(\Gamma_s(G)[P \setminus \mathrm{Sol}(G)]) \geq 4$, a contradiction. Thus $|P| \leq 9$ and hence $p \leq 7$. This shows that $|G|$ divides $2^3.3^2.5.7$.

We consider the following cases.

Case 1. $|\,\mathrm{Sol}(G)| = 4$.

If H is a Sylow p-subgroup of G where $p = 5$ or 7 then $\langle H, \mathrm{Sol}(G)\rangle$ is solvable since H is solvable (by Lemma 3.1). We have $|H \cap \mathrm{Sol}(G)| = 1$ and $|\langle H, \mathrm{Sol}(G)\rangle| = 20, 28$ according as $p = 5, 7$ respectively. Therefore $\Gamma_s(G)[\langle H, \mathrm{Sol}(G)\rangle \setminus \mathrm{Sol}(G)] \cong K_{16}$ or K_{24}. By (1.1) we get $\gamma(\Gamma_s(G)) \geq \gamma(\Gamma_s(G)[\langle H, \mathrm{Sol}(G)\rangle \setminus \mathrm{Sol}(G)]) \geq 13$, which is a contradiction.

Thus $|G|$ is a divisor of 72. Therefore, by Lemma 3.6 we have $\gamma(\Gamma_s(G)) \geq 5$, a contradiction.

Case 2. $|\,\mathrm{Sol}(G)| = 3$.

If H is a Sylow p-subgroup of G where $p = 5$ or 7 then $\langle H, \mathrm{Sol}(G)\rangle$ is solvable. We have $|H \cap \mathrm{Sol}(G)| = 1$ and $|\langle H, \mathrm{Sol}(G)\rangle| = 15, 21$ according as $p = 5, 7$ respectively. Therefore $\Gamma_s(G)[\langle H, \mathrm{Sol}(G)\rangle \setminus \mathrm{Sol}(G)] \cong K_{12}$ or K_{18}. By (1.1) we get $\gamma(\Gamma_s(G)) \geq \gamma(\Gamma_s(G)[\langle H, \mathrm{Sol}(G)\rangle \setminus \mathrm{Sol}(G)]) \geq 6$, which is a contradiction.

Thus $|G|$ is a divisor of 72. Therefore, by Lemma 3.6 we have $\gamma(\Gamma_s(G)) \geq 5$, a contradiction.

Case 3. $|\,\mathrm{Sol}(G)| = 2$.

If H is a Sylow 7-subgroup of G then $\langle H, \mathrm{Sol}(G)\rangle$ is solvable. We have $|H \cap \mathrm{Sol}(G)| = 1$ and $|\langle H, \mathrm{Sol}(G)\rangle| = 14$. So, $\Gamma_s(G)[\langle H, \mathrm{Sol}(G)\rangle \setminus \mathrm{Sol}(G)] \cong K_{12}$. By (1.1) we get $\gamma(\Gamma_s(G)) \geq \gamma(\Gamma_s(G)[\langle H, \mathrm{Sol}(G)\rangle \setminus \mathrm{Sol}(G)]) \geq 6$, which is a contradiction. Let K be a Sylow 3-subgroup of G. If $|K| = 9$ then $\langle K, \mathrm{Sol}(G)\rangle$ is solvable since K is solvable (by Lemma 3.1). We have $|K \cap \mathrm{Sol}(G)| = 1$ and $|\langle K, \mathrm{Sol}(G)\rangle| = 18$. So, $\Gamma_s(G)[\langle K, \mathrm{Sol}(G)\rangle \setminus \mathrm{Sol}(G)] \cong K_{16}$. By (1.1) we get $\gamma(\Gamma_s(G)) \geq \gamma(\Gamma_s(G)[\langle K, \mathrm{Sol}(G)\rangle \setminus \mathrm{Sol}(G)]) = 13$, which is a contradiction.

Thus $|G|$ is a divisor of 120. Therefore, by Lemma 3.6 we have $\gamma(\Gamma_s(G)) \geq 5$, a contradiction.

Case 4. $|\,\mathrm{Sol}(G)| = 1$.

In this case, first we shall show that $7 \nmid |G|$. On the contrary, assume that $7 \mid |G|$. Let n be the number of Sylow 7-subgroups of G. Then $n \mid 2^3.3^2.5$ and $n \equiv 1 \mod 7$. If $n \neq 1$ then $n \geq 8$. Let $H_1, \ldots, H_8$ be the eight distinct Sylow 7-subgroups of G. Then the induced subgraphs $\Gamma_S(G)[H_i \setminus \mathrm{Sol}(G)]$ for each $1 \leq i \leq 8$ contribute $\gamma(\Gamma_S(G)[H_i \setminus \mathrm{Sol}(G)]) = 1$ to the genus of $\Gamma_S(G)$. Thus

$$\gamma(\Gamma_S(G)) \geq \sum_{i=1}^{8} \gamma(\Gamma_S(G)[H_i \setminus \mathrm{Sol}(G)]) = 8,$$

a contradiction. Therefore, Sylow 7-subgroup of G is unique and hence normal. Since we have started with a non-solvable group, by Lemma 3.5, it follows that G has an abelian subgroup of order at least 14. Therefore, by (1.1) we have $\gamma(\Gamma_S(G)) \geq \gamma(K_{13}) = 8$, a contradiction. Hence, $|G|$ is a divisor of $2^3.3^2.5$.

Now, we shall show that $9 \nmid |G|$. Assume that, on the contrary, $9 \mid |G|$. If Sylow 3-subgroup of G is not normal in G, then the number of Sylow 3-subgroups is greater than

or equal to 4. Let H_1, H_2, H_3 be the three Sylow 3-subgroups of G. Then the induced subgraph $\Gamma_S(G)[H_1 \setminus \mathrm{Sol}(G)] \cong K_8$ and so it contributes $\gamma(\Gamma_S(G)[H_1 \setminus \mathrm{Sol}(G)]) = 2$ to the genus of $\Gamma_S(G)$. If $|H_1 \cap H_2| = 1$, then the induced subgraph $\Gamma_S(G)[H_2 \setminus \mathrm{Sol}(G)] \cong K_8$ and so it contributes $+2$ to the genus $\Gamma_S(G)$. Thus

$$\gamma(\Gamma_S(G)) \geq \gamma(\Gamma_S(G)[(H_1 \cup H_2) \setminus \mathrm{Sol}(G)]) = 4$$

which is a contradiction. So assume that $|H_1 \cap H_2| = 3$. Similarly $|H_1 \cap H_3| = 3$ and $|H_2 \cap H_3| = 3$. Let $M = H_2 \setminus H_1$. Then $|M| = 6$. Also note that if $L = H_1 \cup H_2$ and $K = H_3 \setminus L$, then $|K| \geq 4$. Also $H_1 \cap M = H_1 \cap K = M \cap K = \emptyset$.

If $|K| \geq 5$ then H_1 contribute $+2$ to genus of $\Gamma_S(G)$, M and K each contribute $+1$ to genus of $\Gamma_S(G)$. Hence genus of $\Gamma_S(G)$ is greater than or equal to 4, a contradiction.

Assume that $|K| = 4$. In this case $|M \cap H_3| = 2$. Let $x \in M \cap H_3$. Then H_1 contribute $+2$ to genus of $\Gamma_S(G)$, $M \setminus \{x\}$ and $K \cup \{x\}$ each contribute $+1$ to genus of $\Gamma_S(G)$. Hence genus of $\Gamma_S(G)$ is greater than or equal to 4, a contradiction.

These show that Sylow 3-subgroup of G is unique and hence normal in G. Therefore, by Lemma 3.5 and Lemma 3.6, G has an abelian subgroup A of order at least 18. Hence,

$$\gamma(\Gamma_S(G)) \geq \gamma(\Gamma_S(G)[A \setminus \mathrm{Sol}(G)]) \geq \gamma(K_{17}) = 16$$

which is a contradiction.

It follows that $9 \nmid |G|$ and $|G|$ is a divisor of 120. Therefore, by Lemma 3.6 we get $\gamma(\Gamma_S(G)) \geq 5$, a contradiction. Hence, $\gamma(\Gamma_s(G)) \geq 4$ and the result follows. □

The above theorem gives that $\gamma(\Gamma_s(G)) \geq 4$. Usually, genera of solvable graphs of finite non-solvable groups are very large. For example, if G is the smallest non-solvable group A_5 then $\Gamma_s(G)$ has 59 vertices and 571 edges. Also $\gamma(\Gamma_s(G)) \geq 571/6 - 59/2 + 1 = 68$ (follows from [28, Corollary 6–14]). The following theorem shows that the crosscap number of the solvable graph of a finite non-solvable group is greater than 1.

Proposition 3.8. *The solvable graph of a finite non-solvable group is not projective.*

Proof. Suppose G is a finite non-solvable group whose solvable graph is projective. Note that if $\Gamma_s(G)$ has a subgraph isomorphic to K_n then, by (1.2), we must have $n \leq 6$. Let $x \in G$, such that $x, x^2 \notin \mathrm{Sol}(G)$. Then

$$\Gamma_s(G)[x\,\mathrm{Sol}(G) \cup x^2\,\mathrm{Sol}(G)] \cong K_{2|\,\mathrm{Sol}(G)|}.$$

Therefore, $2|\,\mathrm{Sol}(G)| \leq 6$ and hence $|\,\mathrm{Sol}(G)| \leq 3$.

Let $p \mid |G|$ be a prime and P be a Sylow p-subgroup of G. Then $\Gamma_s(G)[P \setminus \mathrm{Sol}(G)] \cong K_{|P \setminus \mathrm{Sol}(G)|}$ since P is solvable. Therefore, $|P \setminus \mathrm{Sol}(G)| = |P| - |P \cap \mathrm{Sol}(G)| \leq 6$ and hence $|P| \leq 9$. This shows that $|G|$ is a divisor of $2^3.3^2.5.7$.

If $7 \mid |G|$ then Sylow 7-subgroup of G is unique and hence normal in G; otherwise, let H and K be two Sylow 7-subgroups of G. Then $|H \cap K| = |H \cap \mathrm{Sol}(G)| = |K \cap \mathrm{Sol}(G)| = 1$. Therefore, $\Gamma_s(G)[(H \cup K) \setminus \mathrm{Sol}(G)]$ has a subgraph isomorphic to $2K_6$. Hence, $\Gamma_s(G)$ has a subgraph isomorphic to $2K_5$, which is a contradiction. Similarly, if $9 \mid |G|$, then the Sylow 3-subgroup of G is normal in G. Therefore, by Lemma 3.5, it follows that $|G| \leq 72$ or $|G|$ is a divisor of $2^3.3.5$. In the both cases, by Lemma 3.6, $\Gamma_s(G)$ has complete subgraphs isomorphic to K_{11}, which is a contradiction. This completes the proof. □

We conclude this section, by an observation and a couple of problems regarding the diameter and connectedness of $\Gamma_s(G)$. Using the following programme in GAP[29], we see that the solvable graph of the groups $A_5, S_5, A_5 \times \mathbb{Z}_2, SL(2,5), PSL(3,2)$ and $GL(2,4)$ are connected with diameter 2. The solvable graphs of S_6 and A_6 are connected with diameters greater than 2.

```
g:=PSL(3,2);
sol:=RadicalGroup(g);
L:=[];
gsol:=Difference(g,sol);
for x in gsol do
 AddSet(L,[x]);
 for y in Difference(gsol,L) do
  if IsSolvable(Subgroup(g,[x,y]))=true then
   break;
  fi;
  i:=0;
  for z in gsol do
   if IsSolvable(Subgroup(g,[x,z]))=true and

   IsSolvable(Subgroup(g,[z,y]))=true
   then
    i:=1;
    break;
   fi;
  od;
  if i=0 then
   Print("Diameter>2");
   Print(x,"   ",y);
  fi;
 od;
od;
```

In this connection, we have the following problems.

Problem 3.1. Is $\Gamma_s(G)$ connected for any finite non-solvable group G?

Problem 3.2. Is there any finite bound for the diameter of $\Gamma_s(G)$ when $\Gamma_s(G)$ is connected?

4. Relations with solvability degree

The solvability degree of a finite group G is defined by the following ratio

$$P_s(G) := \frac{|\{(u,v) \in G \times G : \langle u,v\rangle \text{ is solvable}\}|}{|G|^2}.$$

Using the solvability criterion (see [12, Section 1]),

"A finite group is solvable if and only if every pair of its elements generates a solvable group"

for finite groups we have G is solvable if and only if its solvability degree is 1. It was shown in [20, Theorem A] that $P_s(G) \leq \frac{11}{30}$ for any finite non-solvable group G. In this section, we study a few properties of $P_s(G)$ and derive a connection between $P_s(G)$ and $\Gamma_s(G)$ for finite non-solvable groups G. We begin with the following lemma.

Lemma 4.1. *Let G be a finite group. Then $P_s(G) = \frac{1}{|G|^2} \sum\limits_{u \in G} |\operatorname{Sol}_G(u)|$.*

Proof. Let $\mathcal{S} = \{(u,v) \in G \times G : \langle u,v\rangle \text{ is solvable}\}$. Then

$$\mathcal{S} = \bigcup_{u \in G}(\{u\} \times \{v \in G : \langle u,v\rangle \text{ is solvable}\}) = \bigcup_{u \in G}(\{u\} \times \operatorname{Sol}_G(u)).$$

Therefore, $|\mathcal{S}| = \sum\limits_{u \in G} |\operatorname{Sol}_G(u)|$. Hence, the result follows. □

Corollary 4.2. *$|G|P_s(G)$ is an integer for any finite group G.*

Proof. By Proposition 2.16 of [21] we have that $|G|$ divides $\sum_{u\in G} |\operatorname{Sol}_G(u)|$. Hence, the result follows from Lemma 4.1. □

We have the following lower bound for $P_s(G)$.

Theorem 4.3. *For any finite group G,*

$$P_s(G) \geq \frac{|\operatorname{Sol}(G)|}{|G|} + \frac{2(|G| - |\operatorname{Sol}(G)|)}{|G|^2}.$$

Proof. By Lemma 4.1, we have

$$\begin{aligned} |G|^2 P_s(G) &= \sum_{u\in \operatorname{Sol}(G)} |\operatorname{Sol}_G(u)| + \sum_{u\in G\setminus \operatorname{Sol}(G)} |\operatorname{Sol}_G(u)| \\ &= |G||\operatorname{Sol}(G)| + \sum_{u\in G\setminus \operatorname{Sol}(G)} |\operatorname{Sol}_G(u)|. \end{aligned} \tag{4.1}$$

By Proposition 2.13 of [21], $|C_G(u)|$ is a divisor of $|\operatorname{Sol}_G(u)|$ for all $u \in G$ where $C_G(u) = \{v \in G : uv = vu\}$, the centralizer of $u \in G$. Since $|C_G(u)| \geq 2$ for all $u \in G$ we have $|\operatorname{Sol}_G(u)| \geq 2$ for all $u \in G$. Therefore

$$\sum_{u\in G\setminus \operatorname{Sol}(G)} |\operatorname{Sol}_G(u)| \geq 2(|G| - |\operatorname{Sol}(G)|).$$

Hence, the result follows from (4.1). □

The following theorem shows that $P_s(G) > \Pr(G)$ for any finite non-solvable group where $\Pr(G)$ is the commuting probability of G (see [19]).

Theorem 4.4. *Let G be a finite group. Then $P_s(G) \geq \Pr(G)$ with equality if and only if G is a solvable group.*

Proof. The result follows from Lemma 4.1 and the fact that $\Pr(G) = \frac{1}{|G|^2}\sum_{u\in G} |C_G(u)|$ noting that $C_G(u) \subseteq \operatorname{Sol}_G(u)$ and so $|\operatorname{Sol}_G(u)| \geq |C_G(u)|$ for all $u \in G$.

The equality holds if and only if $C_G(u) = \operatorname{Sol}_G(u)$ for all $u \in G$, that is $\operatorname{Sol}_G(u)$ is a subgroup of G for all $u \in G$. Hence, by Proposition 2.22 of [21], the equality holds if and only if G is solvable. □

Let $|E(\Gamma_s(G))|$ be the number of edges of the non-solvable graph $\Gamma_s(G)$ of G. The following theorem gives a relation between $P_s(G)$ and $|E(\Gamma_s(G))|$.

Theorem 4.5. *Let G be a finite non-solvable group. Then*

$$2|E(\Gamma_s(G))| = |G|^2 P_s(G) + |\operatorname{Sol}(G)|^2 + |\operatorname{Sol}(G)| - |G|(2|\operatorname{Sol}(G)| + 1).$$

Proof. We have

$$2|E(\Gamma_s(G))| = |\{(x, y) \in (G \setminus \operatorname{Sol}(G)) \times (G \setminus \operatorname{Sol}(G)) : \langle x, y\rangle \text{ is solvable}\}| - |G| + |\operatorname{Sol}(G)|.$$

Also

$$\begin{aligned} \mathcal{S} &= \{(x, y) \in G \times G : \langle x, y\rangle \text{ is solvable}\} \\ &= \operatorname{Sol}(G) \times \operatorname{Sol}(G) \quad \sqcup \quad \operatorname{Sol}(G) \times (G \setminus \operatorname{Sol}(G)) \quad \sqcup \quad (G \setminus \operatorname{Sol}(G)) \times \operatorname{Sol}(G) \\ &\quad \sqcup \quad \{(x, y) \in (G \setminus \operatorname{Sol}(G)) \times (G \setminus \operatorname{Sol}(G)) : \langle x, y\rangle \text{ is solvable}\}. \end{aligned}$$

Therefore

$$\begin{aligned} |\mathcal{S}| &= |\operatorname{Sol}(G)|^2 + 2|\operatorname{Sol}(G)|(|G| - |\operatorname{Sol}(G)|) + 2|E(\Gamma_s(G))| + |G| - |\operatorname{Sol}(G)| \\ \implies |G|^2 P_s(G) &= |G|(2|\operatorname{Sol}(G)| + 1) - |\operatorname{Sol}(G)|^2 - |\operatorname{Sol}(G)| + 2|E(\Gamma_s(G))|. \end{aligned}$$

Hence, the result follows. □

We conclude this paper noting that lower bounds for $|E(\Gamma_s(G))|$ can be obtained from Theorem 4.5 using the lower bounds given in Theorem 4.3, Theorem 4.4 and the lower bounds for $\Pr(G)$ obtained in [22].

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Blockwise and Low Density Key Error Correcting Codes

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Abstract
To protect the information from disturbances created by noisy channels, redundant symbols (check symbols) with the information symbols are added. These extra symbols play important role for the efficiency of the communication system. It is always important to know how much these check symbols are required for a code designed for a specific purpose. In this communication, we give lower and upper bounds on check symbols needed to a linear code correcting key errors of length upto p which are confined to a single sub-block. We provide two examples of such linear codes. We, further, obtain those bounds for the case when key error occurs in the whole code length, but the number of disturbing components within key error is upto a certain number. Two examples in this case also are provided.

Keywords- Parity check matrix, Syndromes, Bounds, Key errors.

1. Introduction

The main purpose of the coding theorists was to detect and correct the errors which are produced during communication through noisy channels. Communication channels are affected by many external and internal factors that result different types of errors in the messages sent. The coding theorists are always trying to retrieve the message even it is corrupted by various types of errors. Plenty of works have been done by many in this direction. Sharma and Gaur (2013) discussed and studied a different type of error related to the typing on a key board. Their study was on such errors w.r.t. S-K metric (partition). Later this type of error was named as "***key error***" in the work of Das (2014). When a person types on a key board and if by mistake, he presses a different key in the left or right side of the key, then a word with no or different meaning appears. In this case, a key error is created. The definition of a key error given in Das (2014) is as follows:

"*An i-key error of length p is a vector such that the* i^{th} *component is non-zero and the other non-zero component are confined to immediate p consecutive components in either side of the* i^{th} *component.***"**

For example, key errors of length 2 in a vector of length 7 over $GF(3)$ are

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$$(\underset{\substack{\downarrow\\ \text{entry}\\ \text{error}}}{1}\ 020000),\ (2\ \underset{\substack{\downarrow\\ \text{entry}\\ \text{error}}}{1}\ 22000),\ (20\ \underset{\substack{\downarrow\\ \text{entry}\\ \text{error}}}{2}\ 0100),\ (22\ \underset{\substack{\downarrow\\ \text{entry}\\ \text{error}}}{2}\ 2100),\ (0012\ \underset{\substack{\downarrow\\ \text{entry}\\ \text{error}}}{1}\ 01),$$

$$(00010\ \underset{\substack{\downarrow\\ \text{entry}\\ \text{error}}}{2}\ 1),\ (00012\ \underset{\substack{\downarrow\\ \text{entry}\\ \text{error}}}{1}\ 1),\ (000010\ \underset{\substack{\downarrow\\ \text{entry}\\ \text{error}}}{1}\ )\ \text{etc.}$$

According to the definition, the i^{th} position of the vector is always non-zero and other errors can extend up to p^{th} position on both side of the i^{th} position. This means a key error of length may affect $2p+1$ consecutive component. The i^{th} position is called the entry error position of the i-key error. When the entry error position can take any place within the code length (or sub-block of code length), we simply call the error as *key error*.

Detection of key error is studied by Das (2014) and correction of key error is taken care of by Das (2015). A midway concept between detection and correction (called error location), was initiated by Wolf and Elspas (1963). Location and weight distribution of key error are discussed by Das and Kumar (2020). In continuation with the study of key error, in this paper, we have considered the following two cases:

(i) when the length of the code is divided into some smaller length mutually exclusive sub-blocks and key error occurs within a single sub-block.
(ii) when key error occurs in whole code length, but with low density, i.e., maximum number of disturbed components within key error should be less than a limit, called low density key error.

The case (*i*) is considered keeping in mind the situation when sub-blocks are independent and a corrupted sub-block with key error does not affect other sub-blocks. This is the extension work of location of key error (Das and Kumar, 2020) to correction of key error occurring within a sub-block. In (Das and Kumar, 2020), location of key error was possible, but the correction of such error was not possible. In this paper, we present this case, i.e., to correct key errors of length upto p within any single sub-block of length (say) t. We present Single Blockwise Key Error (of length upto p within a sub-block of length t) Correcting Codes. We denote such codes by $\boldsymbol{SBKE_{p/t}C}$ **codes**. We obtain lower as well as upper bounds on number of check symbols (redundant symbols) of such $\boldsymbol{SBKE_{p/t}C}$ **codes**.

The case (*ii*) is considered when key error occurs in the whole code length and certain number or less components within the key error get disturbed. The possibility of certain number or less components getting disturbed is more likely than all components within key errors. Consideration of such situation is initiated by Wyner (1963). In view of this, we present here the lower as well as the upper bounds on number of check symbols for linear codes that are capable to correct key error of length upto p with hamming weight ω or fewer ($\omega \leq 2p + 1$) in the whole code length. We denote such codes by $\boldsymbol{K_{\omega,p}EC}$ **codes**.

The number of check symbols (or redundant symbols) of a code is important because the rate of information increases if the number of check symbols is lesser and decreases if the number is more. The bounds on number of check symbols (redundant symbols) tell us the limitation and capability of error detection and correction of a code.

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The rest of the paper is written as follows. In Section 2, we derive lower as well as upper bounds on the number of check symbols for a ***SBKE$_{p/t}$C* code** and then we give two examples of such codes. In Section 3, similar bounds for a ***K$_{\omega,p}$EC* code** are obtained. This is also followed by two examples. At the end, conclusion is given.

2. Correction of Key Errors Blockwise

We, in this section, first provide the lower number and then the upper number of check (redundant) symbols required for the linear codes which can correct key errors blockwise. The section ends with two examples of such codes.

Theorem 2.1 *For an (n = ft, n-r) **SBKE$_{p/t}$C code** over finite field F_q having r check symbols, the value of r must be at least*

$$\log\left[1+ f\left\{\frac{q^{2p+1}-q}{1+q}+\left\{(t-2p)(1+q^{2p+1})+q+p\right\}\frac{q-1}{1+q}+\frac{q^{2p+1}-q^3}{(1+q)^2}\right\}\right].$$

Proof. We prove this theorem by enumerating the total number of correctable errors occurring in *f* sub-blocks each of length *t*. From Das (2015), the total number of correctable errors in a *t*-tuple is

$$\frac{q^{2p+1}-q}{1+q}+\left\{(t-2p)(1+q^{2p+1})+q+p\right\}\frac{q-1}{1+q}+\frac{q^{2p+1}-q^3}{(1+q)^2}.$$

.

Since the code has *f* sub-blocks of length *t*, therefore the total number of correctable errors occurring in all *f* sub-blocks is

$$f\left\{\frac{q^{2p+1}-q}{1+q}+\left\{(t-2p)(1+q^{2p+1})+q+p\right\}\frac{q-1}{1+q}+\frac{q^{2p+1}-q^3}{(1+q)^2}\right\}.$$

Hence

$$q^r \geq 1+ f\left\{\frac{q^{2p+1}-q}{1+q}+\left\{(t-2p)(1+q^{2p+1})+q+p\right\}\frac{q-1}{1+q}+\frac{q^{2p+1}-q^3}{(1+q)^2}\right\}.$$

Remark 2.2 For *f* = 1, Theorem 2.1 coincides with Theorem 2.1 (Das, 2015) for code length $n = t$.

For the upper bound, we follow the method of well-known Varshaomov-Gilbert-Sacks bound ((Sacks, 1958) , Theorem 4.7 (Peterson and Weldon, 1972)).

Theorem 2.3 *An (n = ft, n-r) **SBKE$_{p/t}$C code** ($t > 4p+1$) over finite field F_q having r check symbols shall always exist provided*

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$$q^r > \frac{1+q^{2p+1}}{1+q}\left[\left\{1+\frac{q^{2p+1}-q}{1+q}+\left\{(t-4p-1)(1+q^{2p+1})+q+p\right\}\frac{q-1}{1+q}+\frac{q^{2p+1}-q^3}{(1+q)^2}\right\}\right.$$
$$\left.+(f-1)\left\{\frac{q^{2p+1}-q}{1+q}+\left\{(t-2p)(1+q^{2p+1})+q+p\right\}\frac{q-1}{1+q}+\frac{q^{2p+1}-q^3}{(1+q)^2}\right\}\right].$$

Proof. To prove this theorem, we construct a suitable parity check matrix *H* of order $r\times n$ for the ***SBKE$_{p/t}$C*** code in need. Let us assume that the initial $f-1$ sub-blocks of *H* and the initial $\tau-1$ columns $h_1, h_2, h_3, h_4, \ldots, h_{\tau-1}$ of the f^{th} sub-block of *H* are selected appropriately. Then, we add the τ^{th} column h_τ of f^{th} sub-block by the two following conditions:

$$h_\tau \neq (u_1h_{\tau-1}+u_2h_{\tau-2}+\cdots+u_{2p}h_{\tau-2p})+(v_lh_l+v_{l+1}h_{l+1}+\cdots+v_{l+2p}h_{l+2p}), \quad (1)$$

where $u_i, v_i \in GF(q)$; $l+2p<\tau-2p$; the coefficient u_i's are such that $\{h_{\tau-2p}, h_{\tau-2p+1},..., h_{\tau-1}, h_\tau\}$ form a key error of length upto *p* and the coefficient v_i's are such that $\{h_l, h_{l+1},... h_{l+2p-1}, h_{l+2p}\}$ (where $l=1,2,\ldots,\tau-4p-1$) form any key error of length upto *p* in the first $\tau-2p-1$ columns of the f^{th} sub-block.

and

$$h_\tau \neq (u_1h_{\tau-1}+u_2h_{\tau-2}+\cdots+u_{2p}h_{\tau-2p})+(w_lh_l+w_{l+1}h_{l+1}+\cdots+w_{l+2p}h_{l+2p}), \quad (2)$$

where $u_i, w_i \in GF(q)$, the coefficient u_i's are such that $\{h_{\tau-2p}, h_{\tau-2p+1},..., h_{\tau-1}, h_\tau\}$ form a key error of length upto *p* and the coefficient w_i's are such that $\{h_l, h_{l+1},... h_{l+2p-1}, h_{l+2p}\}$ (where $l=1,2,\ldots,t-2p$) form a key error of length upto *p* in any *other* sub-block.

From Das (2015), the total number of coefficients u_i's and v_i's in expressions (1) is given by respectively

$$\frac{1+q^{2p+1}}{1+q}$$

and

$$1+\frac{q^{2p+1}-q}{1+q}+\left\{(\tau-4p-1)(1+q^{2p+1})+q+p\right\}\frac{q-1}{1+q}+\frac{q^{2p+1}-q^3}{(1+q)^2}.$$

So, the total number of u_i's and v_i's on the expression (1) is given by

$$\frac{1+q^{2p+1}}{1+q}\left\{1+\frac{q^{2p+1}-q}{1+q}+\left\{(\tau-4p-1)(1+q^{2p+1})+q+p\right\}\frac{q-1}{1+q}+\frac{q^{2p+1}-q^3}{(1+q)^2}\right\}. \quad (3)$$

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Now, the computation of coefficients w_i's on R. H. S. of (2) is equivalent to finding key errors of length upto p in a vector of length t, which is given by

$$\frac{q^{2p+1}-q}{1+q}+\left\{(t-2p)(1+q^{2p+1})+q+p\right\}\frac{q-1}{1+q}+\frac{q^{2p+1}-q^3}{(1+q)^2}.$$

As the number of sub-blocks is $f-1$, so, the total number of coefficient w_i's in (2) is

$$(f-1)\left\{\frac{q^{2p+1}-q}{1+q}+\left\{(t-2p)(1+q^{2p+1})+q+p\right\}\frac{q-1}{1+q}+\frac{q^{2p+1}-q^3}{(1+q)^2}\right\}. \tag{4}$$

Hence, the total number of u_i's and w_i's in expression (2) is given by

$$(f-1)\left(\frac{1+q^{2p+1}}{1+q}\right)\left\{\frac{q^{2p+1}-q}{1+q}+\left\{(t-2p)(1+q^{2p+1})+q+p\right\}\frac{q-1}{1+q}+\frac{q^{2p+1}-q^3}{(1+q)^2}\right\}. \tag{5}$$

Thus, the total number of linear sums due to expression (1) and (2) is

$$Expr:(3)+Expr:(5)$$

i.e.

$$\left(\frac{1+q^{2p+1}}{1+q}\right)\left[\left\{1+\frac{q^{2p+1}-q}{1+q}+\left\{(\tau-4p-1)(1+q^{2p+1})+q+p\right\}\frac{q-1}{1+q}+\frac{q^{2p+1}-q^3}{(1+q)^2}\right\}\right.$$
$$\left.+(f-1)\left\{\frac{q^{2p+1}-q}{1+q}+\left\{(t-2p)(1+q^{2p+1})+q+p\right\}\frac{q-1}{1+q}+\frac{q^{2p+1}-q^3}{(1+q)^2}\right\}\right].$$

Now, putting this number less than q^r, we get

$$q^r>\left(\frac{1+q^{2p+1}}{1+q}\right)\left[\left\{1+\frac{q^{2p+1}-q}{1+q}+\left\{(\tau-4p-1)(1+q^{2p+1})+q+p\right\}\frac{q-1}{1+q}+\frac{q^{2p+1}-q^3}{(1+q)^2}\right\}\right.$$
$$\left.+(f-1)\left\{\frac{q^{2p+1}-q}{1+q}+\left\{(t-2p)(1+q^{2p+1})+q+p\right\}\frac{q-1}{1+q}+\frac{q^{2p+1}-q^3}{(1+q)^2}\right\}\right].$$

By replacing τ by t, we get the required result.

Remark 2.4 For f = 1, Theorem 2.3 coincides with Theorem 2.2 (Das, 2015) for code length $n = t$.

Now, we give two examples of codes in support of our results derived above. In the first, we give an example in binary case which is followed by an example in ternary case.

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Example 2.5 Taking q = 2, p = 2, t=*11*, f = 4 in Theorem 2.3, we get $r > 10$. This gives rise to a binary (44, 32) linear code. The parity check matrix H of order 12×44 of this code is constructed by the synthesis procedure discussed in Theorem 2.3 and is given below.

$$H = \begin{bmatrix}
10000000000 & 01111111111 & 11111111111 & 11111111111 \\
01000000000 & 01001111011 & 11111100100 & 01100110010 \\
00100000000 & 01100111001 & 11000010011 & 00001000011 \\
00010000000 & 01011011000 & 11000011110 & 11111000100 \\
00001000000 & 01101001110 & 01000101001 & 01011101101 \\
00000100000 & 01000100010 & 00100111110 & 10100111011 \\
00000010000 & 01111100000 & 00110010110 & 11101001110 \\
00000001000 & 01001110001 & 01011010101 & 01001010101 \\
00000000100 & 01100010111 & 00001100010 & 10111010011 \\
00000000010 & 01011011011 & 10000111100 & 00010101010 \\
00000000001 & 01101111000 & 11100111011 & 01111110111 \\
00000000000 & 11000101000 & 11101111110 & 11101111111
\end{bmatrix}$$

All the 364 key errors of length upto p = 2 occurring in the same sub-block or in different sub-block and the corresponding syndromes for this code can be obtained with the help of MS-EXCEL and it is verified that the syndromes are all non-zero and distinct. So, this is a ***SBKE$_{2/11}$C codes in binary case.***

Example 2.6 Taking q = 3, p = 2, f = 2, t=*11* in Theorem 2.3, we get a ternary (22, 11) linear code and its parity check matrix is given by

$$H = \begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0 & 1&1&1&1&1&1&1&1&0&1&1 \\
0&1&0&0&0&0&0&0&0&0&0 & 1&0&0&1&2&1&1&0&1&2&0 \\
0&0&1&0&0&0&0&0&0&0&0 & 1&2&0&0&1&1&2&0&0&1&2 \\
0&0&0&1&0&0&0&0&0&0&0 & 1&0&2&1&0&1&1&0&0&0&0 \\
0&0&0&0&1&0&0&0&0&0&0 & 1&1&0&1&0&0&1&2&2&0&0 \\
0&0&0&0&0&1&0&0&0&0&0 & 1&0&0&0&2&0&0&0&1&0&2 \\
0&0&0&0&0&0&1&0&0&0&0 & 1&2&1&1&1&0&0&2&0&0&0 \\
0&0&0&0&0&0&0&1&0&0&0 & 1&0&0&1&1&1&0&0&0&1&0 \\
0&0&0&0&0&0&0&0&1&0&0 & 1&1&0&0&0&2&0&1&1&1&0 \\
0&0&0&0&0&0&0&0&0&1&0 & 1&0&1&1&0&2&1&0&0&2&1 \\
0&0&0&0&0&0&0&0&0&0&1 & 1&1&0&0&0&1&1&2&2&0&0
\end{bmatrix}.$$

We can verify that all the syndromes of key errors of length upto 2 occurring in same sub-block or different sub-block are non-zero and distinct. So, this is a ternary ***SBKE$_{2/11}$C code.***

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3. Correction of Low Density Key Errors

In Das (2014), correction of key errors of length upto p occurring in the whole code length is studied. Motivated by Wyner (1963), we consider key errors of length upto p with hamming weight ω or fewer in the whole code length and correction of such errors is studied in this section. We obtain analogous bounds as presented in Section 2 and we end this section with two examples of such codes. First, we give the following identity which is used for simplification of calculation of our results. The identity can be proved easily with Pascal Formula (Balakrishnan, 1996).

Identity 3.1 For non-negative integers n, m and p with $n \geq m$,

$$\binom{n}{m}+\binom{n+1}{m}+\binom{n+2}{m}+\cdots+\binom{n+p}{m}=\binom{n+p+1}{m+1}-\binom{n}{m+1}.$$

For the following two results, we assume that $\binom{n}{m}=0$ if n is negative integer or smaller than m.

Theorem 3.1 *For an (n, n-r)* $\boldsymbol{K_{\omega,p}EC}$ *code over finite field* F_q *having r check symbols, the following inequality must be satisfied*

$$q^r \geq 1+\sum_{\alpha=0}^{\omega-2}\sum_{\eta=1}^{p-1}\left[\binom{2\eta+1}{\alpha+1}-\binom{\eta+1}{\alpha+1}\right](q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-1}\left[\binom{p+1}{\alpha+1}-\binom{1}{\alpha+1}\right](q-1)^{1+\alpha}$$
$$+(n-2p)\left\{\sum_{\alpha=0}^{\omega-2}\sum_{\gamma=0}^{p-1}\binom{2\gamma+1}{\alpha}(q-1)^{2+\alpha}+(q-1)\right\}$$
$$+\sum_{\alpha=0}^{\omega-2}\sum_{\mu=0}^{p-2}(p-1-\mu)\binom{2\mu+1}{\alpha}(q-1)^{2+\alpha}+p(q-1).$$

Proof. For the proof, we count the number of key errors of length upto p with hamming weight ω or fewer in the following way.

When the entry error position is from 1st to p^{th} position, the number of key errors of length upto p with hamming weight ω or fewer is calculated as

$$\sum_{\alpha=0}^{\omega-1}\binom{p}{\alpha}(q-1)^{1+\alpha}+\left[\sum_{\alpha=0}^{\omega-2}\binom{p}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-1}\binom{p-1}{\alpha}(q-1)^{1+\alpha}\right]$$
$$+\left[\sum_{\alpha=0}^{\omega-2}\binom{p+1}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-2}\binom{p-1}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-1}\binom{p-2}{\alpha}(q-1)^{1+\alpha}\right]$$

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$$+\left[\sum_{\alpha=0}^{\omega-2}\binom{p+2}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-2}\binom{p}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-2}\binom{p-2}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-1}\binom{p-3}{\alpha}(q-1)^{1+\alpha}\right]$$
$$\vdots$$
$$+\left[\sum_{\alpha=0}^{\omega-2}\binom{2p-2}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-2}\binom{2p-4}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-2}\binom{2p-6}{\alpha}(q-1)^{2+\alpha}+\cdots\right.$$
$$\left.+\sum_{\alpha=0}^{\omega-2}\binom{2}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-1}\binom{1}{\alpha}(q-1)^{1+\alpha}\right],$$

i.e.

$$\sum_{\alpha=0}^{\omega-2}\left[\binom{p}{\alpha}+\binom{p+1}{\alpha}+\binom{p+2}{\alpha}+\cdots+\binom{2p-2}{\alpha}\right](q-1)^{2+\alpha}$$
$$+\sum_{\alpha=0}^{\omega-2}\left[\binom{p-1}{\alpha}+\binom{p}{\alpha}+\binom{p+1}{\alpha}+\cdots+\binom{2p-4}{\alpha}\right](q-1)^{2+\alpha}$$
$$+\sum_{\alpha=0}^{\omega-2}\left[\binom{p-2}{\alpha}+\binom{p-1}{\alpha}+\binom{p}{\alpha}+\cdots+\binom{2p-6}{\alpha}\right](q-1)^{2+\alpha}$$
$$\vdots$$
$$+\sum_{\alpha=0}^{\omega-2}\left[\binom{2}{\alpha}\right](q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-1}\left[\binom{p}{\alpha}+\binom{p-1}{\alpha}+\binom{p-2}{\alpha}+\cdots+\binom{1}{\alpha}\right](q-1)^{1+\alpha}.$$

This, after applying Identity 3.1, simplifies to

$$\sum_{\alpha=0}^{\omega-2}\left[\binom{2p-1}{\alpha+1}-\binom{p}{\alpha+1}\right](q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-2}\left[\binom{2p-3}{\alpha+1}-\binom{p-1}{\alpha+1}\right](q-1)^{2+\alpha}$$
$$+\sum_{\alpha=0}^{\omega-2}\left[\binom{2p-5}{\alpha+1}-\binom{p-2}{\alpha+1}\right](q-1)^{2+\alpha}+\cdots+\sum_{\alpha=0}^{\omega-2}\left[\binom{3}{\alpha+1}-\binom{2}{\alpha+1}\right](q-1)^{2+\alpha}$$
$$+\sum_{\alpha=0}^{\omega-1}\left[\binom{p+1}{\alpha+1}-\binom{1}{\alpha+1}\right](q-1)^{1+\alpha}$$
$$=\sum_{\alpha=0}^{\omega-2}\sum_{\eta=1}^{p-1}\left[\binom{2\eta-1}{\alpha+1}-\binom{\eta+1}{\alpha+1}\right](q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-1}\left[\binom{p+1}{\alpha+1}-\binom{1}{\alpha+1}\right](q-1)^{1+\alpha}. \quad (6)$$

When the entry error position is taken from $(p+1)^{\text{th}}$ upto $(n-p)^{\text{th}}$ position, the number of key errors of length upto p with hamming weight ω or fewer is

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$$(n-2p)\left\{\sum_{\alpha=0}^{\omega-2}\binom{2p-1}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-2}\binom{2p-3}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-2}\binom{2p-5}{\alpha}(q-1)^{2+\alpha}+\cdots\right.$$
$$\left.+\sum_{\alpha=0}^{\omega-2}\binom{1}{\alpha}(q-1)^{2+\alpha}+(q-1)\right\}$$
$$=(n-2p)\left\{\sum_{\alpha=0}^{\omega-2}\sum_{\gamma=0}^{p-1}\binom{2\gamma+1}{\alpha}(q-1)^{2+\alpha}+(q-1)\right\}. \quad (7)$$

Similarly, when the entry error position is considered in the last p components, the number of such key errors is

$$\left[\sum_{\alpha=0}^{\omega-2}\binom{2p-3}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-2}\binom{2p-5}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-2}\binom{2p-7}{\alpha}(q-1)^{2+\alpha}+\cdots\right.$$
$$\left.+\sum_{\alpha=0}^{\omega-2}\binom{3}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-2}\binom{1}{\alpha}(q-1)^{2+\alpha}+(q-1)\right]$$
$$+\left[\sum_{\alpha=0}^{\omega-2}\binom{2p-5}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-2}\binom{2p-7}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-2}\binom{2p-9}{\alpha}(q-1)^{2+\alpha}+\cdots\right.$$
$$\left.+\sum_{\alpha=0}^{\omega-2}\binom{3}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-2}\binom{1}{\alpha}(q-1)^{2+\alpha}+(q-1)\right]$$
$$\vdots$$
$$+\left[\sum_{\alpha=0}^{\omega-2}\binom{5}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-2}\binom{3}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-2}\binom{1}{\alpha}(q-1)^{2+\alpha}+(q-1)\right]$$
$$+\left[\sum_{\alpha=0}^{\omega-2}\binom{3}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-2}\binom{1}{\alpha}(q-1)^{2+\alpha}+(q-1)\right]$$
$$+\left[\sum_{\alpha=0}^{\omega-2}\binom{1}{\alpha}(q-1)^{2+\alpha}+(q-1)\right]+(q-1),$$

i.e.

$$\left[\sum_{\alpha=0}^{\omega-2}\binom{2p-3}{\alpha}(q-1)^{2+\alpha}+2\sum_{\alpha=0}^{\omega-2}\binom{2p-5}{\alpha}(q-1)^{2+\alpha}+3\sum_{\alpha=0}^{\omega-2}\binom{2p-7}{\alpha}(q-1)^{2+\alpha}+\cdots\right.$$

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$$+(p-2)\sum_{\alpha=0}^{\omega-2}\binom{3}{\alpha}(q-1)^{2+\alpha}+(p-1)\sum_{\alpha=0}^{\omega-2}\binom{1}{\alpha}(q-1)^{2+\alpha}+p(q-1)\Bigg]$$

$$=\sum_{\alpha=0}^{\omega-2}\sum_{\mu=0}^{p-2}(p-1-\mu)\binom{2\mu+1}{\alpha}(q-1)^{2+\alpha}+p(q-1). \tag{8}$$

Therefore, the total number of key errors of length upto p with hamming weight ω or fewer is given by

$$Expr:(6)+Expr:(7)+Expr:(8)$$

$$=\sum_{\alpha=0}^{\omega-2}\sum_{\eta=1}^{p-1}\left[\binom{2\eta+1}{\alpha+1}-\binom{\eta+1}{\alpha+1}\right](q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-1}\left[\binom{p+1}{\alpha+1}-\binom{1}{\alpha+1}\right](q-1)^{1+\alpha}$$

$$+(n-2p)\left\{\sum_{\alpha=0}^{\omega-2}\sum_{\gamma=0}^{p-1}\binom{2\gamma+1}{\alpha}(q-1)^{2+\alpha}+(q-1)\right\}$$

$$+\sum_{\alpha=0}^{\omega-2}\sum_{\mu=0}^{p-2}(p-1-\mu)\binom{2\mu+1}{\alpha}(q-1)^{2+\alpha}+p(q-1). \tag{9}$$

For correction, this number must be greater than or equal to q^r. Therefore, we must have

$$q^r\geq 1+\sum_{\alpha=0}^{\omega-2}\sum_{\eta=1}^{p-1}\left[\binom{2\eta+1}{\alpha+1}-\binom{\eta+1}{\alpha+1}\right](q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-1}\left[\binom{p+1}{\alpha+1}-\binom{1}{\alpha+1}\right](q-1)^{1+\alpha}$$

$$+(n-2p)\left\{\sum_{\alpha=0}^{\omega-2}\sum_{\gamma=0}^{p-1}\binom{2\gamma+1}{\alpha}(q-1)^{2+\alpha}+(q-1)\right\}$$

$$+\sum_{\alpha=0}^{\omega-2}\sum_{\mu=0}^{p-2}(p-1-\mu)\binom{2\mu+1}{\alpha}(q-1)^{2+\alpha}+p(q-1). \tag{10}$$

Remark 3.2 For $p = 0$, the maximum value of ω is 1. Then, the key errors of length $p = 0$ with hamming weight $\omega = 1$ or fewer will be the all single errors. Putting $p = 0$ and $\omega = 1$ in the inequality (10), we have

$$q^r\geq 1+n(q-1).$$

This coincides with the famous Hamming's Sphere-Packing bound (Hamming, 1950; also Peterson and Weldon, 1972) for single errors. This is the necessary condition to be satisfied for any single error correcting linear code.

Theorem 3.3 *An (n, n-r)* $\boldsymbol{K_{\omega,p}EC}$ *code* $(\omega\leq 2p+1, n>4p+2)$ *over finite field* F_q *having r check symbols shall always exist provided that*

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$$q^r > \left\{\sum_{\alpha=0}^{\omega-3}\sum_{\eta=1}^{p-1}\left[\binom{2\eta+1}{\alpha+1}-\binom{\eta+1}{\alpha+1}\right](q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-2}\left[\binom{p+1}{\alpha+1}-\binom{1}{\alpha+1}\right](q-1)^{1+\alpha}+1\right\}$$
$$\times\left\{1+\sum_{\alpha=0}^{\omega-2}\sum_{\eta=1}^{p-1}\left[\binom{2\eta+1}{\alpha+1}-\binom{\eta+1}{\alpha+1}\right](q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-1}\left[\binom{p+1}{\alpha+1}-\binom{1}{\alpha+1}\right](q-1)^{1+\alpha}\right.$$
$$+(n-1-4p)\left\{\sum_{\alpha=0}^{\omega-2}\sum_{\gamma=0}^{p-1}\binom{2\gamma+1}{\alpha}(q-1)^{2+\alpha}+(q-1)\right\}$$
$$\left.+\sum_{\alpha=0}^{\omega-2}\sum_{\mu=0}^{p-2}(p-1-\mu)\binom{2\mu+1}{\alpha}(q-1)^{2+\alpha}+p(q-1)\right\}.$$

Proof. For construction of parity check matrix H of order $r\times n$ for the desired $\boldsymbol{K_{\omega,p}EC}$ code, we follow the synthesis procedure as followed in Theorem 2.3. We take any nonzero r-tuple as the first column h_1 of the matrix H and for selection of next $\tau-1$ columns $h_2, h_3, h_4, \ldots., h_{\tau-1}, h_\tau$ one by one, we follow the condition:

$$h_\tau \neq (u_1h_{\tau-1}+u_2h_{\tau-2}+\cdots+u_{2p}h_{\tau-2p})+(v_lh_l+v_{l+1}h_{l+1}+\cdots+v_{l+2p}h_{l+2p}), \tag{11}$$

where $u_i, v_i \in GF(q)$; $l+2p<\tau-2p$; the coefficient u_i's are such that $\{h_{\tau-2p}, h_{\tau-2p+1},.., h_{\tau-1}, h_\tau\}$ form a key error of length upto p with the condition that the number of non-zero coefficients u_i's should not exceed ω and the coefficient v_i's are such that $\{h_l, h_{l+1},\ldots,h_{l+2p}\}$, $(l=1,2,\ldots,\tau-4p-1)$ form a key error of length upto p with hamming weight ω or fewer in the first $\tau-2p-1$ columns of H.

We now calculate the total number of all possible linear sums on the R.H.S. of (11):

The number of coefficients u_i's, including the zero vector, is calculated as

$$\sum_{\alpha=0}^{\omega-2}\binom{p}{\alpha}(q-1)^{1+\alpha}$$
$$+\left[\sum_{\alpha=0}^{\omega-3}\binom{p}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-2}\binom{p-1}{\alpha}(q-1)^{1+\alpha}\right]$$
$$+\left[\sum_{\alpha=0}^{\omega-3}\binom{p+1}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-3}\binom{p-1}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-2}\binom{p-2}{\alpha}(q-1)^{1+\alpha}\right]$$

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$$+\left[\sum_{\alpha=0}^{\omega-3}\binom{p+2}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-3}\binom{p}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-3}\binom{p-2}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-2}\binom{p-3}{\alpha}(q-1)^{1+\alpha}\right]$$

$$\vdots$$

$$+\left[\sum_{\alpha=0}^{\omega-3}\binom{2p-2}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-3}\binom{2p-4}{\alpha}(q-1)^{2+\alpha}+\cdots+\sum_{\alpha=0}^{\omega-3}\binom{2}{\alpha}(q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-2}\binom{1}{\alpha}(q-1)^{1+\alpha}\right]+1,$$

which can be written as (by using Identity 3.1)

$$\sum_{\alpha=0}^{\omega-3}\sum_{\eta=1}^{p-1}\left[\binom{2\eta+1}{\alpha+1}-\binom{\eta+1}{\alpha+1}\right](q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-2}\left[\binom{p+1}{\alpha+1}-\binom{1}{\alpha+1}\right](q-1)^{1+\alpha}+1. \quad (12)$$

The number of the coefficients v_i is equal to the number of key errors of length upto p with hamming weight ω or fewer in a $(\tau-1-2p)$-tuple. This number, including the vector of all zero components, is computed in (9), i.e.

$$\sum_{\alpha=0}^{\omega-2}\sum_{\eta=1}^{p-1}\left[\binom{2\eta+1}{\alpha+1}-\binom{\eta+1}{\alpha+1}\right](q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-1}\left[\binom{p+1}{\alpha+1}-\binom{1}{\alpha+1}\right](q-1)^{1+\alpha}$$

$$+(\tau-1-4p)\left\{\sum_{\alpha=0}^{\omega-2}\sum_{\gamma=0}^{p-1}\binom{2\gamma+1}{\alpha}(q-1)^{2+\alpha}+(q-1)\right\}$$

$$+\sum_{\alpha=0}^{\omega-2}\sum_{\mu=0}^{p-2}(p-1-\mu)\binom{2\mu+1}{\alpha}(q-1)^{2+\alpha}+p(q-1). \quad (13)$$

Therefore, the total number of linear sums on R.H.S. of (11) is equal to

$Expr:(12)\times Expr:(13)$

$$=\left\{\sum_{\alpha=0}^{\omega-3}\sum_{\eta=1}^{p-1}\left[\binom{2\eta+1}{\alpha+1}-\binom{\eta+1}{\alpha+1}\right](q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-2}\left[\binom{p+1}{\alpha+1}-\binom{1}{\alpha+1}\right](q-1)^{1+\alpha}+1\right\}$$

$$\times\left\{1+\sum_{\alpha=0}^{\omega-2}\sum_{\eta=1}^{p-1}\left[\binom{2\eta+1}{\alpha+1}-\binom{\eta+1}{\alpha+1}\right](q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-1}\left[\binom{p+1}{\alpha+1}-\binom{1}{\alpha+1}\right](q-1)^{1+\alpha}\right.$$

$$+(\tau-1-4p)\left\{\sum_{\alpha=0}^{\omega-2}\sum_{\gamma=0}^{p-1}\binom{2\gamma+1}{\alpha}(q-1)^{2+\alpha}+(q-1)\right\}$$

$$\left.+\sum_{\alpha=0}^{\omega-2}\sum_{\mu=0}^{p-2}(p-1-\mu)\binom{2\mu+1}{\alpha}(q-1)^{2+\alpha}+p(q-1)\right\}. \quad (14)$$

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Since the available number of *r*-tuples is q^r, so the addition of the τ^{th} column h_τ to *H* is possible provided

$$q^r > Expr:(14).$$

For a code of length *n*, we need to replace τ by *n* and we get

$$q^r > \left\{\sum_{\alpha=0}^{\omega-3}\sum_{\eta=1}^{p-1}\left[\binom{2\eta+1}{\alpha+1}-\binom{\eta+1}{\alpha+1}\right](q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-2}\left[\binom{p+1}{\alpha+1}-\binom{1}{\alpha+1}\right](q-1)^{1+\alpha}+1\right\}$$
$$\times\left\{1+\sum_{\alpha=0}^{\omega-2}\sum_{\eta=1}^{p-1}\left[\binom{2\eta+1}{\alpha+1}-\binom{\eta+1}{\alpha+1}\right](q-1)^{2+\alpha}+\sum_{\alpha=0}^{\omega-1}\left[\binom{p+1}{\alpha+1}-\binom{1}{\alpha+1}\right](q-1)^{1+\alpha}\right.$$
$$+(n-1-4p)\left\{\sum_{\alpha=0}^{\omega-2}\sum_{\gamma=0}^{p-1}\binom{2\gamma+1}{\alpha}(q-1)^{2+\alpha}+(q-1)\right\}$$
$$\left.+\sum_{\alpha=0}^{\omega-2}\sum_{\mu=0}^{p-2}(p-1-\mu)\binom{2\mu+1}{\alpha}(q-1)^{2+\alpha}+p(q-1)\right\}. \quad (15)$$

Remark 3.4 For $p = 0$ and $\omega = 1$, the inequality (15) reduces to

$$q^r > 1+(n-1)(q-1)=\sum_{\alpha=0}^{1}\binom{n-1}{\alpha}(q-1)^{\alpha}.$$

This coincides with the famous Varsharmov-Gilbert-Sacks bound (Sacks, 1958; Peterson and Weldon, 1972) for single errors. This is the sufficient condition for existing single error correcting linear code.

In this case also, we provide two examples of codes: one for binary case and one for ternary case.

Example 3.5 Taking $n = 11$, $q = 2$, $p = 2$ and $\omega = 3$ in Theorem 3.3, we get a binary (11, 3) linear code whose parity check matrix *H* is constructed by the method discussed in the theorem:

$$H=\begin{bmatrix}1&0&0&0&0&0&0&0&1&1&1\\0&1&0&0&0&0&0&0&1&1&0\\0&0&1&0&0&0&0&0&1&0&0\\0&0&0&1&0&0&0&0&1&1&1\\0&0&0&0&1&0&0&0&1&1&0\\0&0&0&0&0&1&0&0&1&0&0\\0&0&0&0&0&0&1&0&1&1&1\\0&0&0&0&0&0&0&1&1&1&0\end{bmatrix}.$$

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All 62 key errors of length upto $p = 2$ with hamming weight 3 or fewer and their corresponding syndromes code can be obtained with the help of MS-EXCEL and the syndromes are found to be nonzero and distinct. Therefore, the above code is a binary $\boldsymbol{K_{3,2}EC}$ *code*.

Example 3.6 $n = 11$, $q = 3$, $p = 2$ and $\omega = 3$ in Theorem 3.3 give rise to a (11, 3) linear code. The parity check matrix H of this code is given by

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 \end{bmatrix}.$$

Syndromes of all key errors of length upto 2 with hamming weight 3 or fewer are found to be nonzero and distinct. Therefore, the above code is a ternary (11, 3) $\boldsymbol{K_{3,2}EC}$ *code*.

4. Conclusion

This paper derives lower as well as upper number of check symbols needed to exist linear codes that correct all the key errors occurring within a sub-block. Further, such bounds for codes correcting low density key errors are also obtained. The numerical examples are given to justify the results. They give us the surety that the information rate for the codes is always achievable at least a certain limit. Correcting key errors in more than one sub-block (i.e., in multiple sub-blocks) remains to be studied.

Conflict of Interest
The authors declare that they have no conflict of interest.

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Fourth order compact scheme for the Navier-Stokes equations on time deformable domains

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Abstract

In this work, we report the development of a spatially fourth order temporally second order compact scheme for incompressible Navier-Stokes (N-S) equations in time-varying domain. Sen [J. Comput. Phys. 251 (2013) 251-271] put forward an implicit compact finite difference scheme for the unsteady convection-diffusion equation. It is now further extended to simulate fluid flow problems on deformable surfaces using curvilinear moving grids. The formulation is conceptualized in conjunction with recent advances in numerical grid deformations techniques such as inverse distance weighting (IDW) interpolation and its hybrid implementation. Adequate emphasis is provided to approximate grid metrics up to the desired level of accuracy and freestream preserving property has been numerically examined. As we discretize the non-conservative form of the N-S equation, the importance of accurate satisfaction of geometric conservation law (GCL) is investigated. To the best of our knowledge, this is the first higher order compact method that can directly tackle non-conservative form of N-S equation in single and multi-block time dependent complex regions. Several numerical verification and validation studies are carried out to illustrate the flexibility of the approach to handle high-order approximations on evolving geometries.

Keywords: Higher order compact, Incompressible Navier-Stokes equations, Geometric conservation, Moving grids, Deformable domain

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1. Introduction

Many problems in science and engineering involve solving partial differential equations (PDEs) in complex geometries that might even evolve in time. Here, it is necessary to properly account for the time variation of the solution domain and could require a new discretization at each time step. In this context improving the accuracy of a solution with simultaneous reduction of computational costs has been an ever-lasting goal in the computational fluid dynamics (CFD) community. Higher order methods enjoy unique interest largely because of their ability to produce highly accurate solutions often at a comparably lower computational cost. A detailed discussion of the advantages and challenges associated with higher order schemes can be found in [1]. Nevertheless, for efficient capture of small amplitude high wave-number fluctuations numerical schemes with higher truncation order and lower dissipation error are preferred. Fourth order compact schemes are thus inherently advantageous to correctly predict high frequency waves with a fewer number of grid points. These schemes carry lesser dispersion and anisotropy errors. Moreover, in compact schemes, the computational cost of inverting matrices is low as their smaller stencils translate to a banded system with smaller bandwidth. Typically these schemes are used to examine flow fields on simple static domains discretized by Cartesian meshes. Pioneering investigation using high order compact method on a time-varying domain was carried out by Visbal and Gaitonde [2]. Here mapping is constructed between a fixed reference domain and the real time-varying geometry. Subsequently, the original conservation law is first transformed to the reference configuration and then solved using a high order compact difference scheme in time dependent curvilinear coordinates. Apart from exploring the virtues of compact discretization the process has several advantages. Steger [3] whose early work pertains to the second order accurate discretization of conservative N-S equations listed some of those leverages. These include the boundary surfaces in the physical plane being mapped onto rectangular surfaces in the transformed plane, incorporation of unsteady motion into the governing equation, and clustering of grid points in regions with high flow gradients. Visbal and Gaitonde [2] worked with the conservative form of the N-S equations and following them many important works have discretized conservative forms [4, 5, 6, 7] to simulate flow using moving and deforming grids. In this approach, efficient discretization strategies are designed to approximate first order derivatives. The second order viscous terms are often tackled by the repeated application of the approximation procedure developed for the first order derivatives or at times by central differencing.

Working with a structured grid, finite volume approximation of first order advective terms in conservative formulation for variable meshes generally requires due diligence. The extra care required for efficient implementation and to avoid loss of accuracy is well pointed out by Hirt and Nichols [8] and Li *et al.* [9]. For general grid motion, two geometrical relations called GCL ensure that each cell in the flow region is closed and safeguard conservation of volume for time-varying cells. As GCL represents geometric identities, its differential and integral forms are trivial, but unfortunately, in discretized form, automatic satisfaction of the GCL is not guaranteed. A

detailed argument on conservation laws and their finite volume and finite difference formulation with emphasis on the specialized procedure adopted for grid metric calculation can be found in the work of Sjögreen *et al.* [10] and Abe *et al.* [11]. Nevertheless, a brief discussion on the development of GCL and its limitation in ensuring only up to second or third order accuracy for space and time using these metric formulations was discussed by Persson *et al.* [12].

Although there is a growing interest in higher order compact finite difference methods for fluid problems another challenge hindering its growth in time deforming domain is the absence of an effective grid deformation algorithm. It is well acknowledged that space-time accurate solutions of the N-S equations require good quality structured grid with smooth grid metrics as the solution domain continues to evolve. Maintaining smoothness and appropriate spacing, even when the initial mesh is comprehensive, are of primary importance in the numerical simulation of evolving physical phenomena. Failure to resolve important length scales of the flow at each time step might contribute to aliasing error. Traditional mesh deformation techniques based on transfinite interpolation (TFI) [13], although quite fast, are not always sufficient to obtain a reasonable quality of the deformed mesh. The generality of the transformed equations could be amply realized only with a fairly automatic method of generating smoothly varying grids that allow clustering and fit into arbitrary domains. In this context, contemporary grid deformations techniques using radial basis function (RBF) and inverse distance weighting (IDW) could play a pivotal role. Modern mesh deformation techniques based on RBFs [14, 15] produce high quality mesh but are computationally expensive. Another recently introduced explicit grid movement strategy established by Witteveen and Bijl [16] that utilizes weighted sums of the reciprocal distance of the boundary node displacements to deform interior nodes was found to be quite effective and fast. This method is popularly known as the IDW method. Subsequently, this procedure was significantly improved by Luke *et al.* [17]. Towards this end, a comprehensive study involving modern grid deformation techniques was carried out by Sen *et al.* [18]. In that study, the authors were able to bring an improvement over the current state-of-the-art by combining approaches of fundamental mesh deformation methods. It was shown that hybrid methodology can deform a grid with high skew quality metric value in comparatively lesser CPU time. The procedure was indeed found to be effective in simulating fluid–structure interaction (FSI) problems relying on large-eddy simulation (LES) [19, 20].

The purpose of this work is to present a fourth order compact finite difference method for simulating viscous incompressible fluids in general deformable boundaries. The formulation discretizes the non-conservative form of the incompressible N-S equations and is different from the strategy adopted for the conservative form [2]. The fourth order method advocated here is a pinnacle of a series of independent yet intertwined studies starting with Sen [21] and going onto Sen [22], Sen and Sheu [23] and finally Sen *et al.* [18]. It is a realization of compact discretization strategy for parabolic convection-diffusion equation in time-varying boundaries. The schemes, where transport variable and its first derivatives are considered as the unknowns, coalesce virtues of compact discretization and Padé approximation and carry good numerical dispersion and dissipation

characteristics. Working with the current methodology it seems that satisfaction of GCL might not be necessary and does not play a significant role. As the grid evolves, the use of modern hybrid grid deformation techniques [17, 18] in the absence of an analytic process helps maintain excellent mesh quality. Temporally second order accurate, the scheme is probably the first higher order compact discretization of non-conservative N-S equations in deformable geometries as we failed to locate any advancement towards such an approximation in the literature. Without any prejudice, we shall like to mention the work of Chen and Xie [28] where a second order accurate discretization based on the vorticity-vector potential formulation was discussed for incompressible flows in time-dependent curvilinear coordinates.

In the next section, we present the underlying mathematical background associated with various stages necessary for the development of the scheme. Subsequently in section 3, numerical verification and validation studies are carried out. Finally in section 4 conclusions are jotted down.

2. Mathematical formulation

2.1. Governing equation

We consider fluid to occupy a planner domain $\boldsymbol{x} \in \Omega(t)$ at time $t \in (0, T]$. The motion is assumed to be governed by the unsteady incompressible N-S equations. In the Eulerian frame using Cartesian coordinate the nondimensional primitive variable form of this equation together with the equation of continuity is

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} - \dfrac{1}{Re}\Delta \boldsymbol{u} = -\nabla p, & \boldsymbol{x} \in \Omega(t) \quad (1a) \\ \nabla \cdot \boldsymbol{u} = 0, & \boldsymbol{x} \in \Omega(t) \quad (1b) \end{cases}$$

where $\boldsymbol{u}(\boldsymbol{x}, t)$ denotes the velocity and $p(\boldsymbol{x}, t)$ is the pressure. This flow is characterized by the dimensionless Reynolds number Re. Apart from initial conditions, the above equations require appropriate boundary conditions on $\partial\Omega(t)$. The elliptic pressure Poisson equation

$$\begin{aligned} -\Delta p &= \nabla \cdot ((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) \\ &= \nabla \boldsymbol{u} : (\nabla \boldsymbol{u})^T, \quad \boldsymbol{x} \in \Omega(t) \end{aligned} \tag{2}$$

which requires an additional boundary condition

$$\boldsymbol{n} \cdot \nabla p = \boldsymbol{n} \cdot \left(\frac{1}{Re}\Delta \boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \right), \quad \boldsymbol{x} \in \partial\Omega(t) \tag{3}$$

is an able substitute of Eq. (1b). An alternative formulation, economical in two-dimension (2D) includes streamfunction (ψ) and vorticity (ω) and is often referred to as $\psi-\omega$ formulation. It is

$$\begin{cases} \dfrac{\partial \omega}{\partial t} + (\boldsymbol{u}\cdot\nabla)\omega - \dfrac{1}{Re}\Delta\omega = 0, & \boldsymbol{x}\in\Omega(t) \quad (4a) \\ -\Delta\psi = \omega, & \boldsymbol{x}\in\Omega(t) \quad (4b) \end{cases}$$

where $\omega = (\nabla\times\boldsymbol{u})\cdot\hat{k}$ is the out-of-plane component of vorticity and $\boldsymbol{u} = \nabla\times\Psi$ with $\Psi = (0,0,\psi)$. A prototype equation that can be used to describe both primitive and $\psi-\omega$ forms of N-S systems is the convection-diffusion equation

$$\frac{\partial\phi}{\partial t} - a(\boldsymbol{x},t)\Delta\phi + \boldsymbol{c}(\boldsymbol{x},t)\cdot\nabla\phi = s(\boldsymbol{x},t), \quad (\boldsymbol{x},t)\in\Omega(t)\times(0,T] \tag{5}$$

with the initial condition

$$\phi(\boldsymbol{x},0) = \phi_0(\boldsymbol{x}), \qquad \boldsymbol{x}\in\Omega(0) \tag{6}$$

and boundary condition

$$b_1(\boldsymbol{x},t)\phi + b_2(\boldsymbol{x},t)\frac{\partial\phi}{\partial\boldsymbol{n}} = g(\boldsymbol{x},t), \qquad \boldsymbol{x}\in\partial\Omega(t), \quad t\in(0,T]. \tag{7}$$

Here $\phi(\boldsymbol{x},t)$ is the transport variable with convection velocity $c(\boldsymbol{x},t) = (c_1(\boldsymbol{x},t), c_2(\boldsymbol{x},t))^T$, and $a(\boldsymbol{x},t) > 0$ having unit boundary normal vector $\boldsymbol{n}$.

2.2. Time-dependent coordinate transformation

We introduce an iso-parametric invertible time dependent mapping

$$x = x(\xi,\eta,\tau), \quad y = y(\xi,\eta,\tau), \quad t = \tau \tag{8}$$

to transform the spatial domain $\Omega(t)$ to a topologically equivalent domain $\mathcal{D}$ thereby resulting in the generation of curvilinear meshes. The Jacobian of the transformation is

$$\mathcal{J} = \frac{\partial(x,y,t)}{\partial(\xi,\eta,\tau)} = \begin{bmatrix} x_\xi & x_\eta & x_\tau \\ y_\xi & y_\eta & y_\tau \\ 0 & 0 & 1 \end{bmatrix}. \tag{9}$$

The Jacobian of the inverse transform then corresponds to

$$\mathcal{J}^{-1} = \frac{\partial(\xi,\eta,\tau)}{\partial(x,y,t)} = \begin{bmatrix} \xi_x & \xi_y & \xi_t \\ \eta_x & \eta_y & \eta_t \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{J}\begin{bmatrix} y_\eta & -x_\eta & -J_1 \\ -y_\xi & x_\xi & -J_2 \\ 0 & 0 & J \end{bmatrix} \tag{10}$$

where $J = \left|\dfrac{\partial(x,y)}{\partial(\xi,\eta)}\right| \neq 0$ and $J_1 = \left|\dfrac{\partial(x,y)}{\partial(\tau,\eta)}\right|$, $J_2 = \left|\dfrac{\partial(x,y)}{\partial(\xi,\tau)}\right|$. Note that J_1 and J_2 are influenced by grid velocity in addition to grid metrics. Directly using the chain rule the convection-diffusion equation (5) along with auxiliary conditions (6) and (7) can be rewritten as

$$\begin{cases} \partial_\tau \phi(\xi,\eta,\tau) + A\phi(\xi,\eta,\tau) = s(\xi,\eta,\tau), & (\xi,\eta,\tau) \in \mathcal{D} \times (0,T] \quad (11a) \\ \phi(\xi,\eta,0) = \phi_0(\xi,\eta), & (\xi,\eta) \in \mathcal{D} \quad (11b) \\ \vartheta_1(\xi,\eta,\tau)\phi + \vartheta_2(\xi,\eta,\tau)\partial_{\boldsymbol{\nu}}\phi = g(\xi,\eta,\tau), & (\xi,\eta) \in \partial\mathcal{D},\ \tau \in (0,T]. \quad (11c) \end{cases}$$

with partial differential operator $[A]$ being

$$[A] \equiv [-\alpha_1 \partial_{\xi\xi} - \beta \partial_{\xi\eta} - \alpha_2 \partial_{\eta\eta} + \chi_1 \partial_\xi + \chi_2 \partial_\eta]. \tag{12}$$

Eq. (11a) possesses positive definiteness of the diffusion matrix *i.e.* $\alpha_1 > 0$, $\alpha_2 > 0$, $|\beta|^2 \leq 4\alpha_1\alpha_2$ $\forall\ (\xi,\eta,\tau) \in \mathcal{D} \times (0,T]$ where

$$\begin{cases} \alpha_1 = \dfrac{a}{J^2}(x_\eta^2 + y_\eta^2), & (13a) \\ \alpha_2 = \dfrac{a}{J^2}(x_\xi^2 + y_\xi^2), & (13b) \\ \beta = -\dfrac{2a}{J^2}(x_\xi x_\eta + y_\xi x_\eta), & (13c) \\ \chi_1 = \dfrac{1}{J}(-J_1 + c_1 y_\eta - c_2 x_\eta) - \dfrac{a}{J^3}\Big(J_\eta(x_\xi x_\eta + y_\xi y_\eta) \\ \quad - J_\xi(x_\eta^2 + y_\eta^2) + J(x_\eta x_{\xi\eta} + y_\eta y_{\xi\eta} - x_\xi x_{\eta\eta} - y_\xi y_{\eta\eta}\Big), & (13d) \\ \chi_2 = \dfrac{1}{J}(-J_2 - c_1 y_\xi + c_2 x_\xi) - \dfrac{a}{J^3}\Big(J_\xi(x_\xi x_\eta + y_\xi y_\eta) \\ \quad - J_\eta(x_\xi^2 + y_\xi^2) + J(x_\xi x_{\xi\eta} + y_\xi y_{\xi\eta} - x_\eta x_{\xi\xi} - y_\eta y_{\xi\xi}\Big). & (13e) \end{cases}$$

2.3. Geometric conservation law

As discretization of the governing equation is carried out using a temporally varying grid the mesh geometry changes with time. To better focus the role of GCL, convection and diffusion terms of the 2D convection-diffusion equation with its coefficients kept fixed are expressed as inviscid and viscous fluxes, respectively. The following equation

$$\frac{\partial \phi}{\partial t} - \frac{\partial}{\partial x}\left(a\frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial y}\left(a\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial x}\left(c_1\phi\right) + \frac{\partial}{\partial y}\left(c_2\phi\right) = s \tag{14}$$

can be further written as

$$\frac{\partial \phi}{\partial t} - \frac{\partial f_v}{\partial x} - \frac{\partial g_v}{\partial y} + \frac{\partial f_i}{\partial x} + \frac{\partial g_i}{\partial y} = s, \tag{15}$$

where the fluxes are defined as $f_i = c_1\phi$, $g_i = c_2\phi$, $f_v = a\frac{\partial \phi}{\partial x}$, $g_v = a\frac{\partial \phi}{\partial y}$. The time dependent mapping provided in Eq. (8) allows one to transform the above equation to yield a strongly conservative formulation using the chain rule. The transformed governing equation in the computational plane is presented using the newly defined solution and fluxes and it takes the following form

$$\begin{aligned}\frac{\partial \phi^c}{\partial \tau} - \frac{\partial f_v^c}{\partial \xi} - \frac{\partial g_v^c}{\partial \eta} + \frac{\partial f_i^c}{\partial \xi} + \frac{\partial g_i^c}{\partial \eta} &= s + \phi\left(\frac{\partial J}{\partial \tau} - \frac{\partial J_1}{\partial \xi} - \frac{\partial J_2}{\partial \eta}\right) \\ &+ \left(c_1\phi - a\frac{\partial \phi}{\partial x}\right)\left(\frac{\partial (J\xi_x)}{\partial \eta} + \frac{\partial (J\eta_x)}{\partial \eta}\right) \\ &+ \left(c_2\phi - a\frac{\partial \phi}{\partial x}\right)\left(\frac{\partial (J\xi_y)}{\partial \eta} + \frac{\partial (J\eta_y)}{\partial \eta}\right)\end{aligned} \tag{16}$$

where $\phi^c = J\phi$, $f_i^c = (c_1\xi_x + c_2\xi_y + \xi_t)\phi^c$, $f_i^c = (c_1\eta_x + c_2\eta_y + \eta_t)\phi^c$, $f_v^c = J(f_v\xi_x + g_v\xi_y)$, $g_v^c = J(f_v\eta_x + g_v\eta_y)$. From Eq. (16), it is clear that for the system to remain conservative it is imperative that

$$\begin{cases} \dfrac{\partial J}{\partial \tau} - \dfrac{\partial J_1}{\partial \xi} - \dfrac{\partial J_2}{\partial \eta} = 0, & \text{(17a)} \\ \dfrac{\partial (J\xi_x)}{\partial \eta} + \dfrac{\partial (J\eta_x)}{\partial \eta} = 0, & \text{(17b)} \\ \dfrac{\partial (J\xi_y)}{\partial \eta} + \dfrac{\partial (J\eta_y)}{\partial \eta} = 0. & \text{(17c)} \end{cases}$$

The above set of equations ensure strong conservation for flow equations and they are referred to as GCL equations. Adherence to these equations rule out any apparent numerical source term in Eq. (16) and hence are important to accurately represent the transformed governing equation. Thus in any discretization of conservative form, these identities are to be satisfied numerically to avoid numerical instability. This has led to specialized methods for computing grid metrics that duly accommodate GCL relations [10, 11]. On the other hand, the transformed equation cast in non-conservative form as reported in Eq. (11a) *vis-a-vis* Eq. (5) might be immune to such numerical considerations and could turn out to be the one that is much more advantageous. Nevertheless, the computation of temporal and spatial grid metrics up to the respective desired levels of accuracy remain vital.

2.4. Compact discretization

The following central difference operators

$$\delta_\xi \phi_{i,j} \equiv \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2h}, \quad \delta_\eta \phi_{i,j} \equiv \frac{\phi_{i,j+1} - \phi_{i,j-1}}{2k},$$

$$\delta_\xi^2 \phi_{i,j} \equiv \frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{h^2}, \quad \delta_\eta^2 \phi_{i,j} \equiv \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{k^2},$$

are used for ease in description. Here h and k are spatial interval in ξ and η-direction, respectively, in the structured computational mesh and $\phi_{i,j} = \phi(\xi_i, \eta_j)$, with $\xi_i = \xi_{\min} + ih$, $\eta_j = \eta_{\min} + jk$, $0 \le i \le N_\xi$ and $0 \le j \le N_\eta$.

The second order unary and mixed partial derivatives appearing in the diffusive terms in the operator $[A]$ of Eq. (12) can be compactly approximated with fourth-order accuracy following their development in the work of Sen [21, 22]. These approximations amenable with variable coefficients possess good numerical characteristics and are given by

$$\begin{cases} \partial_{\xi\xi}\phi_{i,j} = 2\delta_\xi^2 \phi_{_{i,j}} - \delta_\xi \phi_{\xi_{_{i,j}}} + O(h^4), & \text{(18a)} \\ \partial_{\eta\eta}\phi_{i,j} = 2\delta_\eta^2 \phi_{_{i,j}} - \delta_\eta \phi_{\eta_{i,j}} + O(k^4), & \text{(18b)} \\ \partial_{\xi\eta}\phi_{i,j} = \delta_\xi \phi_{\eta_{i,j}} + \delta_\eta \phi_{\xi_{_{i,j}}} - \delta_\xi \delta_\eta \phi_{i,j} + O(h^2 k^2). & \text{(18c)} \end{cases}$$

We thus define spatially fourth order accurate discrete operator $A_{h,k}$ as

$$\begin{aligned} A_{h,k}\phi_{i,j} = (-2\alpha_{1_{i,j}}\delta_\xi^2 - 2\alpha_{2_{i,j}}\delta_\eta^2 + \beta_{_{i,j}}\delta_\xi\delta_\eta)\phi_{_{i,j}} \\ +(\alpha_{1_{i,j}}\delta_\xi - \beta_{_{i,j}}\delta_\eta + \chi_{1_{i,j}})\phi_{\xi_{i,j}} + (\alpha_{2_{i,j}}\delta_\eta - \beta_{_{i,j}}\delta_\xi + \chi_{2_{i,j}})\phi_{\eta_{i,j}}. \end{aligned} \quad \text{(19)}$$

The above operators further require compatible fourth order accurate approximations of the gradients of ϕ and are accomplished by applying Padé approximations

$$\begin{cases} (1 + \frac{h^2}{6}\delta_\xi^2)\phi_{\xi_{i,j}} = \delta_\xi \phi_{i,j} + O(h^4), & \text{(20a)} \\ (1 + \frac{k^2}{6}\delta_\eta^2)\phi_{\eta_{i,j}} = \delta_\eta \phi_{i,j} + O(k^4). & \text{(20b)} \end{cases}$$

Finally, second order accurate temporal discretization of the Eq. (11a) is realized using Crank-Nicolson method. Thus, a fully discrete scheme for grid point (i, j) at time level (n) is

$$\left(1 + \frac{\delta\tau}{2}A_{h,k}^{(n+1)}\right)\phi_{i,j}^{(n+1)} = \left(1 - \frac{\delta\tau}{2}A_{h,k}^{(n)}\right)\phi_{i,j}^{(n)} + \frac{\delta\tau}{2}\left(s_{i,j}^{(n+1)} + s_{i,j}^{(n)}\right). \quad \text{(21)}$$

The above formulation can be effectively implemented to discretize parabolic systems presented via Eq. (1a) and Eq. (4a) with convection velocity set to $\boldsymbol{c} = \boldsymbol{u}$. For steady elliptic equation

$$-\Delta\phi = s(x, y) \tag{22}$$

representing pressure Poisson and stream function equations (2) and (4b) respectively at each temporal step the discrete form relates to

$$\tilde{A}_{h,k}\phi_{i,j} = s_{i,j} \tag{23}$$

where discrete operator $\tilde{A}_{h,k}$ is

$$\begin{aligned}\tilde{A}_{h,k}\phi_{i,j} =&(-2\tilde{\alpha}_{1_{i,j}}\delta_\xi^2 - 2\tilde{\alpha}_{2_{i,j}}\delta_\eta^2 + \tilde{\beta}_{i,j}\delta_\xi\delta_\eta)\phi_{i,j} \\ &+ (\tilde{\alpha}_{1_{i,j}}\delta_\xi - \tilde{\beta}_{i,j}\delta_\eta + \tilde{\chi}_{1_{i,j}})\phi_{\xi_{i,j}} + (\tilde{\alpha}_{2_{i,j}}\delta_\eta - \tilde{\beta}_{i,j}\delta_\xi + \tilde{\chi}_{2_{i,j}})\phi_{\eta_{i,j}}\end{aligned} \tag{24}$$

with

$$\tilde{\alpha}_1 = \frac{1}{J^2}(x_\eta^2 + y_\eta^2), \quad \tilde{\alpha}_2 = \frac{1}{J^2}(x_\xi^2 + y_\xi^2), \quad \tilde{\beta} = -\frac{2}{J^2}(x_\xi x_\eta + y_\xi x_\eta),$$

$$\tilde{\chi}_1 = -\frac{1}{J^3}\bigg(J_\eta(x_\xi x_\eta + y_\xi y_\eta) - J_\xi(x_\eta^2 + y_\eta^2) + J(x_\eta x_{\xi\eta} + y_\eta y_{\xi\eta} - x_\xi x_{\eta\eta} - y_\xi y_{\eta\eta}\bigg),$$

$$\tilde{\chi}_2 = -\frac{1}{J^3}\bigg(J_\xi(x_\xi x_\eta + y_\xi y_\eta) - J_\eta(x_\xi^2 + y_\xi^2) + J(x_\xi x_{\xi\eta} + y_\xi y_{\xi\eta} - x_\eta x_{\xi\xi} - y_\eta y_{\xi\xi}\bigg).$$

2.5. Computation of grid metrics

For the discretizations presented in Eqs. (21) and (24) to be able to retain spatial fourth order precision it is important to compute grid metrics to the same order of accuracy. In the absence of analytic expression grids in the physical plane are required to be generated numerically and hence discrete computation of grid metrics becomes imperative. As symmetric-conservative metric evaluation procedure ensuring automatic satisfaction of GCL identities is not necessitated for our discretized non-conservative system, computation of grid metrics is rather straightforward once the deforming grid is generated for a particular time step. To begin with, first order derivatives of the transformation are estimated using Padé approximations at the interior points. It is well known in the literature that for a pth order interior scheme, boundary accuracy can be $(p-1)$st order without compromising global accuracy. Hence we work with one sided third order wide stencil approximations at boundary points. A similar process is repeated for computing second order derivatives appearing in Eq. (13). With the grid metrics available at all nodal points, the Jacobian can be explicitly found, following which gradients of the Jacobian can be estimated employing the Padé approximation once again. Calculation of components of grid velocity having higher accuracy demands the storage of successive previous grids and is memory intensive. Except for one verification case simulations in this study are carried out by storing only one preceding grid thereby restricting computations to the first order accuracy although the overall discretization is temporally second order accurate.

2.6. Grid movement

The numerical grid movement is carried out using TFI and IWD procedure in the absence of analytic expression. For a few cases, the hybrid procedure was found to be best suited. The main steps associated with TFI and IDW interpolation are delineated below, details of which could be found in [13, 17, 18].

Consider that there are a Nb and Ni number of boundary and interior nodes respectively in the physical domain $\partial\Omega \cup \Omega$. Given the displacements of the boundary points $\boldsymbol{d}(\boldsymbol{x}_b)$, TFI transmutation $\boldsymbol{d}(\boldsymbol{x})$ at an interior point $\boldsymbol{x}$ is given by

$$\boldsymbol{d}(\boldsymbol{x}) = \boldsymbol{F_1} \oplus \boldsymbol{F_2} = \boldsymbol{F_1} + \boldsymbol{F_2} - \boldsymbol{F_1 F_2} \tag{25}$$

where $\boldsymbol{F}_i$'s are univariate projectors and $\boldsymbol{F}_i\boldsymbol{F}_j$ refer to their tensor product. Univariate projectors are obtained from $\boldsymbol{d}(\boldsymbol{x}_b)$ using transfinite formula based on arc-length to maintain the grid distribution laws of the initial grid details of which can be found in [13, 18].

In contrast, IDW essentially interpolates translational and rotational displacements of the boundary nodes to the interior nodes. If $\boldsymbol{M}_b$ is the resultant rotation matrix and $\boldsymbol{t}_b$ is the associated translation vector for a boundary point having coordinate $\boldsymbol{x}_b$ at any temporal step of the evolving grid, then the displacement field $\boldsymbol{d}(\boldsymbol{x}_b)$ for that time step at that point could be written as

$$\boldsymbol{x}_b + \boldsymbol{d}(\boldsymbol{x}_b) = \boldsymbol{M}_b \boldsymbol{x}_b + \boldsymbol{t}_b \; . \tag{26}$$

The previous position vector $\boldsymbol{x}_b$, subsequent position vector $\boldsymbol{x}_b + \boldsymbol{d}(\boldsymbol{x}_b)$ and the translation vector $\boldsymbol{t}_b$ can be used to define quaternions $R_{p_b} = [0, \boldsymbol{x}_b]$, $R_{s_b} = [0, \boldsymbol{x}_b + \boldsymbol{d}(\boldsymbol{x}_b)]$ and $T_b = [0, \boldsymbol{t}_b]$. These three quaternions are consequently used to define rotation quaternions Q_b at each boundary point and are given by

$$R_{s_b} = Q_b \, R_{p_b} \, Q_b^* + T_b \; . \tag{27}$$

As rotation quaternions and the translation quaternions are estimated for all boundary grid points they are transferred to the interior grids by using the IDW interpolation methodology. This procedure interpolates a function at a given point $\boldsymbol{x}$ by taking a weighted average of known functional values. For example, the displacements at the interior nodes can be calculated as

$$\boldsymbol{d}(\boldsymbol{x}) = \frac{\sum\limits_{n=1}^{Nb} w_n(\boldsymbol{x}) \, \boldsymbol{d}(\boldsymbol{x}_b^n)}{\sum\limits_{n=1}^{Nb} w_n(\boldsymbol{x})} \; , \tag{28}$$

where

$$w_n(\boldsymbol{x}) = A_n \left[\left(\frac{L_{\text{def}}}{\parallel \boldsymbol{x} - \boldsymbol{x}_b^n \parallel} \right)^a + \left(\frac{\alpha L_{\text{def}}}{\parallel \boldsymbol{x} - \boldsymbol{x}_b^n \parallel} \right)^b \right] , \tag{29}$$

is the weighting function. Here A_n is the area weight of the nth boundary node. The best values for parameters a and b as 3 and 5, respectively. Luke *et al.* [17] defined L_{def} and α as

$$L_{\text{def}} = \max_{n=1}^{Nb} \parallel \boldsymbol{x}_b^n - \boldsymbol{x}_{\text{centroid}} \parallel \qquad \text{where} \qquad \boldsymbol{x}_{\text{centroid}} = \frac{1}{Nb+Ni} \sum_{n=1}^{Nb+Ni} \boldsymbol{x}^n \,, \tag{30}$$

and

$$\alpha = \frac{\eta}{L_{\text{def}}} \max_{n=1}^{Nb} \parallel \boldsymbol{d}(\boldsymbol{x}_b^n) - \boldsymbol{d}_{\text{mean}} \parallel \quad \text{with} \quad \boldsymbol{d}_{\text{mean}} = \sum_{n=1}^{Nb} a_n \, \boldsymbol{d}(\boldsymbol{x}_b^n) \tag{31}$$

respectively. Further suggested values are $a_n = A_n \Big/ \sum_{n=1}^{Nb} A_n$ and $\eta = 5$. In this work A_n is kept uniformly at one. It is to be noted that Eq. (31) is an option and could be altered by a fixed value of α appropriate to the deformation sought.

The above procedure applied to interpolate boundary rotation and translation quaternions is quite effective to deform mesh in situations involving a combination of translation and rotation and results in the generation of the smoothly varying grid. Here adequate care needs to be taken to normalize components of the rotation quaternions at the interior mesh points before finally estimating the displacement vectors at those grids.

3. Numerical examples

3.1. Convection-diffusion of Gaussian pulse

Verification of the newly developed scheme is carried out by simulating convection-diffusion of an initial Gaussian pulse

$$\phi_0(x, y) = e^{-a[(x-x_c)^2+(y-y_c)^2]} \tag{32}$$

of unit height centred at (x_c, y_c).

3.1.1. Moving and deforming domain

To an initial $[0, 2] \times [0, 2]$ domain discretized using Cartesian grid time-varying deformations are prescribed on all sides. On the left and right boundaries, identical displacements using $A(t) \sin(\pi\eta)$ are enforced. The top boundary is misshapen according to $2A(t) \sin(0.5\pi\xi)$ whereas the bottom boundary is deformed using bi-quadratic polynomial $2A(t)\xi^2(2-\xi)^2$. Time varying magnitude of deformation is set at $A(t) = 0.2 \sin(0.5\pi\tau/T)$. Moreover, all boundary points uniformly follow the pulse centred at $(0.5, 0.5)$ with translational velocity $(0.7, 0.7)\tau/T$ resulting in a continuous displacement of the domain of interest. In the absence of any analytic approach, numerical grid deformation is envisaged in this test case. TFI is preferred as the problem involves the convex domain.

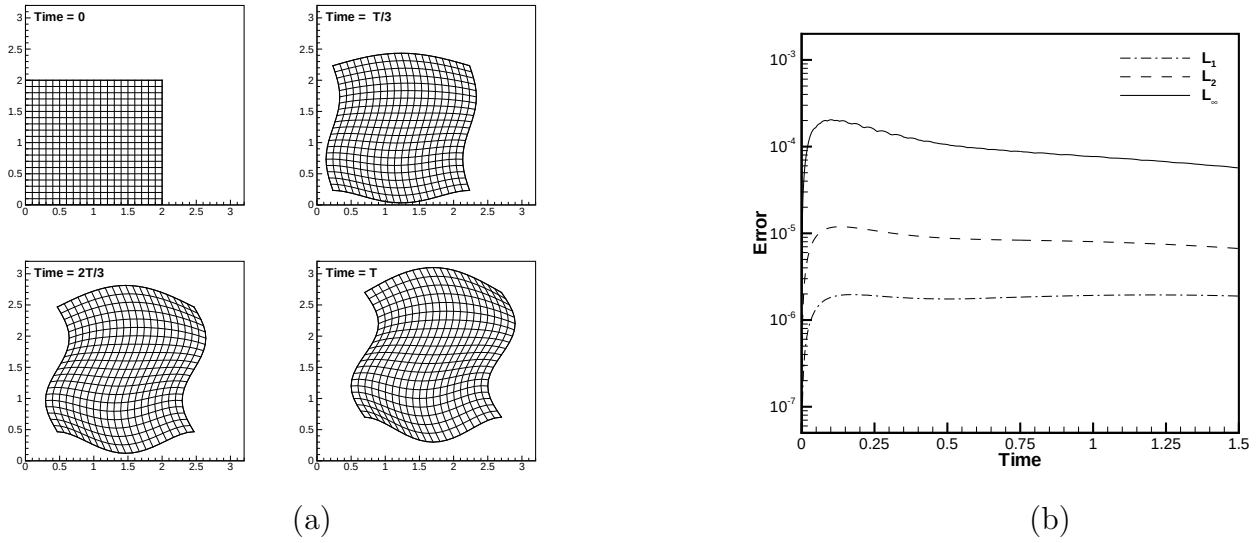
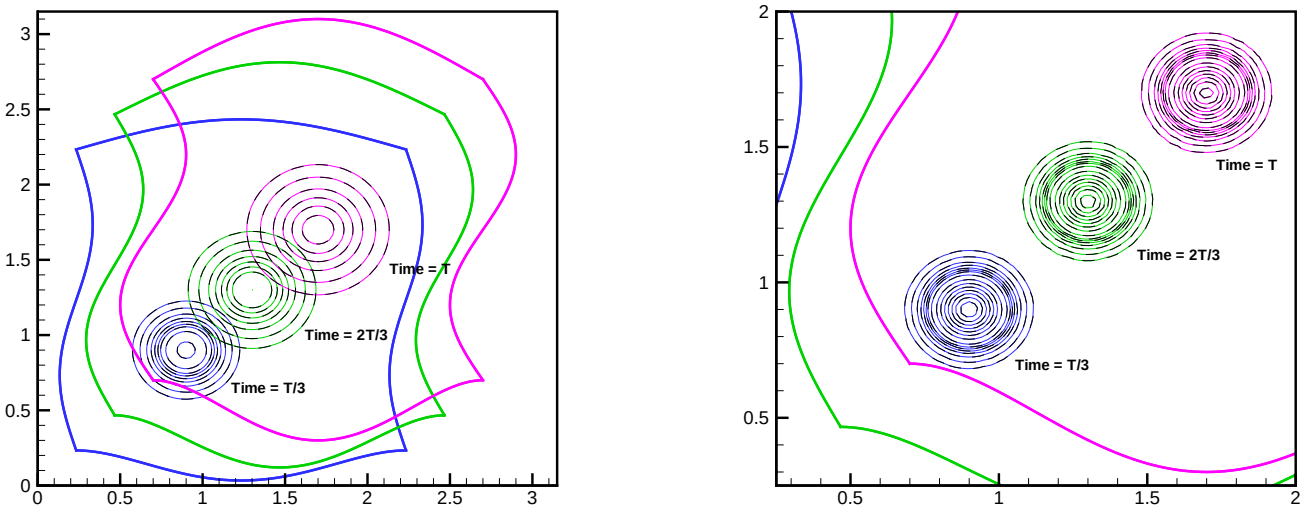


Figure 1: *Problem 1: (a) Deformed and translating gird at four different time. (b) Time evolution of errors corresponding to three different norms.*

Four stages of the resultant time-varying grid are presented in Fig. 1(a). In this figure, a significant deterioration of the orthogonality and translation of the grid can be noticed. To check convergence of the scheme we compute upto $T = 1.5$ with $h^2 = k^2 = (1/40)^2 = \delta\tau$ taking diffusion coefficient and convection coefficients fixed at $a = 100$ and $c_1 = 80 = c_2$ respectively. This choice corresponds to cell Peclet number $Pe = 2$ in the computational plane. Isomorphic temporally decaying nature of error computed using three different norms could be found in Fig. 1(b). For this problem grid metrics are calculated using a fourth order central scheme at interior points and one sided wide stencil schemes at boundary points. As this problem is designed for verification we store three consecutive grids and estimate grid velocities with second order accuracy. A numerical verification of the spatial order of accuracy in evolving domain is carried out in Table 1 using three different grids. Computations indeed support the theoretical order of convergence especially at the early time but are seen to report reduced order with time advancing. We further compare the numerical and analytic solutions in Fig. 2(a). The numerical solution obtained using 81×81 grid and $\delta\tau = 6.25 \times 10^{-4}$ is almost indistinguishable from the analytical solution. Investigation on the temporal order of accuracy of the scheme could be found in Table 2. For this estimation, additional grid translation is withdrawn and a relative finer grid spacing $h = k = 0.05$ is employed to keep spatial error low. Three different temporal step sizes $\delta\tau = 0.05$, 0.025 and 0.0125 are

Table 1: *Problem 1:* L_1-, L_2-, L_∞- *norm error and spatial order of convergence with* $\delta\tau = h^2 = k^2$.

Time		21×21	Order	41×41	Order	81×81
$\tau = 0.5$	L_1	6.612×10^{-4}	4.85	2.283×10^{-5}	3.71	1.750×10^{-6}
	L_2	2.275×10^{-3}	4.32	1.137×10^{-4}	3.69	8.786×10^{-6}
	L_∞	1.891×10^{-2}	3.72	1.435×10^{-3}	3.76	1.056×10^{-4}
$\tau = 1.0$	L_1	3.279×10^{-4}	4.46	1.490×10^{-5}	2.95	1.924×10^{-6}
	L_2	1.123×10^{-3}	4.07	6.706×10^{-5}	3.06	8.018×10^{-6}
	L_∞	1.007×10^{-2}	3.81	7.180×10^{-4}	3.22	7.689×10^{-5}



(a) (b)

Figure 2: *Problem 1: Comparison of numerical solution (solid lines) and exact solution (black dashed lines) at three different times for (a)* $Pe = 2$ *and (b)* $Pe = 200$.

used and the results obtain are shown in Table 2. Numerical computations are seen to support the theoretical second order temporal accuracy of the scheme. To further compare our results we compute with higher convection velocity $c_1 = 10000 = c_2$ using spatial grid spacing $h = 0.05 = k$

Table 2: *Problem 1: L_1-, L_2-, L_∞- norm error and temporal order of convergence.*

Time		$\delta\tau = 0.05$	Order	$\delta\tau = 0.025$	Order	$\delta\tau = 0.0125$
$\tau = 0.5$	L_1	1.366×10^{-3}	1.85	3.778×10^{-4}	1.64	1.211×10^{-4}
	L_2	6.557×10^{-3}	1.79	1.890×10^{-3}	1.50	6.672×10^{-4}
	L_∞	7.609×10^{-2}	1.82	2.150×10^{-2}	1.48	7.683×10^{-3}
$\tau = 1.0$	L_1	1.042×10^{-3}	1.80	2.992×10^{-4}	1.54	1.031×10^{-4}
	L_2	4.050×10^{-3}	1.73	1.219×10^{-3}	1.43	4.513×10^{-4}
	L_∞	3.799×10^{-2}	1.77	1.117×10^{-2}	1.40	4.240×10^{-3}

and $\delta\tau = 2.5\times10^{-4}$ corresponding to $Pe = 200$. A numerical solution along with the exact solution computed with $T = 0.012$ can be found in Fig. 2(b). Again a good comparison could be observed.

3.1.2. Randomized mesh

Here convection-diffusion of the Gaussian pulse centred at $(0.5, 0.5)$ through a randomized mesh is considered. Starting with an initial uniform mesh a new randomized mesh is generated at every time step by 20% relative perturbation of the grid spacing in both x and y-directions. The resultant mesh, not only exhibits rapid and persistent non-smooth metric variations but is also different in each time step and might be distinct from the static randomized mesh considered in the work of Visbal and Gaitonde [2]. Randomized variation of nominal mesh at two subsequent time steps are shown in Fig. 3(a) and (b). Following [2] metric continuity along all sides of the domain is maintained by leaving a few near boundary points unperturbed. The complete mesh can be found in Fig. 3(c). Given the current formulation, where regularity and gradual variation of the grid is implicit, this mesh is an extreme test for correct simulation. We calculate for small Peclet number $Pe = 0.25$ by choosing $a = 100$, $c_1 = 80 = c_2$ and $\delta\tau = 1 \times 10^{-3}$. Numerical solutions obtained are presented in Fig. 3(d) for three different temporal stages along with the exact solution. From this figure, it is clear that the current formulation is quite effective in simulating the motion of the pulse although some phase difference can be noticed.

3.2. Freestream preservation in deforming domain

Steger [3] in his study with conservative formulation found that freestream preservation characteristics of the compact methods are similar to that of the standard central difference schemes. Further Visbal and Gaitonde reported that compact schemes account for negligible metric cancellation errors when an identical discretization procedure is adopted for fluxes and the metrics [2]. Nevertheless, working with conservative N-S equations in moving and deforming meshes the GCL identities as delineated in Eq. (17) are of paramount interest to eliminate metric cancellation

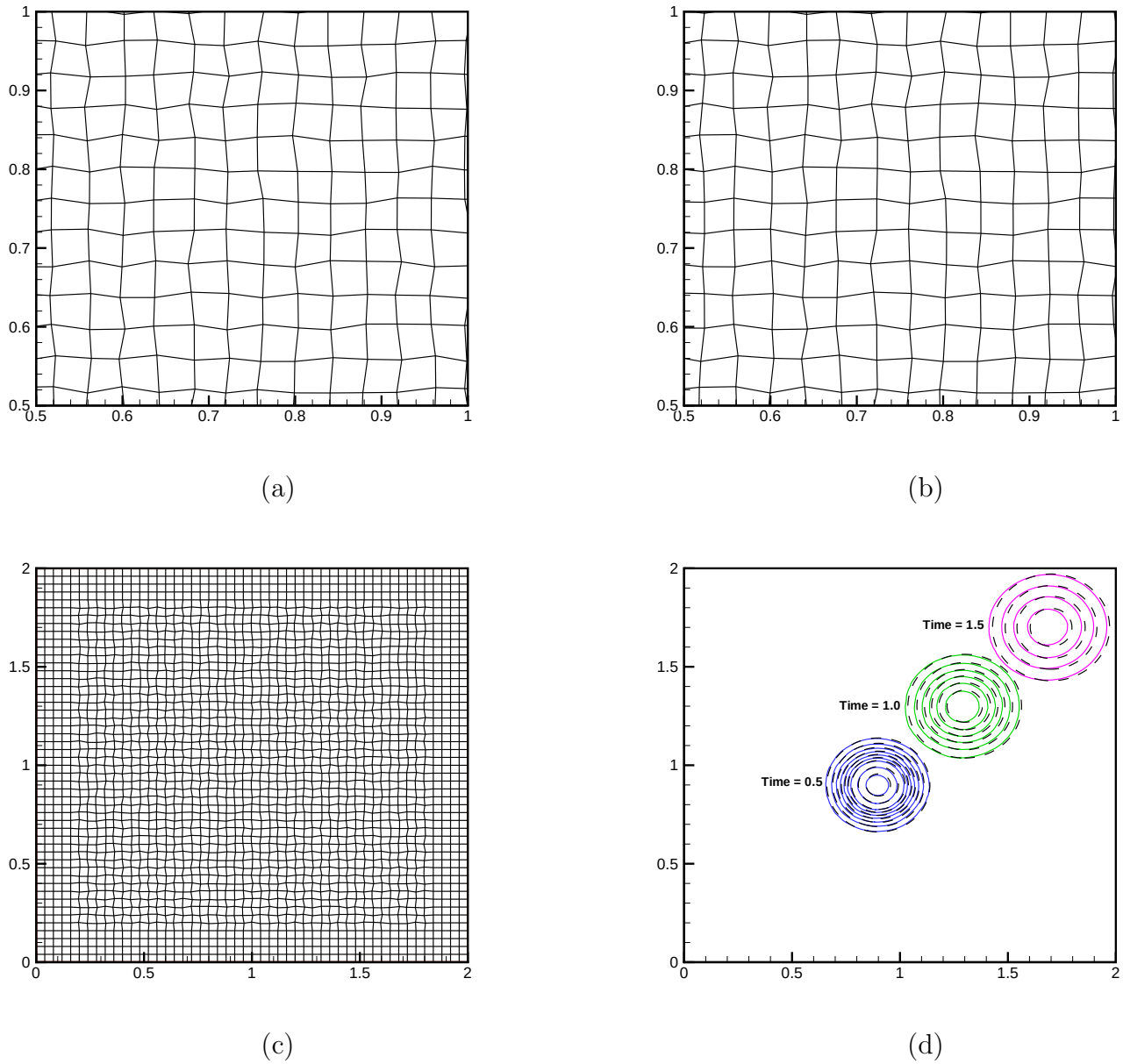


Figure 3: *Problem 1: (a) and (b) Randomized mesh at two subsequent time steps. (c) Nominal* 51×51 *mesh. (d) Comparison of numerical solution (solid lines) and exact solution (black dashed lines) at three different time.*

Table 3: *Problem 2: Freestream preservation errors for 2D wavy mesh*

Grid	25×25		65×65	
	Error $\|\|v\|\|_\infty$	Error $\|\|p\|\|_\infty$	Error $\|\|v\|\|_\infty$	Error $\|\|p\|\|_\infty$
With GCL	4×10^{-30}	4×10^{-30}	3×10^{-29}	1×10^{-29}
Without GCL	5×10^{-30}	6×10^{-30}	9×10^{-29}	8×10^{-29}

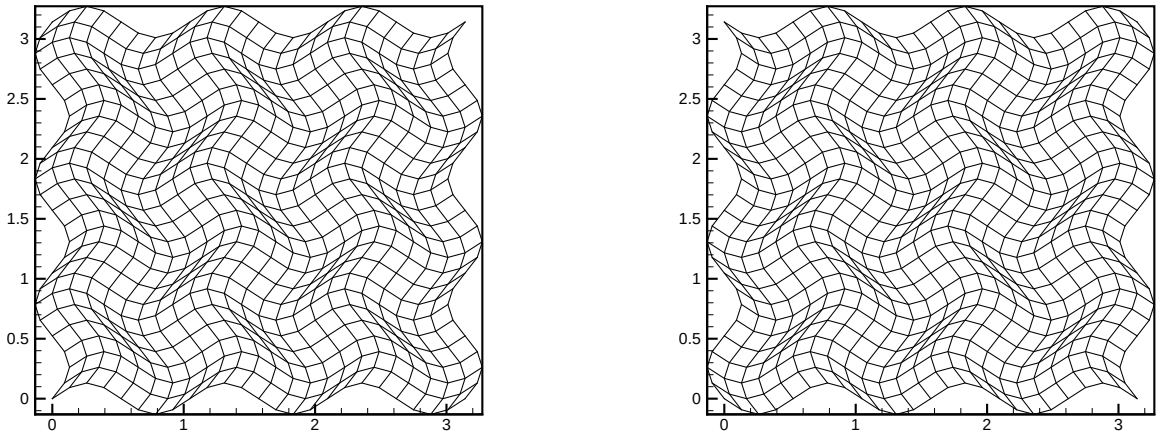
errors and thereby ensure freestream preservation. In the context of non-conservative formulation, it is important to document freestream preservation characteristics of compact schemes on highly distorted curvilinear meshes using a rather straightforward approach to calculating the grid metrics. A comparative study is also envisaged by enforcing the metric identities following the newly developed symmetric-conservative metric evaluation procedure of Abe *et al.* [11]. Authors in [11] analytically re-expressed coordinate transformation metrics and the procedure was found to ensure automatic satisfaction of GCL identities in 2D and is given by,

$$\begin{cases} J = [(x_\xi y)_\eta - (x_\eta y)_\xi + (xy_\eta)_\xi - (xy_\xi)_\eta]/2, & \text{(33a)} \\ J_1 = [(x_\tau y)_\eta - (x_\eta y)_\tau + (xy_\eta)_\tau - (xy_\tau)_\eta]/2, & \text{(33b)} \\ J_2 = [(x_\xi y)_\tau - (x_\tau y)_\xi + (xy_\tau)_\xi - (xy_\xi)_\tau]/2. & \text{(33c)} \end{cases}$$

Inviscid uniform flow with $u = 1$, $v = 0$ and pressure field obtained from pressure Poisson equation is computed using a wavy curvilinear time deforming grid. The grid is generated and subsequently deformed using analytic expression [2]

$$\begin{cases} x = \xi + A_\xi \sin(2\pi\varpi\tau) \left(\dfrac{\xi_{\max} - \xi_{\min}}{N_\xi} \right) \sin \left(n_\xi \pi \dfrac{\eta - \eta_{\min}}{\eta_{\max} - \eta_{\min}} \right), & \text{(34a)} \\ y = \eta + A_\eta \sin(2\pi\varpi\tau) \left(\dfrac{\eta_{\max} - \eta_{\min}}{N_\eta} \right) \sin \left(n_\eta \pi \dfrac{\xi - \xi_{\min}}{\xi_{\max} - \xi_{\min}} \right), & \text{(34b)} \end{cases}$$

with $A_\xi = 1 = A_\eta$, $\varpi = 0.25$, $\xi_{\min} = 0 = \eta_{\min}$, $\xi_{\max} = \pi = \eta_{\max}$. Computations are carried out using two different grids 25×25 with $n_\xi = 4 = n_\eta$ and 65×65 with $n_\xi = 6 = n_\eta$ taking $\delta\tau = 0.025$ and 0.0025 respectively upto $T = 2.5$ following [2]. Presence of N_ξ and N_η act as stabilizing skewness of the grid with mesh refinement. Nominal grids at two different times are presented in Fig. 4. For this test coordinate transformation metrics are approximated using two different approaches. In the first case compact fourth order Padé discretizations are used. Additionally, the Jacobian of the transformation J, as well as J_1 and J_2 are arrived at by using GCL identities given by the Eq. (33) to enforce metric cancellation. In the second approach, straightforward fourth order central differencing is employed. The results obtained are presented in Table 3. From the table, it is clear that free stream is preserved for the current formulation even without GCL



(a) (b)

Figure 4: *Problem 2: Nominal curvilinear wavy meshes at time (a) $\tau = 1.0$ and (b) $\tau = 3.0$.*

identities being enforced. This correlates well with our theoretical understanding developed in section 2.3.

3.3. N-S equation with analytic solution

Taylor in 1923 established an exact solution of the incompressible N-S equations [25]. Since then it has been used to test numerical schemes applied to fluid flow problems. In this study, we further adapt this problem for a time-varying domain. We consider the flow decayed by viscosity problem governed by the N-S equations having analytical solutions

$$\begin{cases} u_1(x,y,t) = -\cos(Nx)\sin(Ny)e^{-2N^2t/Re}, & \text{(35a)} \\ u_2(x,y,t) = \sin(Nx)\cos(Ny)e^{-2N^2t/Re}, & \text{(35b)} \\ p(x,y,t) = -\dfrac{1}{4}\left(\cos(2Nx)+\cos(2Ny)\right)e^{-4N^2t/Re}. & \text{(35c)} \end{cases}$$

Here N is a positive integer and controls the numbers of vortices generated in the flow domain. The problem is considered in a time-varying physical domain $\Omega(t)$ that could be readily mapped

to a fixed and regular parametric domain $\mathcal{D} = [0, \pi] \times [0, \pi]$ by setting

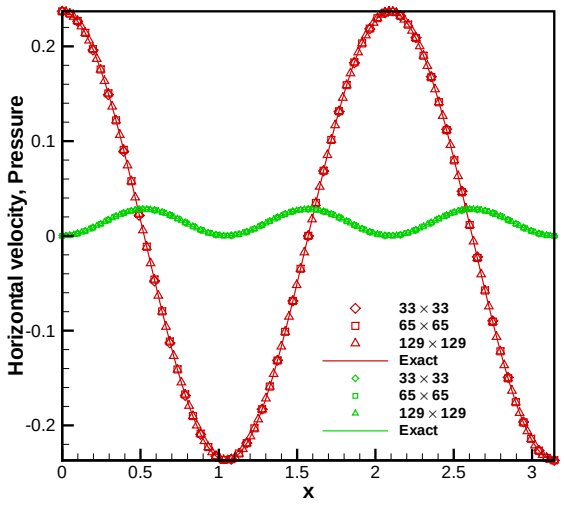
$$\begin{cases} x = \xi + A_\xi \sin(2\pi\varpi\tau) \sin(n_\xi \eta), & \text{(36a)} \\ y = \eta + A_\eta \sin(2\pi\varpi\tau) \sin(n_\eta \xi). & \text{(36b)} \end{cases}$$

The above transformation leads to a computationally somewhat more challenging grid for finer meshes *vis-a-vis* the transformation proposed by Visbal and Gaitonde [2] and used in the previous section. This is due to the invariance of maximum deformation with respect to grid spacing leading to increasingly skewed mesh. Here we set $A_\xi = \pi/64 = A_\eta$, $\varpi = 0.25$ and $n_\xi = 6 = n_\eta$. Further, we take $N = 3$, which leads to the creation of $N^2 = 9$ vortices in the velocity field and $2N^2 = 18$ vortices in the pressure field. Computations are done for $Re = 100$ with three different grids 33×33, 65×65 and 129×129 and the spatial order of accuracy of horizontal velocity component and pressure estimated with L^2-norm is verified in Table 4. Vertical velocity is not presented as it has features similar to horizontal velocity. Order of accuracy around three can be noticed in Table 4 and is found to decrease with a decrease in grid spacing. This might correlate to increased skewness of the refined mesh. We investigate the temporal accuracy of the scheme in conjunction

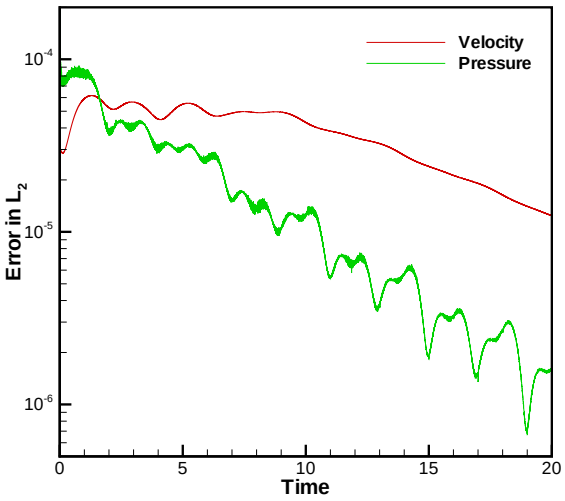
Table 4: *Problem 3: L_2- norm error and spatial order of convergence with $\delta\tau = h^2 = k^2$.*

Time		33×33	Order	65×65	Order	129×129
$\tau = 5.0$	u_1	3.386×10^{-3}	3.15	3.813×10^{-4}	2.80	5.493×10^{-5}
	p	1.712×10^{-3}	3.10	1.990×10^{-4}	2.69	3.091×10^{-5}
$\tau = 8.0$	u_1	3.825×10^{-3}	3.29	3.920×10^{-4}	3.00	4.908×10^{-5}
	p	5.362×10^{-4}	3.04	6.528×10^{-5}	2.26	1.360×10^{-5}
$\tau = 10.0$	u_1	3.431×10^{-3}	3.29	3.486×10^{-4}	3.02	4.306×10^{-5}
	p	6.397×10^{-4}	2.99	8.056×10^{-5}	2.66	1.272×10^{-5}

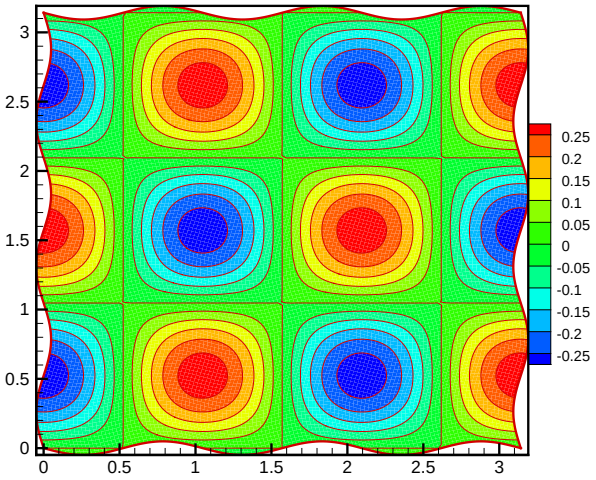
with this prototype N-S equation in Table 5 where computations were carried out using 65×65 grid using three different time steps. Our computation report temporal accuracy between one and two for this nonlinear problem. In Fig. 5(a) grid independence of our computation is shown. That the exact solution is approached with grid refinement can be seen in this figure. Convergence of velocity and pressure field in L_2- norm is depicted in Fig. 5(b). Although temporal convergence of velocity is linear after some time, pressure is seen to follow an oscillatory path. Finally horizontal velocity contours and pressure contours at time $\tau = 7.0$ are presented in Figs. 5(c) and 5(d) respectively. The results indeed correlate well with the exact solution.



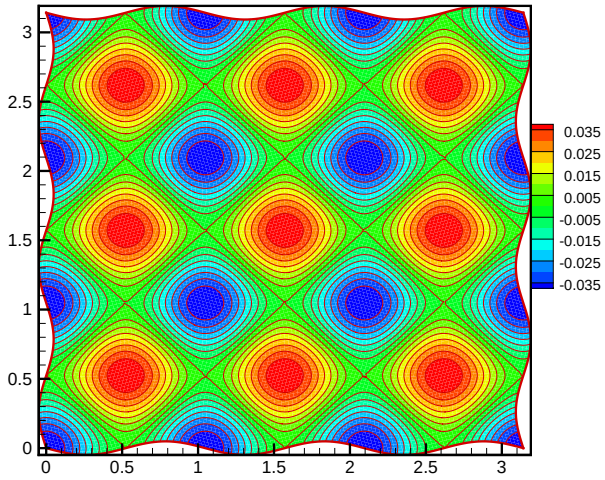
(a)



(b)



(c)



(d)

Figure 5: *Problem 3: (a) Grid independence of* u_1 *and* p *at* $\tau = 8.0$*, (b) convergence of* u_1 *and* p *in* L^2*- norm, (c)* u_1 *contour and (d)* p *contour at* $\tau = 7.0$*.*

Table 5: *Problem 3: L_2- norm error and temporal order of convergence.*

Time		$\delta\tau = 0.04$	Order	$\delta\tau = 0.02$	Order	$\delta\tau = 0.01$
$\tau = 5.0$	u_1	4.684×10^{-3}	1.35	1.842×10^{-3}	1.26	7.670×10^{-4}
	p	1.922×10^{-3}	1.39	7.354×10^{-4}	1.21	3.183×10^{-4}
$\tau = 8.0$	u_1	3.999×10^{-3}	1.36	1.558×10^{-3}	1.30	6.330×10^{-4}
	p	1.120×10^{-3}	1.19	4.921×10^{-4}	1.12	2.257×10^{-4}
$\tau = 10.0$	u_1	3.329×10^{-3}	1.35	1.306×10^{-3}	1.27	5.430×10^{-4}
	p	7.492×10^{-4}	1.17	3.326×10^{-4}	1.02	1.637×10^{-4}

3.4. Driven cavity problem

The 2D driven cavity problem defined on a unit square is often used as the benchmark test case for incompressible flow solvers is considered to demonstrate the accuracy of the proposed discretization on curvilinear time deforming grid. We carry out numerical simulations on both Cartesian uniform and time varying stretched curvilinear grids. Results obtained are essentially indistinguishable and are further compared with benchmark solutions to amply demonstrate the accuracy of the procedure advocated here. Such an indirect approach to validate the accuracy of a discretization strategy could be traced in the works of Wesseling *et al.* [26] and Ge and Sotiropoulos [27] where the curvilinear grids are time independent and in the work of Chen and Xie [28] where the strategy was furthered to a time varying grid.

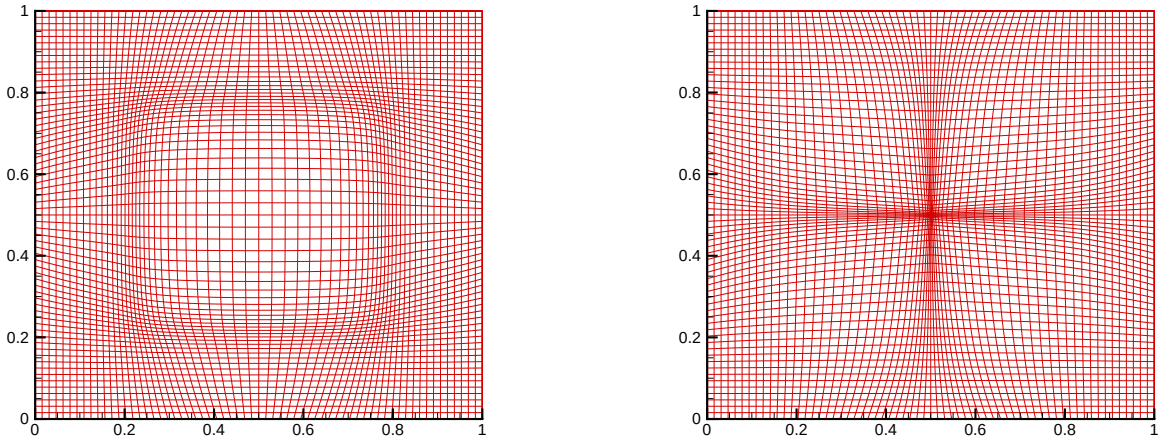
Employing primitive variable form of the Navier-Stokes equation steady-state solution is arrived at for diverse Re. The time deforming curvilinear grid over a rectangular domain $\Omega = [\xi_{min}, \xi_{max}] \times [\eta_{min}, \eta_{max}]$ is constructed combining polynomial stretching with trigonometric temporal variation and is given by

$$\begin{cases} x = \left(\dfrac{\xi - \xi_{min}}{\xi_{max} - \xi_{min}}\right) + A_\xi \left(\dfrac{\xi - \xi_{min}}{\xi_{max} - \xi_{min}}\right)^6 \left(\dfrac{\xi - 0.5(\xi_{min} + \xi_{max})}{\xi_{max} - \xi_{min}}\right) \left(\dfrac{\xi_{max} - \xi}{\xi_{max} - \xi_{min}}\right)^6 \\ \qquad \left(\dfrac{\eta - \eta_{min}}{\eta_{max} - \eta_{min}}\right) \left(\dfrac{\eta_{max} - \eta}{\eta_{max} - \eta_{min}}\right) \sin(2\pi\tau), \qquad (37a) \\ y = \left(\dfrac{\eta - \eta_{min}}{\eta_{max} - \eta_{min}}\right) + A_\eta \left(\dfrac{\eta - \eta_{min}}{\eta_{max} - \eta_{min}}\right)^6 \left(\dfrac{\eta - 0.5(\eta_{min} + \eta_{max})}{\eta_{max} - \eta_{min}}\right) \left(\dfrac{\eta_{max} - \eta}{\eta_{max} - \eta_{min}}\right)^6 \\ \qquad \left(\dfrac{\xi - \xi_{min}}{\xi_{max} - \xi_{min}}\right) \left(\dfrac{\xi_{max} - \xi}{\xi_{max} - \xi_{min}}\right) \sin(2\pi\tau). \qquad (37b) \end{cases}$$

In our computations unit square is contemplated taking $\xi_{min} = 0 = \eta_{min}$ and $\xi_{max} = 1 = \eta_{max}$

with $A_\xi = 15000 = A_\eta$. Evolution of a 65×65 grid at time 0.25 and 0.75 corresponding to extreme stretching and clustering are presented in Figs. 6(a) and 6(b) respectively. As the grid deform continuously with a unit period, recurrence of such extremal twist along with loss of grid orthogonality could be noticed in the interior of the cavity at each nondimensional time.

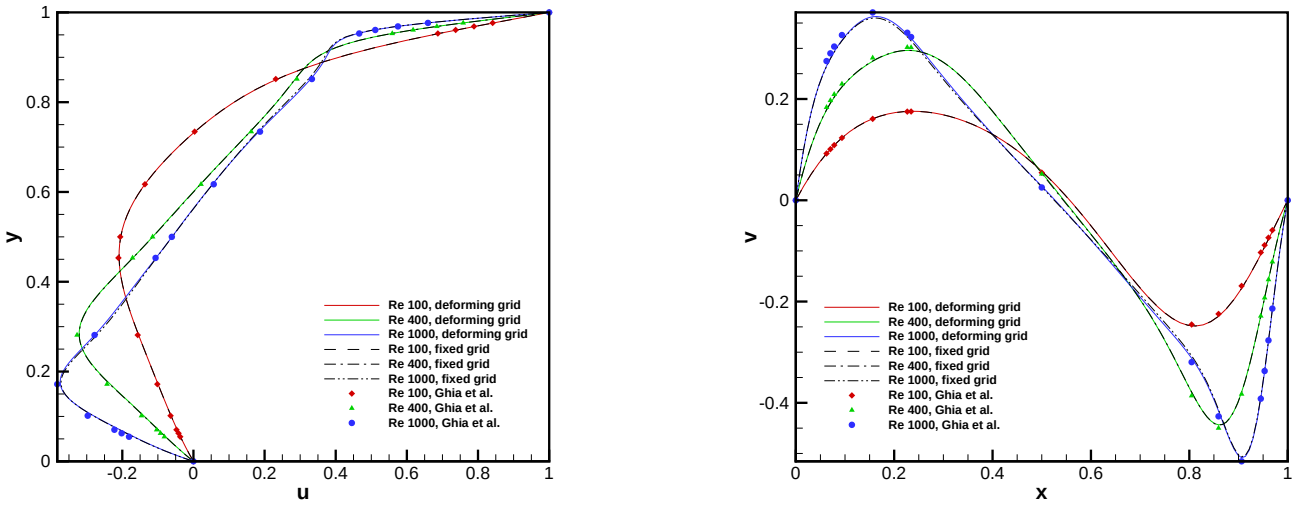
We carried out computations on static uniform Cartesian mesh apart from the continually distorting curvilinear grid for $Re = 100$, 400 and 1000. Time marched steady state solutions are compared to test the ability of the scheme to yield accurate solutions as the underlying grid alternate. Figs. 7(a) and 7(b) illustrate the horizontal and vertical velocity profile along the vertical and horizontal center lines respectively. The velocity field obtained on these two different grid systems are indistinguishable from each other. Further, the profiles compare well to the benchmark result of Ghia *et al.* [29]. What is more encouraging, however, is that the equi-pressure contours for $Re = 400$ and 1000 as seen in Figs. 7(c) and 7(d) respectively, computed on deforming and static 129×129 grids are very close to each other.



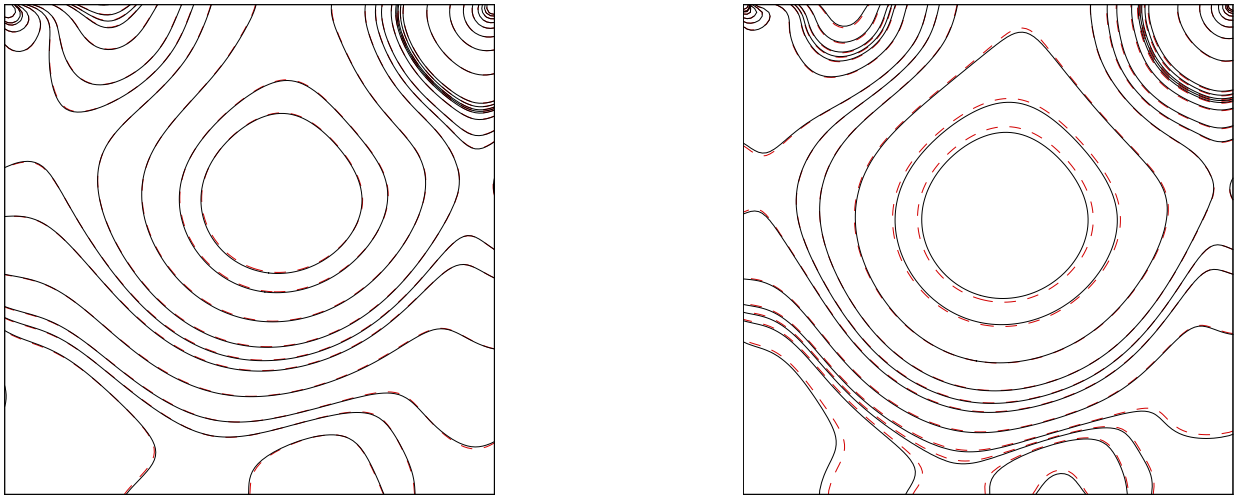
(a) (b)

Figure 6: *Problem 4: Deforming* 65×65 *mesh at time (a)* $\tau = 0.25$ *and (b)* $\tau = 0.75$.

We also present results for $Re = 3200$ and 5000 computed using 257×257 and 321×321 time wavering grids respectively in Fig. 8. All simulations are carried out with $\delta\tau = 0.0025$. Figs. 8(a) and 8(b), which illustrate the velocity profile along the vertical and horizontal center lines for $Re = 3200$ and 5000 respectively, the velocity field obtained on deforming grid again match well with the values provided in [29]. Fig. 8(c) depicts a snapshot of the flow field for $Re = 3200$. Based

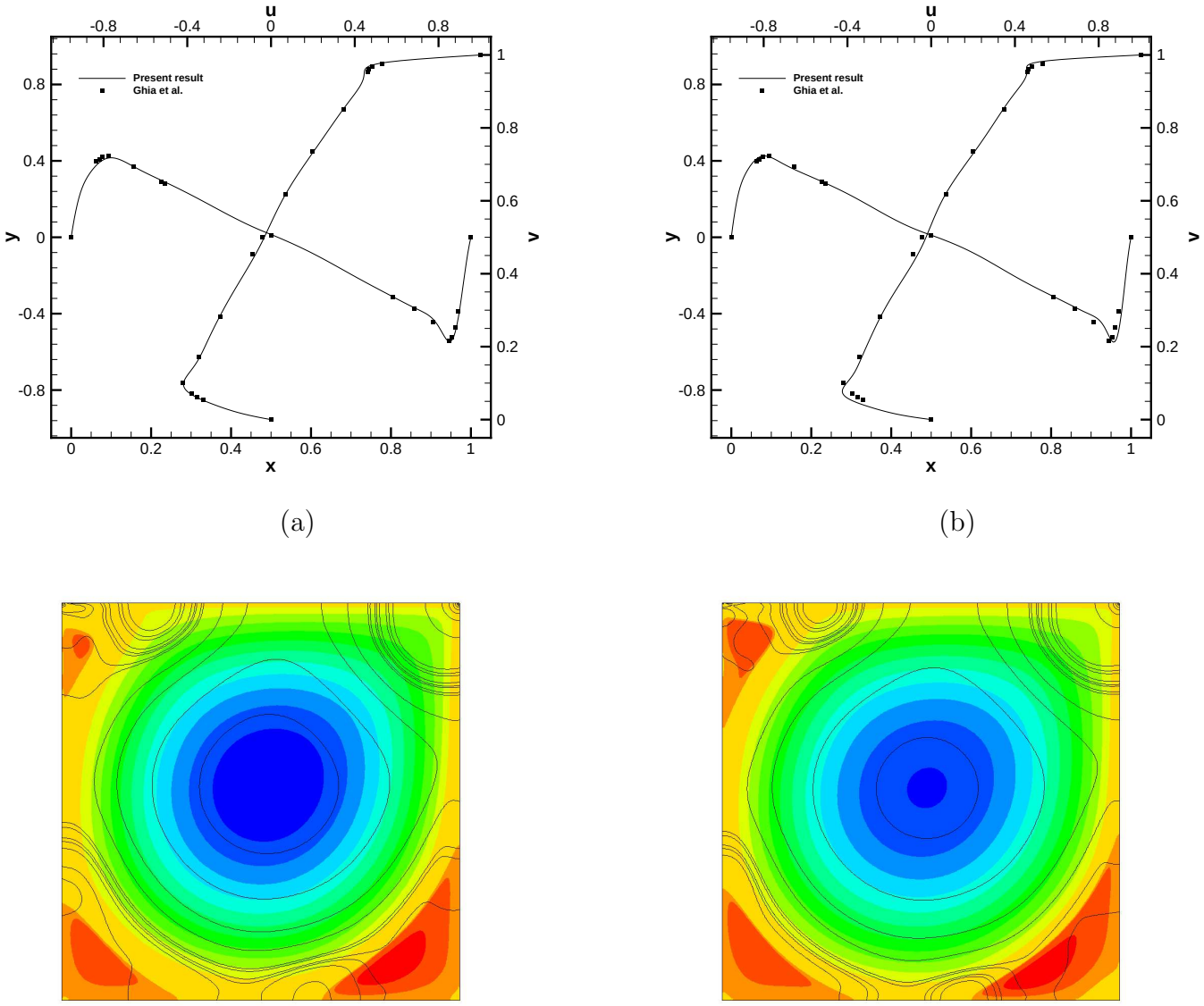


(a) (b)



(c) (d)

Figure 7: *Problem 4: Comparison of velocity profile with benchmark for (a) u-component along vertical centreline, (b) v-component along horizontal centreline. Pressure contours computed with deforming grid (solid line) and static grid (dashed line) for (c) $Re = 400$, (d) $Re = 1000$.*



(a) (b)

(c) (d)

Figure 8: *Problem 4: Profiles of horizontal and vertical velocities along vertical and horizontal lines respectively through geometric center of cavity for (a) $Re = 3200$, (b) $Re = 5000$. Plots of pressure contour and post processed streamlines for (c) $Re = 3200$, (d) $Re = 5000$.*

on the contour plots of the pressure and the streamlines, the separated flow regions are correctly predicted and compares well with the results available in the literature. Similarly, in Fig. 8(d) computed flow field and pressure contours at steady state for $Re = 5000$ are correctly captured.

3.5. Deforming cavity problem

We next simulate incompressible flow inside a closed but deforming boundary. This specially designed case is used to investigate our approach on the flows with the deformation of the boundary. The difference between this case and the previous lid-driven cavity problem is that the lower wall of the cavity is deforming and the domain is meshed using the prescribed function. Here we adopt the work of Chen and Xie [28] and stipulate contortion of the lower boundary of the cavity as

$$y(x,t) = A\left(e^{a(x-c_1)^2} - e^{a(x-c_2)^2}\right)\sin(2\pi f t), \tag{38}$$

where following [28] we take amplitude and frequency of wall deformation as $A = 0.1$ and $f = 0.2$ respectively. Further, positions of maximum deformations are taken symmetrically at $c_1 = 0.375$ and $c_2 = 0.625$ with constant for range of deformation $a = 60$. The curvilinear coordinates and the geometrical configuration of the physical domain at nondimensional time $T/4$ and $3T/4$ where T is the period of domain alteration is presented in Figs. 9(a) and 9(b) respectively.

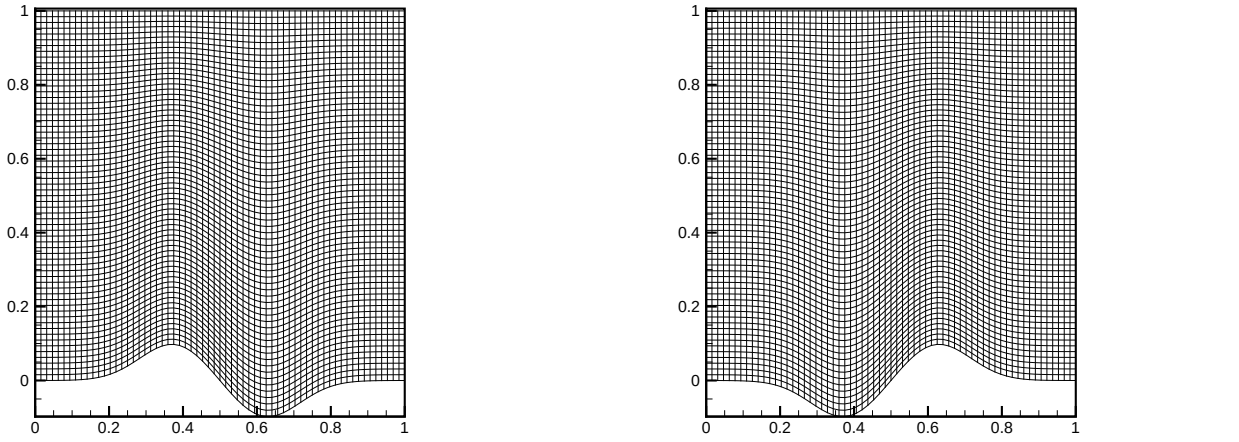
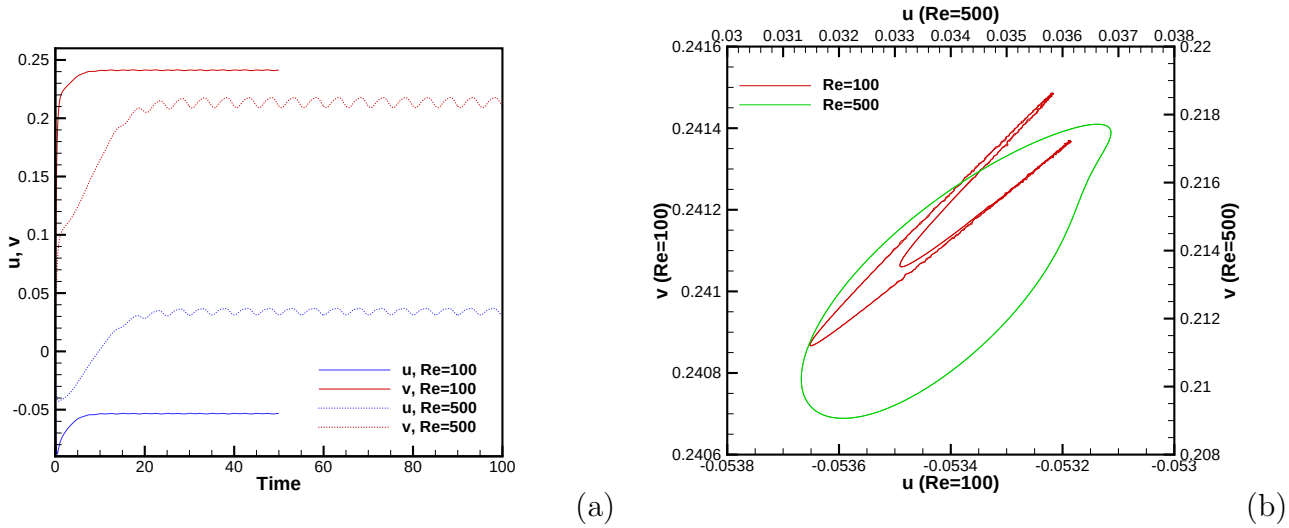


Figure 9: *Problem 5: Configuration of the deformable cavity and* 65×65 *mesh at time (a)* $T/4$ *and (b)* $3T/4$.



(a) (b)

Figure 10: *Problem 5: (a) Time evolution of velocity components and (b) phase portrait at monitoring point* $(2/16, 13/16)$.

In the parametric domain, the flow field is computed for $Re = 100$ and 500 on 65×65 and 129×129 grids respectively with $\delta\tau = 0.0025$. For both these Re values steady-state is reached for lid-driven cavity problem in square cavity discussed in the previous section. But in this section with the domain constantly deforming following a cyclical function in time it is impossible to arrive at steady motion [28]. We thus inquire about the periodic solution and monitor the time evolution of velocity at a chosen point $(2/16, 13/16)$. The choice of monitoring point away from the deforming lower boundary is partly influenced by the work of Bruneau and Saad [30] which was used to analyze the periodic solution for classical driven cavity problem. Time development of horizontal and vertical velocity components at the monitoring point is presented in Fig. 10(a). It is seen that the flow away from the lower boundary after starting from rest settles to periodic variation. This is emphasized for $Re = 500$ whereas for $Re = 100$ variations are much subdued and is captured by phase portrait in Fig. 10(b).

Fig. 11 shows the instantaneous velocity and pressure field for $Re = 500$ at different time phases within one period. From the snapshots, it is clear that although overall periodic, in the region far from the deforming surface the flow is similar to the steady case perceptible in classical cavity problem. From the plot of velocity components, it is amply clear that with changing bottom topography of the cavity secondary vortices on each side of the domain undergoes continuous change in shape and size. Nevertheless, it can be inferred that these vortices remain stable and

attached to the lower boundary. The periodic appearance of a third secondary vortex is also noticed from Figs. 11(j) and 11(k).

3.6. Oscillating cylinder in cross flow

In this example, numerical simulation of the flow past a circular cylinder that is able to oscillate transversely in a uniform background flow is presented. In a seminal work, Williamson and Roshko in 1988 [31] documented the influence of bluff body oscillation on vortex shedding. They experimentally established several distinct wake structures which they classified in terms of the number of vortices shed per oscillation cycle. Afterward, this involved fluid–structure interaction problem has drawn much attention and is considered a typical test case for numerical techniques. In the present study main motivation is to validate some important phenomena hitherto detailed in the literature. In this process, one of the crucial steps is to continually and adequately transform the body fitted grid. As discussed earlier recent advances in IDW interpolation help us generate a good quality time deforming grid. The process starts with an initial grid generated by encompassing the cylinder of unit diameter at origin details of which could be found in the author's earlier work [22]. As shown in schematic diagram Fig. 12 the flow domain consists of two separate blocks joined at the abutting boundary DA. Such a multi-block grid helps us not only to cluster an adequate number of mesh points in the vicinity of the cylinder surface but also to capture flow wake up to a reasonable distance while being economical with the distribution of nodal points. With the computational domain held fixed, adequate capture of transverse oscillation of the cylinder makes it essential to acceptably adjust the grid at each time step. Since the motion of the cylinder is in the transverse direction we resist calculating rotation quaternions and spontaneously engage IDW interpolation methodology. Two representative grids with cylinder displaced 1.2 times its diameter from the mean position at the origin are depicted in Fig. 13. Here every fifth grid line is shown. For the generated meshes skew metric is an important characteristic [32] and in the current context ensures the quality of grid deformation. Having implemented IDW interpolation we are able to maintain skew value close to unity in large regions of the domain and the same deviate marginally only at a few pockets as seen in Fig. 13.

Having discussed grid generation we turn our attention to briefly describe problem formulation and associated boundary conditions. For this test case $\psi - \omega$ formulation is tabbed and the flow commences impulsively. Here a rigid circular cylinder of diameter D is assumed to be elastically mounted. The cylinder is restrained to executes harmonic motion with amplitude A_c and frequency f_c transverse to the incoming flow having free-stream velocity U. Computations are done for $Re = 200$ and 392 where $Re = UD/\nu$, ν being kinematic viscosity. These choices are influenced by the availability of results in the literature.

It is well known that the wake region of the flow past a stationary circular cylinder exhibit second Hopf bifurcation at some critical Re value (Re_{cr}). In the literature, this threshold value is scattered in the range $43 \leq Re_{cr} \leq 50$ details of which could be found elsewhere [33]. Flows above

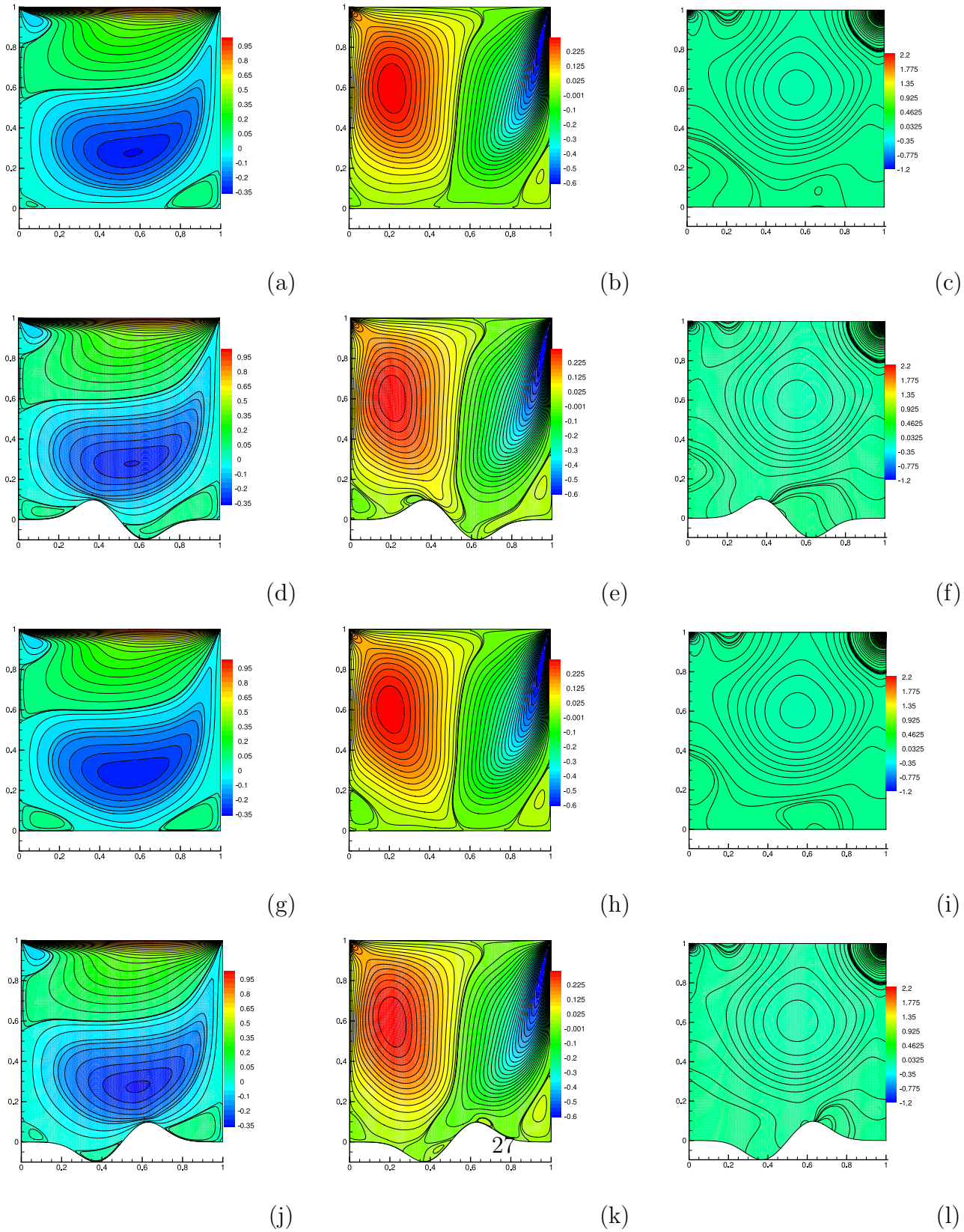


Figure 11: *Problem 5: Evolution of the horizontal velocity (left), vertical velocity (middle) and pressure (right) during one period for* $Re = 500$. *From top to bottom rows correspond to times* 0, $T/4$, $2T/4$ *and* $3T/4$.

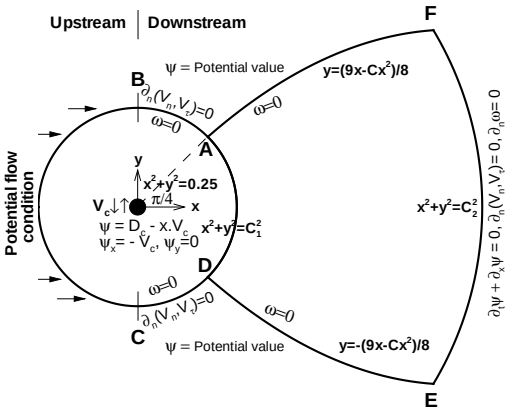


Figure 12: *Problem 6: Schematic diagram exhibiting domain and boundary conditions for cylinder oscillating in cross-flow.*

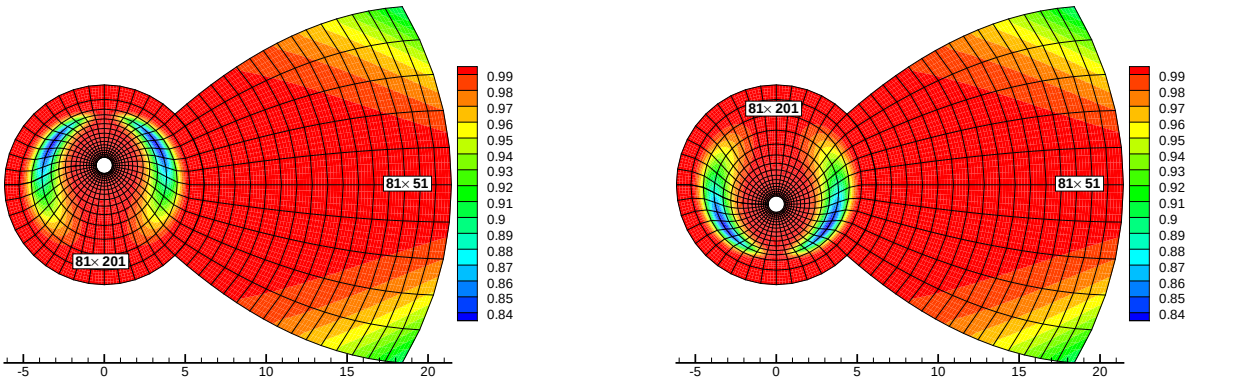


Figure 13: *Problem 6: Grid along with skew metrics for cylinder positioned (a) up and (b) down 1.2 times diameter from the mean position. Every fifth grid line is shown.*

Re_{cr} are characterized by von Kármán vortex street and exhibits certain periodicities identified by Strouhal number (St) and defined as $St = Df_n/U$ where f_n is the frequency of periodic variation of lift exerted on a stationary cylinder. Following Williamson and Roshko [31] we take $St = 0.200$.

In our computations, with cylinder performing forced oscillations, excitation amplitude and frequency are characterized by adimensional amplitude $A_r = A_c/D$ and frequency ratio $f_r = f_c/f_n$. Further oscillatory velocity transverse to the flow is imposed as $V_c = 2\pi f_c A_c \cos(2\pi f_c \tau)$. With $(u, v) = (\psi_y, -\psi_x) = (0, V_c)$ it is easy to see that on the cylinder surface $\psi = D_c - xV_c$ where D_c is the time varying transverse displacement of the cylinder. Again with vorticities produced at no slip boundaries correct and accurate imposition of vorticity boundary conditions is of critical importance. In our previous work [23] we have advocated the global integral vorticity condition which is primitive. This procedure was successfully realized as varied explicit boundary vorticity conditions. Further, we were also able to establish a connection between vorticity integral philosophy and a few well known vorticity boundary conditions. Insight acquired lead us to use Briley's vorticity boundary approximation [34] on the surface of the cylinder as an optimum option for this test case.

Referred to Fig. 12 we impose uniform flow conditions in the upstream leading to $\psi = Uy$ and $\omega = 0$. Similar expressions are also imposed on the boundaries **BA**, **AF**, **CD**, and **DE**. Here it is important to mention that although $(u = U, v = 0)$ is imposed on boundaries **BC**, **DE**, and **AF** we work with zero Neumann condition $\partial_n(V_n, V_\tau) = (0, 0)$ for normal (V_n) and tangential (V_τ) velocity components on **BA** and **CD** as discussed in earlier work [33]. Along the downstream boundary **EF** we impose convective boundary condition for ψ and Neumann conditions for ω, V_n, and V_τ as shown in the schematic diagram Fig. 12. Approximations for gradients of ω at all boundary arcs are obtained by using one sided approximations following the work of Tian *et al.* [35] except at **EF** where the normal gradient of ω is set to zero.

The drag force (F_D) often normalized by the dynamic free stream pressure $\frac{1}{2}\rho U^2$ referred to as the drag coefficient (C_D) might be represented as

$$C_D = \frac{F_D}{\frac{1}{2}\rho U^2 D}. \tag{39}$$

Following Fornberg [38], F_D evaluated by a line integral around the body,

$$C_D = \frac{1}{Re}\int_{-1}^{1} (\omega_\xi - \pi\omega)\sin(\pi\eta)d\eta. \tag{40}$$

In a similar fashion lift coefficient (C_L) could be written as

$$C_L = \frac{1}{Re}\int_{-1}^{1} (\omega_\xi - \pi\omega)\cos(\pi\eta)d\eta. \tag{41}$$

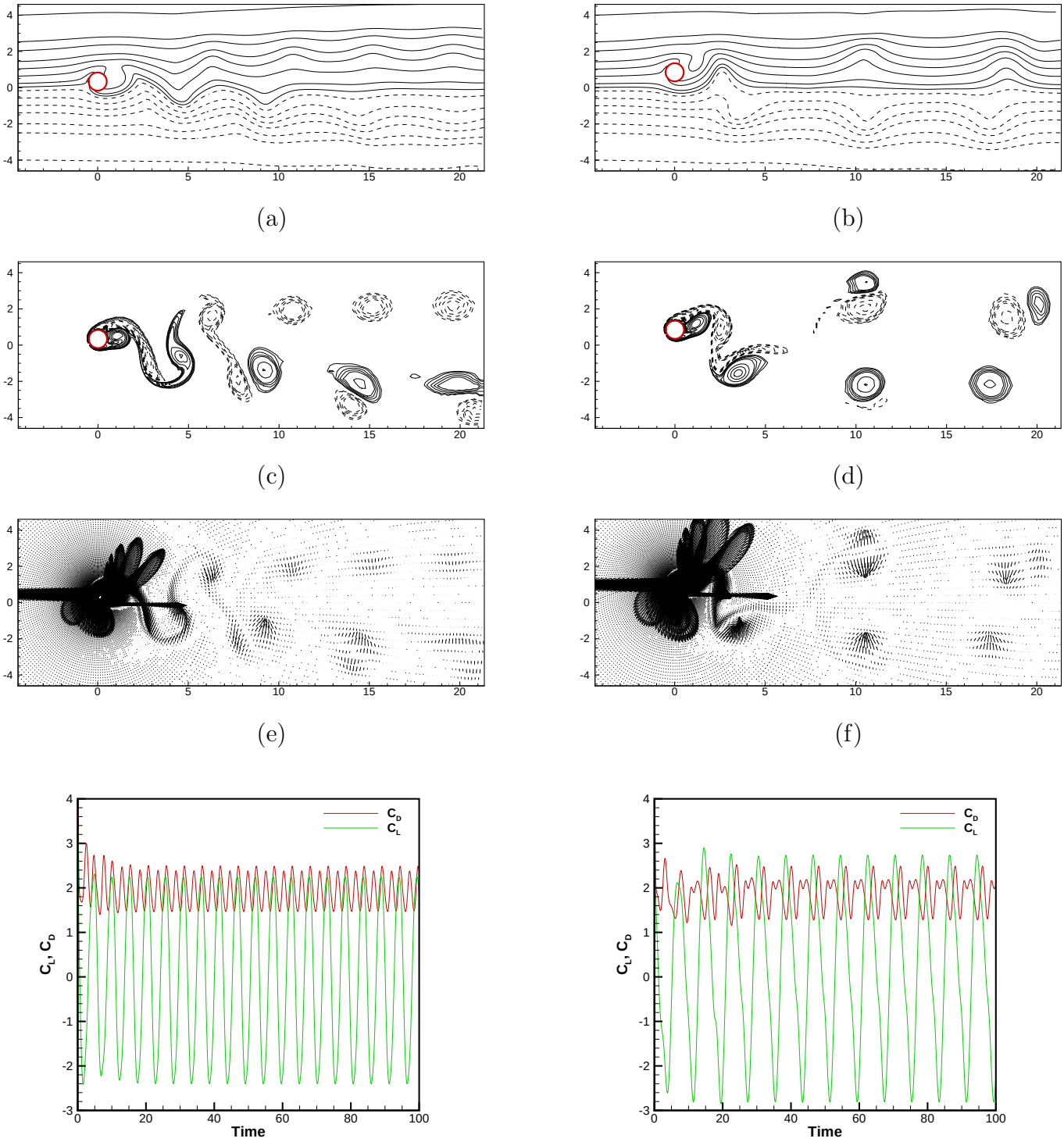


Figure 14: *Problem 6: Streamlines (top row), Vorticity contours (second row), Lamb vector plot (third row) and Variation of drag and lift (bottom row) for* $Re = 200$ *at (i)* $A_r = 0.7$, $f_r = 0.95$ *(left) and (ii)* $A_r = 1.2$, $f_r = 0.625$ *(right). Negative contours appear in dashed lines.*

Four distinct cases with varied flow parameters are studied here. In the first two cases, we work with $Re = 200$ and choose (A_r, f_r) as $(0.7, 0.95)$ and $(1.2, 0.625)$ respectively. Both these cases are partially motivated by the availability of data in the literature [36] and the pressing nature of the specifications. At $A_r = 0.7$ we have a comparatively lesser displacement of the cylinder from its mean position at origin but with $f_r = 0.95$ we are close to the region of primary synchronization. At such a choice $P + S$ wake mode is reported by Leontini *et al.* [36] in their extensive 2D simulations of an oscillating circular cylinder. From Fig. 14(c) it is clear that even with a modest computational setup our scheme is able to accurately capture $P + S$ mode consisting of a pair and single vortex per oscillation cycle. The corresponding streamlines are shown in Fig. 14(a). Lamb vector $\boldsymbol{\omega} \times \boldsymbol{u}$ is analogous to the Coriolis force observed in a rotating frame of reference. It is of crucial importance in vorticity dynamics and drives the particles to move around the vorticity direction [37]. The Lamb vector plot of the flow field presented in Fig. 14(c) identifies asymmetric Karman vortex shedding where the cylinder sheds a paired counter rotating and a single vortex in each cycle. Variation of drag and lift coefficients are presented in Fig 14(g). It is seen that after impulsive start the flow settles down to a periodic state at around nondimensional time 30. In the second case with $(A_r, f_r) = (1.2, 0.625)$ the amplitude of oscillation is increased and the cylinder oscillates away from primary synchronization. According to Leontini *et al.* [36], this choice of parameters leads to a region well over the boundary defining the transition from the $2S$ to $P + S$ wake modes for $Re = 200$. Our computations corroborate the same as seen in Fig. 14(d). This is further accentuated in the Lamb vector plot in Fig. 14(f). Fig. 14(h) demonstrates that with increased amplitude of oscillation variation of lift is more pronounced. With cylinder oscillating further away from primary synchronization variation of drag attains three points of maxima instead of the customary two during each time period.

In the second set of studies we work with $Re = 392$ and frequency ratios are chosen close to unity (0.971 and 1.075). Amplitude ratios are fixed at 0.4 and 0.5 for the first and second cases respectively. These instances are partly motivated by the experimental work of Williamson and Roshko [31] where authors have documented $2S$ mode of vortex synchronization. Our computation captures periodic flow with $2S$ vortex shedding mode as seen in Figs. 15(a), 15(c) and 15(e). In the fundamental lock-in region, the flow indeed settles to a periodic state at a nondimensional time close to 40. But with a frequency ratio slightly above unity, the flow requires much larger time and gradually attains the periodic state at nondimensional time 100 as seen for the second case in Fig. 15(h). Note that lift force also increases with a higher amplitude ratio at 0.5. It is interesting to note that our computation report change in the instantaneous wake structure in the distance downstream. The shed vortices are found to organize themselves into two rows instead of one and are amply clear from the Lamb vector plot in Fig. 15(f). Such a phenomena is reported in the computational study of Leontini *et al.* [36] as well for $Re = 200$ at $f_r = 1.01$, $A_r = 0.5$. Thus it could be inferred our newly developed method is able to capture delicate flow phenomena akin to the oscillating cylinder.

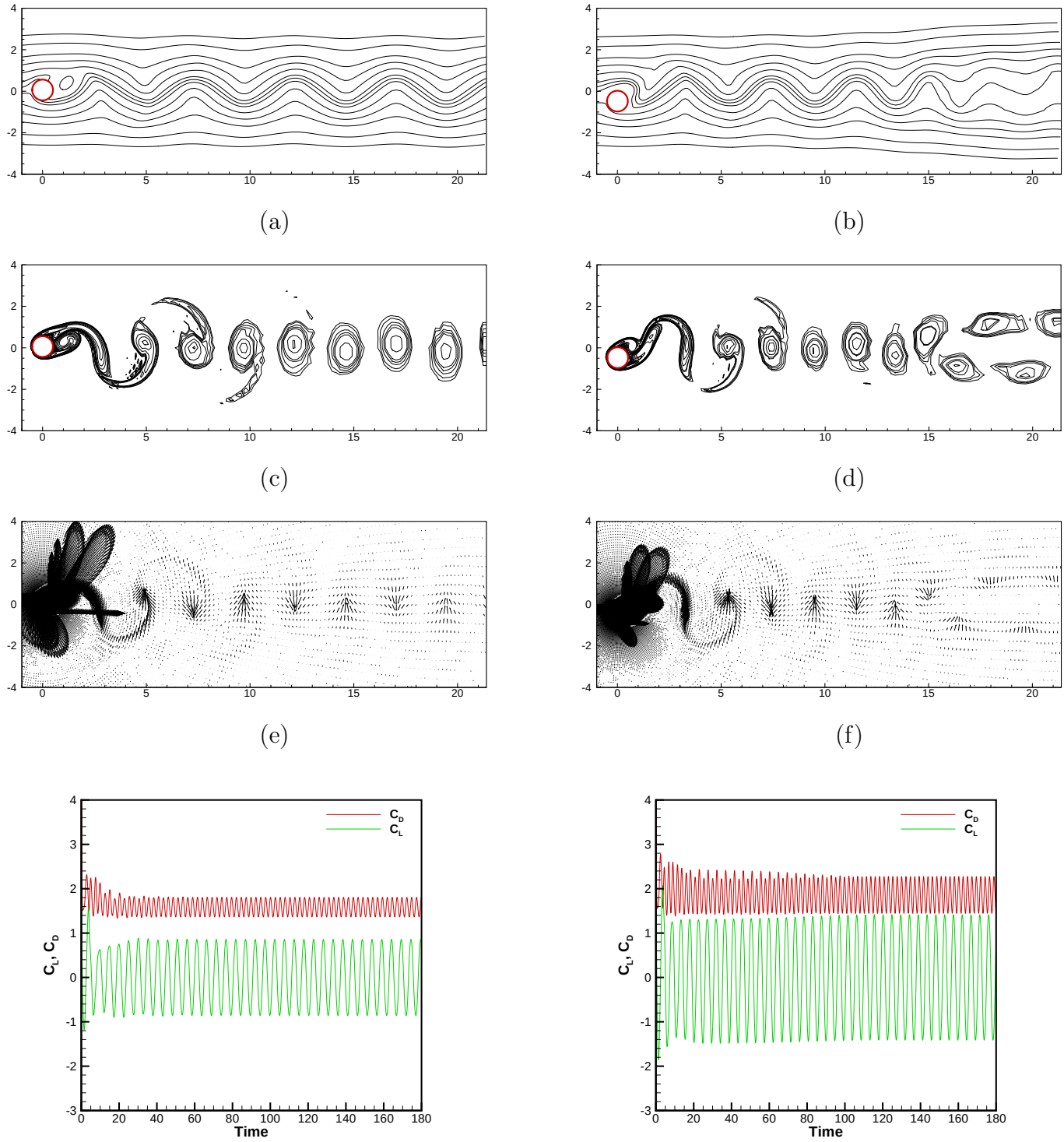


Figure 15: *Problem 6: Streamlines (top row), Vorticity contours (second row), Lamb vector plot (third row) and Variation of drag and lift (bottom row) for* $Re = 392$ *at (i)* $A_r = 0.4$, $f_r = 0.971$ *(left) and (ii)* $A_r = 0.5$, $f_r = 1.075$ *(right).*

3.7. Pitching and heaving airfoil

In this section, we study two airfoils of chord length c pitching and heaving simultaneously subjected to a unit free-stream velocity. The first airfoil is elliptic with a minor to major axes ratio $1:5$ having a pitch centre located at its geometric centre whereas the second airfoil is NACA0012 with a pitch centre at $c/3$ from the leading edge on the axis of symmetry. Following Liang *et al.* [5] for both the airfoils the time dependent rotation motion (pitch) $\alpha(t)$ is prescribed as

$$\alpha(t) = \alpha_{\max} \sin(2\pi f_p t + \pi/2) \tag{42}$$

and the translation motion (heave) is given by

$$h(t) = h_{\max} \sin(2\pi f_h t). \tag{43}$$

Here we take $\alpha_{\max} = \pm\pi/6$, $h_{\max} = 0.25$ and $f_p = f_h = 0.4$. Four typical positions of NACA0012 during one complete period are shown Fig. 16(a). A similar motion could be traced for elliptic airfoil as well.

A schematic diagram depicting the position of NACA0012 airfoil at a given time along with prescribed velocity boundary conditions is shown in Fig. 16(b). For correct flow simulation, it is imperative to prescribe accurate conditions on the surface of the airfoil. In this context, we shall like to provide a sketch of steps required to arrive at streamfunction values on the surface of the airfoil. The motion of any point P on the surface of the airfoil can be described as a result of the motion of the pitch centre C and the motion relative to C. From Eq. (43) horizontal and vertical velocity components at C are 0 and $\dot{h}$ respectively whereas radial and cross-radial components of velocity at P relative to C are $\dot{r}_p = 0$ and $r_p\dot{\theta}_p = r_p\dot{\alpha}$ respectively. Thus horizontal and vertical velocity components relative to C of the point P are $r_p\dot{\alpha}\sin(\theta_p) = \dot{\alpha}(Y_p - h)$ and $r_p\dot{\alpha}\cos(\theta_p) = \dot{\alpha}X_p$. Thus resultant velocity at P is $u = \dot{\alpha}(Y_p - h)$, $v = \dot{h} + \dot{\alpha}X_p$. In the transformed (ξ, η) coordinate system using

$$\psi_\xi = (y_\xi u - x_\xi v), \quad \psi_\eta = (y_\eta u - x_\eta v)$$

it is easy to see that

$$\psi = \frac{\dot{\alpha}}{2}(Y_p^2 - X_p^2) - \dot{\alpha}hY_p - \dot{h}X_p. \tag{44}$$

Simulations precede by generating an O-type mesh of radius $\approx 12c$ encircling the airfoil. Undeformed mesh is generated following the conformal transformation

$$z = \lambda + c_1\frac{a}{\lambda} + c_2\frac{a^2}{\lambda^2} + z_0, \quad \lambda = ae^{i\omega}, \quad \omega = \xi + i\eta, \quad a = \frac{1}{2} - c_1 \tag{45}$$

with $0 \le \xi \le 1.2\pi$, $-\pi \le \eta \le \pi$ as advocated by Sen *et al.* [39] and presented in Fig. 16(c). For elliptic airfoil we take $c_1 = 0.2$ and $c_2 = 0.0$ whereas for NACA0012 the corresponding values

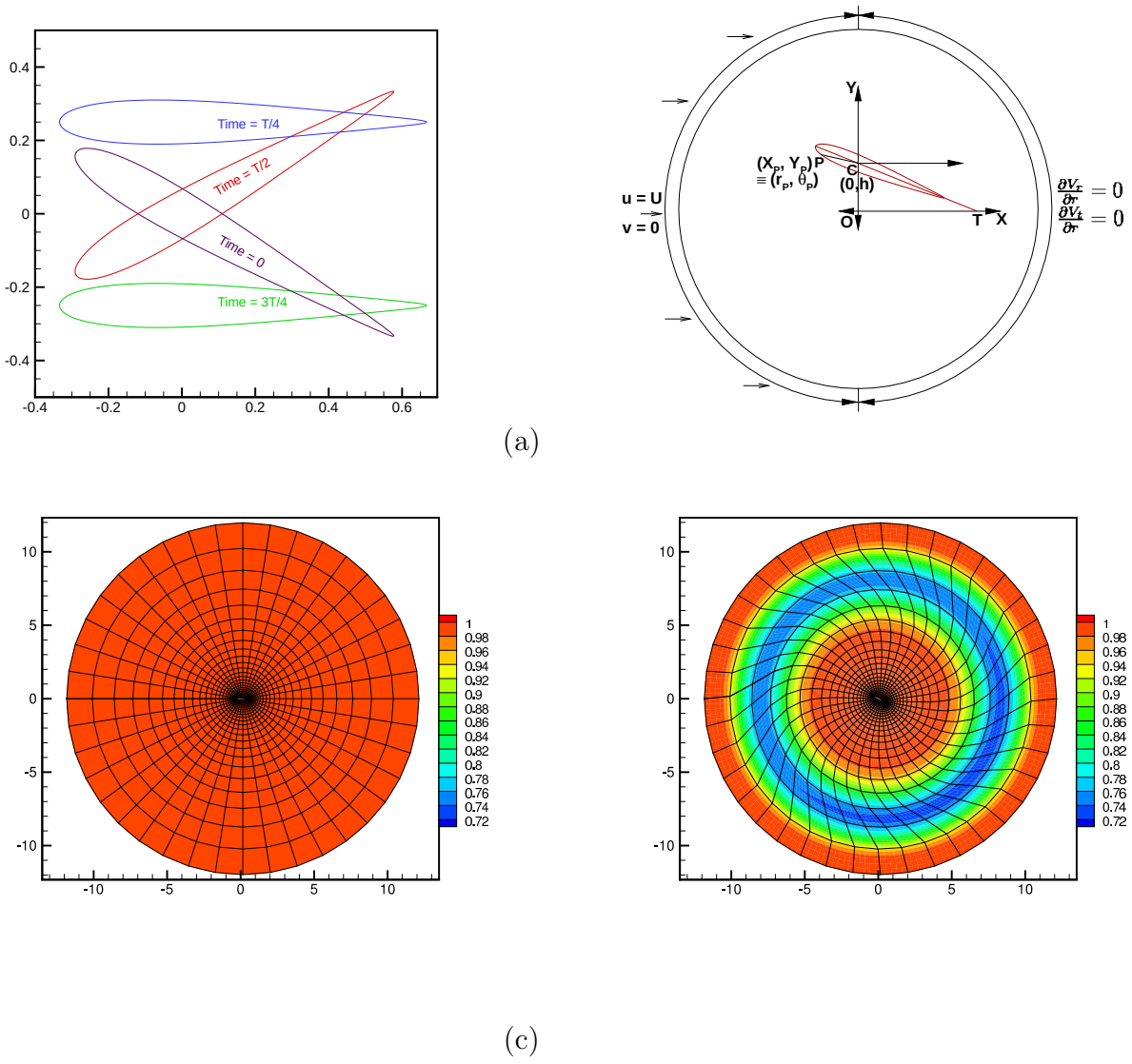


Figure 16: *Problem 7: (a) Four typical positions of NACA0012 during a complete period. (b) Schematic diagram. (c) Reference grid. (d) Deformed mesh at maximum pitching angle. In (c) and (d) every fifth grid line and skew quality metric are shown around NACA0012.*

are 0.2238 and 0.0165. z_0 is appropriately chosen to align the pitch centre with the origin. In this initial mesh, the axis of the airfoil lies along the x-axis. Subsequently, mesh adaptation is carried out using IDW remeshing technique. In this context, up to 10% grid in the immediate vicinity of the airfoil is stipulated to move analogous to the airfoil. Thus conformal nature of the deformed grid is retained adjacent to the airfoil. As this problem involve the pitching of airfoils it is imperative to determine rotation quaternions and interpolate as mentioned earlier. A hybrid version of this strategy is recently used with great success by Apostolatos *et al.* [19] and De Nayer *et al.* [20]. An efficient realization in the current study might be inferred from Fig. 16(d) where a deformed grid corresponding to maximum pitching angle along with associated skew quality metric defined in Knupp [32] is presented. It is heartening to note that restricted skewness is lost in a limited region of the computational domain. For both cases, computations are carried out using 121×201 grid with $\delta\tau = 0.0025$ for chord length based Reynolds number of 200. Choice of Re is motivated by our desire to compute within the laminar flow regime.

Having obtained boundary conditions for streamfunction on the surface of the airfoil we shall like to document the use of potential flow conditions in the upstream as shown in Fig. 16(b). We impose $\psi = Uy$, $\eta \in \{-\pi, -\pi/4\} \cup \{\pi/4, \pi\}$ and convective boundary condition elsewhere. Vorticity on the surface of the airfoil is computed using the recently developed philosophy of Sen and Sheu [23]. Specifically, we employ a cell centre approach to write

$$\omega_{0,j} = -\omega_{1,j} + \frac{2}{J_{0,j} + J_{1,j}} \left(\frac{2}{h}(\psi_{\xi_{0,j}} - \psi_{\xi_{1,j}}) + \frac{1}{2k}(\psi_{\eta_{0,j-1}} + \psi_{\eta_{1,j-1}} - \psi_{\eta_{0,j+1}} - \psi_{\eta_{1,j+1}}) \right) \quad \forall\, j. \tag{46}$$

All remaining vorticity conditions are arrived at using identical philosophy as discussed in the previous section.

Time variations of C_D and C_L are presented in Fig. 17. For the flow past blunt bodies, these coefficients at the surface of the body are two important parameters. The time evolution of these two characteristic parameters illustrates the variation of the flow field. Hence we present below details of their calculation.

From Eq. (45) it is clear that in the undeformed position parametric equation of airfoil is

$$\begin{cases} x = -c_2 + (a + c_1)\cos(\eta) + c_2 \cos(2\eta) + x_0, & \text{(47a)} \\ y = (a - c_1)\sin(\eta) - c_2 \sin(2\eta) + y_0, & \text{(47b)} \end{cases}$$

with $-\pi \leq \eta \leq \pi$. As the airfoil execute pitching and heaving motion its transformed position at any subsequent time τ concerning the absolute frame of reference might be expressed as

$$\boldsymbol{X} = \mathcal{A}\boldsymbol{x} + \boldsymbol{H} \tag{48}$$

where $\boldsymbol{X} = (X, Y)^T$, $\boldsymbol{x} = (x, y)^T$, $\boldsymbol{H} = (0, h(\tau))^T$ and $\mathcal{A} = \begin{bmatrix} \cos\alpha(\tau) & -\sin\alpha(\tau) \\ \sin\alpha(\tau) & \cos\alpha(\tau) \end{bmatrix}$. The unit

normal vector to the airfoil at time τ is thus given by

$$\boldsymbol{n}(\eta,\tau) = (l,m)^T = \left(\frac{dY}{d\eta}, -\frac{dX}{d\eta}\right)^T \Bigg/ \sqrt{\left(\frac{dX}{d\eta}\right)^2 + \left(\frac{dY}{d\eta}\right)^2}. \tag{49}$$

Following Eldredge [40] the aerodynamics force on the airfoil is

$$\boldsymbol{F} = \mu \oint_{\mathcal{C}} \left((\boldsymbol{X} - \boldsymbol{X}_c) \times \frac{\partial \boldsymbol{\omega}}{\partial \boldsymbol{n}} - \boldsymbol{n} \times \boldsymbol{\omega} \right) ds + \rho \bar{A} \dot{\boldsymbol{H}} \tag{50}$$

where $\mathcal{C}$ is any contour surrounding the airfoil and s is measured around the contour $\mathcal{C}$ in the counterclockwise sense. Here μ is dynamic viscosity, ρ density, $\bar{A}$ surface area of an airfoil, $\boldsymbol{X}_c$ co-ordinate of the centre of rotation and $\dot{\boldsymbol{H}}$ is the translation velocity of the airfoil. As noted by Eldredge [40] the term outside integration is the inertial term which along with the first term of the integrand account for the pressure effect. On the other hand, the second integrand captures the viscous shear stress effect. We thus arrive at dimensionless components

$$\begin{aligned}
\frac{\boldsymbol{F}}{\rho U^2 c} &= \frac{1}{Re} \oint_{\mathcal{C}} \left((\boldsymbol{X} - \boldsymbol{X}_c) \times \frac{\partial \boldsymbol{\omega}}{\partial \boldsymbol{n}} - \boldsymbol{n} \times \boldsymbol{\omega} \right) ds + \frac{\bar{A}}{U^2 c} \dot{\boldsymbol{H}} \\
&= \left(\frac{1}{Re} \oint_{\mathcal{C}} \left((Y - Y_c) \frac{\partial \omega}{\partial \boldsymbol{n}} - m\omega \right) ds, \frac{1}{Re} \oint_{\mathcal{C}} \left(-(X - X_c) \frac{\partial \omega}{\partial \boldsymbol{n}} + l\omega \right) ds + \frac{\bar{A}}{U^2 c} \dot{h} \right) \\
&= (F_{X,a}, F_{Y,a}).
\end{aligned} \tag{51}$$

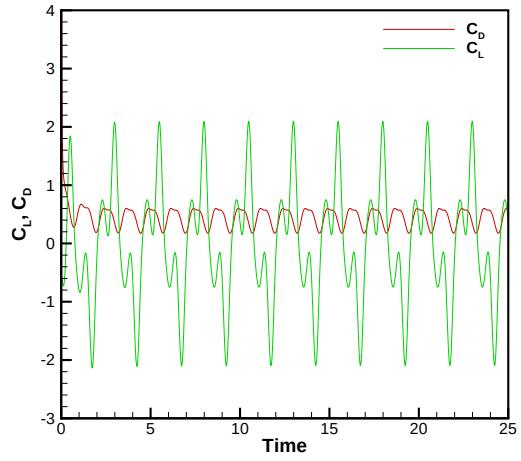
$F_{X,a}$ and $F_{Y,a}$ are dimensionless aerodynamic force components along co-ordinates axes in an absolute frame of reference. The drag and lift coefficients C_D and C_L respectively are calculated using the forces determined in the moving frame of reference [41, 42] and are thus given by

$$\begin{aligned}
(C_D, C_L)^T &= \mathcal{A}^T (F_{X,a}, F_{Y,a})^T \\
&= (F_{X,m}, F_{Y,m}).
\end{aligned} \tag{52}$$

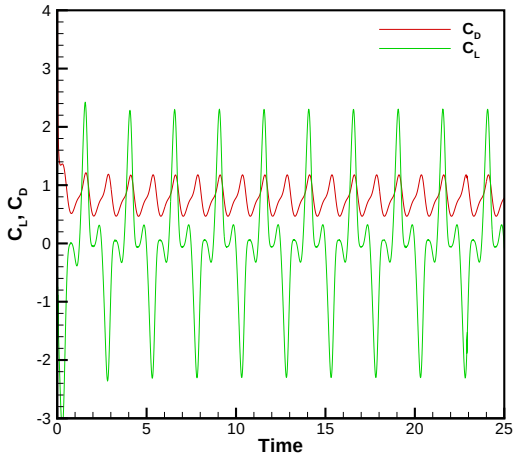
Here $F_{X,m}$ and $F_{Y,m}$ are horizontal and vertical force components respectively in the moving frame attached to the airfoil.

From Fig. 17 we observe that flow field quickly attain periodic nature for both elliptic and NACA0012 airfoil. The rhythmic nature of aerodynamic coefficients is in consonance with pitching and heaving frequency. It is seen that the average drag coefficient of NACA0012 is higher than its elliptic counterpart, whose surface area is twice that of NACA0012. Time averaged C_D values for NACA0012 and elliptic airfoil are 0.783 and 0.435 respectively. On the other hand lift coefficient of NACA0012 has a 15% higher amplitude at 2.416 compared to 2.096 for the elliptic airfoil.

We present an excerpt of the vorticity field at four optimal positions of velocity and displacement representing a full cycle of the motion in Fig. 18. Similar stroking phases of both aerofoils are laid



(b)



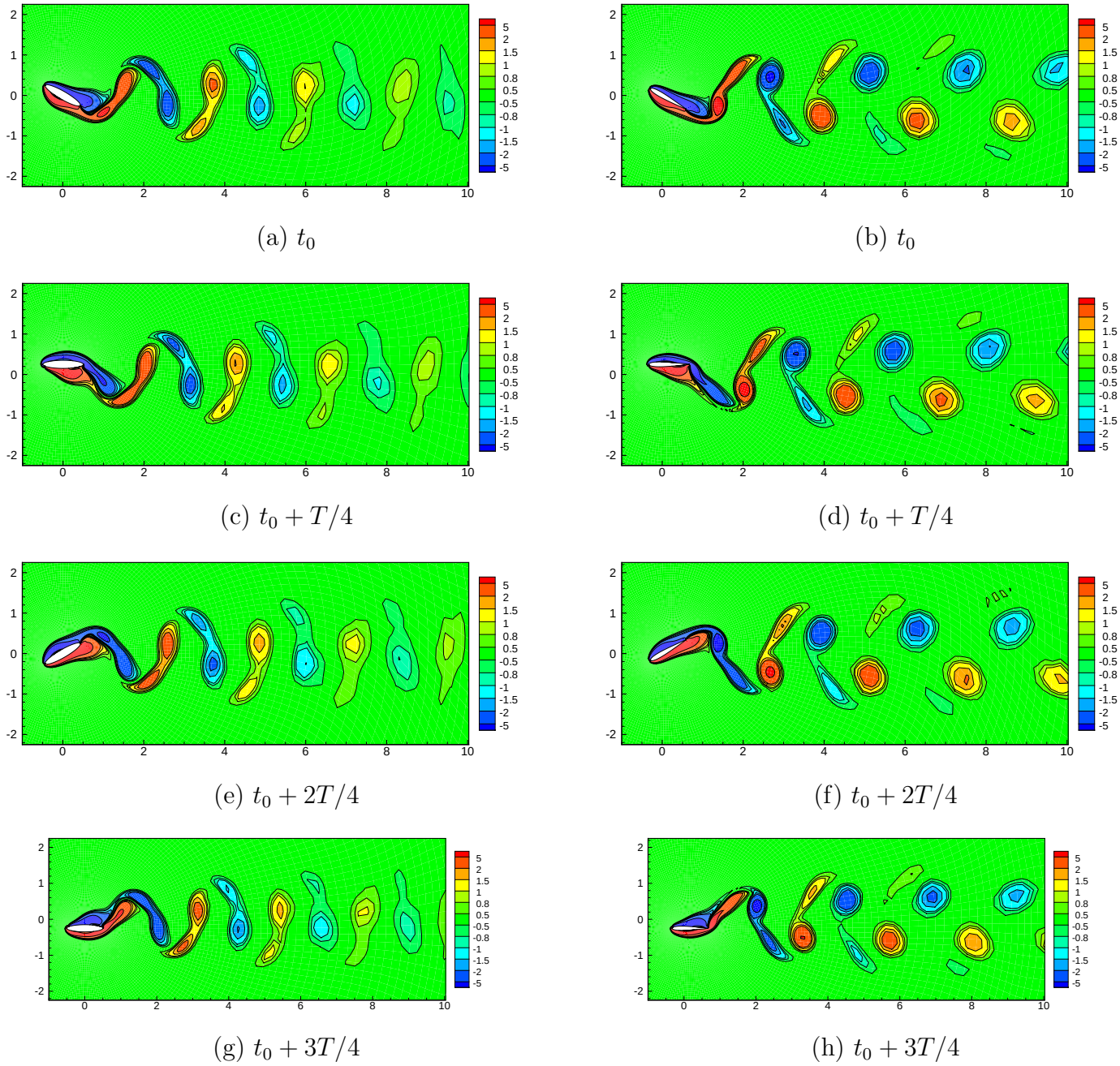
(b)

Figure 17: *Problem 7: Time evolution of drag and lift coefficients: (a) Elliptic airfoil, (b) NACA0012.*

out next to each other for sake of comparison. Based on the vorticity field it is obvious that vortices generated at the sharp trailing edge of NACA0012 are stronger *vis-a-vis* elliptic airfoil and relate well to the higher drag value documented in Fig. 17. Further for NACA0012 shed vortices are found to arrange themselves into two rows based on rotational orientation away from the central line. The downstream vortices behind the elliptic airfoil are elongated, causing the shedding to be weaker and follow $2S$ shedding mode. Fig. 19 shows contour plots of the instantaneous streamlines and streamwise velocity at two different stages of the shedding cycle. These stages correspond to the optimum angle of attack and are alternate mirror-image phases of the flow. Based on the streamwise velocity and the vorticity seen earlier it is obvious that solely in the vicinity of the trailing edge instantaneous phenomena in terms of flow separation and vortex generation take place. For elliptic airfoil, the wake shows reduced unsteadiness and a distinct narrowing *vis-a-vis* NACA0012. Note that vorticity contour comparisons do not provide the same information, as shown in Fig. 19 with the help of streamline contours.

4. Conclusions

A fourth order compact numerical scheme for solving unsteady incompressible flow on time-dependent curvilinear coordinates is presented. The technique developed approximates the convection-diffusion equation on irregular moving geometry and is amenable to both primitive variable and



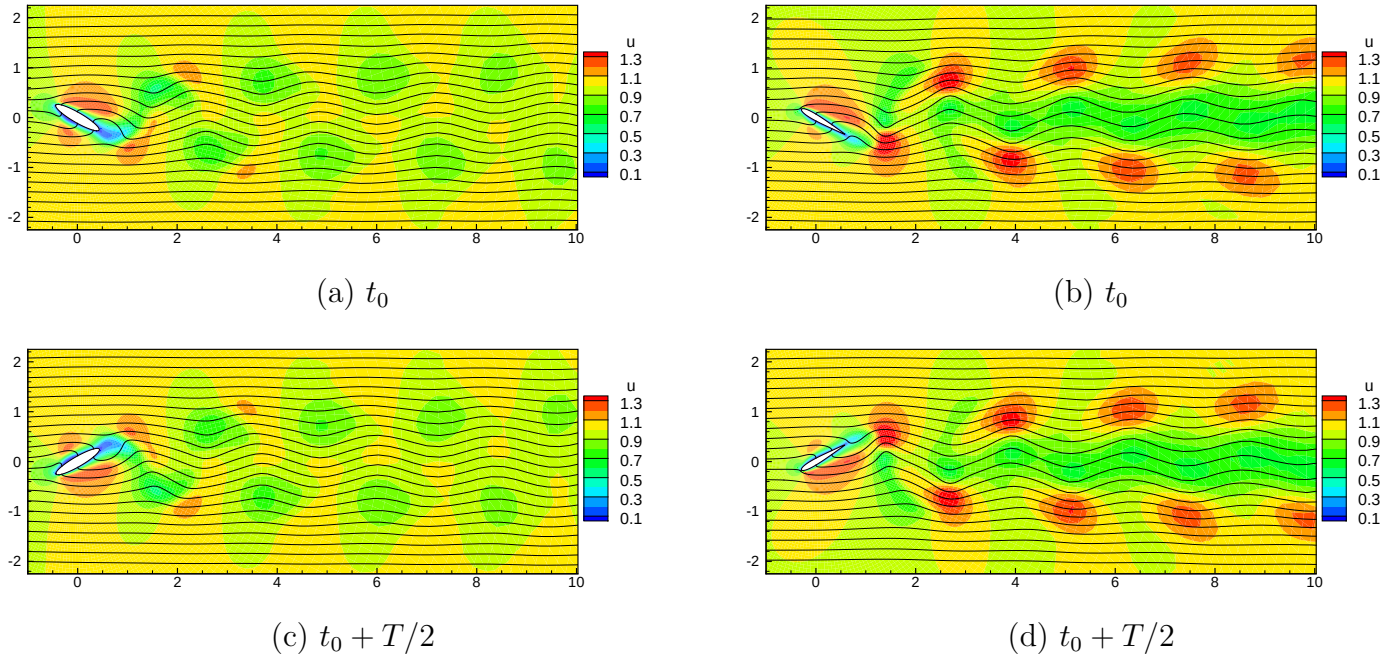
(a) t_0 (b) t_0

(c) $t_0 + T/4$ (d) $t_0 + T/4$

(e) $t_0 + 2T/4$ (f) $t_0 + 2T/4$

(g) $t_0 + 3T/4$ (h) $t_0 + 3T/4$

Figure 18: *Problem 7: One cycle of the pitching and heaving airfoil at* $Re = 200$*: elliptic airfoil (left) and NACA0012 (right) at similar stroking phases.*



(a) t_0 (b) t_0

(c) $t_0 + T/2$ (d) $t_0 + T/2$

Figure 19: *Problem 7: Streamwise velocity u and streamline contours during one cycle of oscillation at $Re = 200$: elliptic airfoil (left) and NACA0012 (right) at similar stroking phases.*

streamfunction-vorticity formulations of the Navier-Stokes (N-S) equations. The method uses compact computational stencil and hence discretization involves neighboring grid points only. In computations of the unsteady convection-diffusion equation and the incompressible N-S equation, the spatial and temporal accuracy of the method is demonstrated. Estimation of grid metrics in this context is rather straightforward as the current study reveals that the satisfaction of GCL is not imperative. In this work, the appropriate implementation of the scheme was achieved in tandem with an efficient grid deformation procedure. A successful simulation of the FSI test cases is a fitting proof of this. Most likely, this is the first attempt at a compact, higher-order computation of non-conservative N-S equations in a continually deforming region. The numerical method described in this study might be useful for the theoretical and numerical investigation of fluid dynamics in the presence of deformable surfaces.

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Convex and quasiconvex functions on trees and their applications

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ABSTRACT

We introduce convex and quasiconvex functions on trees and prove that for a tree the eccentricity, transmission and weight functions are strictly quasiconvex. It is shown that the Perron vector of the distance matrix is strictly convex whereas the Perron vector of the distance signless Laplacian is quasiconvex for a tree. In the class of all trees with a given number of pendent vertices, we prove that the distance Laplacian and distance signless Laplacian spectral radius are both maximized at a dumbbell. Among all trees with fixed maximum degree, we prove that the broom is the unique tree that maximizes the distance Laplacian and distance signless Laplacian spectral radius. We find the unique graph that maximizes the distance spectral radius in the class of all unicyclic graphs of girth g on n vertices. Also we find the unique graph that maximizes the distance signless Laplacian and the distance Laplacian spectral radius in the class of all unicyclic graphs on n vertices.

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1. Introduction

All our graphs are finite, undirected, connected and simple. Let G be a graph on vertices $\{1, 2, \ldots, n\}$. At times, we use $V(G)$ and $E(G)$ to denote the set of vertices and the set of edges of G, respectively. For $i, j \in V(G)$, the *distance* between i and j, denoted by $d_G(i, j)$ or simply d_{ij}, is the length of a shortest path from i to j in G. The *distance matrix* of G, denoted by $\mathcal{D}(G)$ is the $n \times n$ matrix with (i, j)-th entry d_{ij}.

Like the Laplacian and signless Laplacian matrix of a graph, Aouchiche and Hansen[1] first introduced the distance Laplacian and distance signless Laplacian of a graph. The transmission, denoted by $Tr(v)$ of a vertex v is the sum of the distances from v to all other vertices in G. Let $Tr(G)$ be the $n \times n$ diagonal matrix with i-th diagonal entry $Tr(i)$. The distance Laplacian and the distance signless Laplacian of G are defined as the matrix $\mathcal{D}^L(G) = Tr(G) - \mathcal{D}(G)$ and $\mathcal{D}^Q(G) = Tr(G) + \mathcal{D}(G)$, respectively, see [1].

For a column vector $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ we have

$$x^T\mathcal{D}^L(G)x = \sum_{1\le i<j\le n} d_{ij}(x_i - x_j)^2 \text{ and } x^T\mathcal{D}^Q(G)x = \sum_{1\le i<j\le n} d_{ij}(x_i + x_j)^2. \quad (1.1)$$

It follows that $\mathcal{D}^L(G)$ and $\mathcal{D}^Q(G)$ are positive semidefinite. Thus the eigenvalues of distance Laplacian and distance signless Laplacian matrix of a graph are all real. The largest eigenvalue of the distance, distance Laplacian and the distance signless Laplacian matrix are known as the distance spectral radius, distance Laplacian spectral radius and distance signless Laplacian spectral radius, respectively. We use $\rho^{\mathcal{D}}$, ρ^L and $\rho^{\mathcal{Q}}$ to denote the distance, distance Laplacian and distance signless Laplacian spectral radius, respectively.

Note that for a connected graph G, both $\mathcal{D}(G)$ and $\mathcal{D}^Q(G)$ are irreducible nonnegative matrices. Thus by the Perron–Frobenius Theorem, both $\rho^{\mathcal{D}}$ and $\rho^{\mathcal{Q}}$ are simple, and there is a positive eigenvector of $\mathcal{D}(G)$ and $\mathcal{D}^Q(G)$ corresponding to $\rho^{\mathcal{D}}$ and $\rho^{\mathcal{Q}}$, respectively. Such eigenvectors corresponding to $\rho^{\mathcal{D}}$ and $\rho^{\mathcal{Q}}$ are called *Perron vector* of $\mathcal{D}(G)$ and $\mathcal{D}^Q(G)$, respectively. By an eigenvector we mean a unit eigenvector.

Recently, the study of distance, distance Laplacian and distance signless Laplacian spectral radius has attracted many researchers. The problem of finding all graphs with maximal or minimal distance spectral radius among certain class of graphs has been studied extensively. For related results, one may refer to [2,4–6,9] and the references therein.

Subhi and Powers in [8] proved that for $n \ge 3$ the path P_n has the maximum distance spectral radius among trees on n vertices. Stevanović and Ilić in [7] generalized this result, and proved that among trees with fixed maximum degree Δ, the broom graph has maximum distance spectral radius and proved that the star S_n is the unique graph with minimal distance spectral radius among all trees on n vertices.

If vertices i and j are adjacent, we write $i \sim j$. Vertices i, j, k in graph G are said to be consecutive if j is adjacent to both i and k. An edge independent set of a graph G is a set of edges such that any two distinct edges of the set are not incident on a common vertex.

The matching number of G, denoted by $m(G)$, is the maximum of the cardinalities of all edge independent sets. For a connected graph G of order n, its matching number $m(G)$ satisfies $1 \leq m(G) \leq \lfloor \frac{n}{2} \rfloor$. By $\mathcal{T}_n^m$ we denote the class of all trees on n vertices with matching number m.

If u, v are vertices in a graph G, by $G-u$ and $G-\{u,v\}$ we mean the graphs obtained from G by deleting the vertex u and the edge $\{u,v\}$ respectively. Also by $G+\{u,v\}$ we mean the graph obtained from G by adding the edge $\{u,v\}$. A *quasipendent vertex* is a vertex which is adjacent to a pendent vertex. If v is a vertex of a tree T, then the components of $T-v$ are called the *branches* of T at v.

The *dumbbell* $D(n,a,b)$ consists of the path P_{n-a-b} together with a independent vertices adjacent to one pendent vertex of P and b independent vertices adjacent to the other pendent vertex, where $a,b \geq 1$. By $\mathcal{T}_{n,k}$ we denote the class of all trees on n vertices and k pendent vertices.

We now outline the contents of this paper. In section 2 of this article, we introduce some convex and quasiconvex functions on trees and prove that for a tree the eccentricity, transmission and weight functions are strictly quasiconvex. In section 3, we prove that the Perron vector of distance matrix of a tree is strictly convex and the Perron vector of distance signless Laplacian matrix of a tree is strictly quasiconvex. In section 4, we have obtained the unicyclic graph that maximizes the distance spectral radius among all unicyclic graphs of fixed girth and order. In section 5, we prove that the tree that maximizes the distance Laplacian spectral radius among all trees with a given number of pendent vertices is a dumbbell. Moreover, we obtain the tree that maximizes the distance Laplacian spectral radius among all trees with fixed maximum degree. We obtain the unique unicyclic graph that maximizes the distance Laplacian spectral radius among all unicyclic graph. In section 6, we prove that the tree that maximizes the distance signless Laplacian spectral radius among all trees with a given number of pendent vertices is a dumbbell. Moreover, we obtain the tree that maximizes the distance signless Laplacian spectral radius among all trees with fixed maximum degree. We obtain the unique unicyclic graph that maximizes the distance signless Laplacian spectral radius among all unicyclic graphs.

2. Convex and quasiconvex functions on trees

Definition. Let T be a tree with $V(T)=\{1,\ldots,n\}$, $n \geq 3$, and let $f : V(T) \longrightarrow [0,+\infty)$. We say that f is *convex* if for any distinct $i,j,k \in V(T)$ with $i \sim j$, $j \sim k$ we have $2f(j) \leq f(i)+f(k)$ and *strictly convex* if $2f(j) < f(i)+f(k)$. We say that f is *quasiconvex* if for any distinct $i,j,k \in V(T)$ with $i \sim j$, $j \sim k$ we have $f(j) \leq \max\{f(i),f(k)\}$ and *strictly quasiconvex* if $f(j) < \max\{f(i),f(k)\}$.

Lemma 2.1. *Let T be a tree with $V(T)=\{1,\ldots,n\}$ and let $f : V(T) \longrightarrow [0,+\infty)$ be either strictly convex or strictly quasiconvex. Then f attains its minimum, either at a unique vertex, or at two adjacent vertices. Furthermore:*

(i) *If f attains its minimum at the unique vertex i, then for any path $i = i_1 - i_2 - \cdots - i_k$, starting at i, $f(i_1) < f(i_2) < \cdots < f(i_k)$. (We say that f is strictly increasing along any path starting at i.)*

(ii) *If f attains its minimum at the two adjacent vertices i and j, then f is strictly increasing along any path starting at i, and not containing j, or starting at j, and not containing i.*

Proof. First suppose that f is strictly convex. Observe that if $i, j, k \in V(T)$ with $i \sim j$, $j \sim k$, then $f(i) < f(j)$ and the strict convexity of f imply that $f(j) < f(k)$. Let $S \subset V(T)$ be the set of vertices on which f attains its minimum. Let $i, j \in S$ and suppose i and j are not adjacent. Let $i = i_1 - i_2 - \cdots - i_k = j$ be an ij-path. By the observation made earlier, $f(i) = f(i_1) < f(i_2)$ implies that $f(i_2) < \cdots < f(i_k) = f(j)$ and hence $f(i) < f(j)$, which is a contradiction. Thus any two vertices in S must be adjacent. Since T has no cycles, it follows that S has either 1 or 2 vertices. The second part is proved similarly, using the observation made in the beginning. This completes the proof.

If f is strictly quasiconvex, then again if $i, j, k \in V(T)$ with $i \sim j$, $j \sim k$, and $f(i) < f(j)$, then it follows that $f(j) < f(k)$. The rest of the proof is similar. □

Theorem 2.2. *Let T be a tree with $V(T) = \{1, \ldots, n\}$, let $g : [0, +\infty) \longrightarrow [0, +\infty)$ be a strictly increasing convex function. Let $x_1, \ldots, x_n$ be positive numbers and let $f(i) = \sum_{j=1}^{n} g(d_{ij})x_j$, $i = 1, \ldots, n$. Then f is strictly convex.*

Proof. Let i, j, k be vertices of T such that $i \sim j$, $j \sim k$. Let $e_1 = \{i, j\}$, $e_2 = \{j, k\}$ be edges of T. Let T_1, T_2 and T_3 be the components of $T \setminus \{e_1, e_2\}$, containing i, j and k, respectively. We have

$$f(i) = \sum_{\ell \in T_1} g(d_{i\ell})x_\ell + \sum_{\ell \in T_2} g(d_{j\ell} + 1)x_\ell + \sum_{\ell \in T_3} g(d_{k\ell} + 2)x_\ell \tag{2.2}$$

$$f(j) = \sum_{\ell \in T_1} g(d_{i\ell} + 1)x_\ell + \sum_{\ell \in T_2} g(d_{j\ell})x_\ell + \sum_{\ell \in T_3} g(d_{k\ell} + 1)x_\ell \tag{2.3}$$

$$f(k) = \sum_{\ell \in T_1} g(d_{i\ell} + 2)x_\ell + \sum_{\ell \in T_2} g(d_{j\ell} + 1)x_\ell + \sum_{\ell \in T_3} g(d_{k\ell})x_\ell \tag{2.4}$$

Since g is convex, we have

$$g(d_{i\ell}) + g(d_{i\ell} + 2) \geq 2g(d_{i\ell} + 1) \tag{2.5}$$

and

$$g(d_{k\ell} + 2) + g(d_{k\ell}) \geq 2g(d_{k\ell} + 1). \tag{2.6}$$

It follows from (2.2)–(2.6) that

$$f(i)+f(k)\geq \sum_{\ell\in T_1} 2g(d_{i\ell}+1)x_\ell+\sum_{\ell\in T_2} 2g(d_{j\ell}+1)x_\ell+\sum_{\ell\in T_3} 2g(d_{k\ell}+1)x_\ell$$

and since g is strictly increasing we have

$$\begin{aligned} f(i)+f(k)-2f(j) &\geq \sum_{\ell\in T_2} 2(g(d_{j\ell}+1)-g(d_{j\ell}))x_\ell \\ &> 0. \end{aligned}$$

That completes the proof. □

2.1. Sum of the largest p distances

We introduce some notations. Let p be a fixed positive integer, $1\leq p\leq n$. If $x,y\in R^n$, we define $x\otimes_p y$ to be the sum of the p largest components in $\{x_1y_1,\ldots,x_ny_n\}$. If A is an $n\times n$ matrix and if $x\in R^n$, then $A\otimes_p x$ is defined in the natural way, that is, the i-th component of $A\otimes_p x$ is given by $(a_{i1},\ldots,a_{in})\otimes_p x$, $i=1,\ldots,n$. Note that when $p=n$, $x\otimes y=\sum_{i=1}^n x_iy_i$, whereas when $p=1$, then $x\otimes y=\max_i x_iy_i$.

If A is an $n\times n$ matrix with $a_{ij}\geq 0$ for all i, j and if $g:[0,+\infty)\longrightarrow[0,+\infty)$ is a function, then we define $g(A)$ to be the $n\times n$ matrix with (i,j)-entry $g(a_{ij})$.

Let T be a tree with $V(T)=\{1,\ldots,n\}$ and let $y\in R^n$ be a vector with nonnegative coordinates. We say that y is convex (quasiconvex) if $f(i)=y_i$, $i=1,\ldots,n$ is a convex (quasiconvex) function with respect to the tree T. Similar definition is used with respect to strict convexity (quasiconvexity).

Theorem 2.3. *Let T be a tree with $V(T)=\{1,\ldots,n\}$, let $g:[0,+\infty)\longrightarrow[0,+\infty)$ be a strictly increasing function. Let $x\in R^n$ be a vector with positive coordinates. Let p be a positive integer, $1\leq p\leq n$. Then $g(D)\otimes_p x$ is strictly quasiconvex.*

Proof. Let i, j, k be vertices of T such that $i\sim j$, $j\sim k$. We claim that for any $\ell\in V(T)$ we have

$$g(d_{j\ell})x_\ell<\max(g(d_{i\ell})x_\ell,g(d_{k\ell})x_\ell). \tag{2.7}$$

Let $e_1=\{i,j\}$, $e_2=\{j,k\}$ be edges of T. Let T_1, T_2 and T_3 be the components of $T\setminus\{e_1,e_2\}$, containing i, j and k, respectively. If $\ell\in T_1$, then $d_{i\ell}x_\ell=(d_{j\ell}-1)x_\ell$ and $d_{k\ell}x_\ell=(d_{j\ell}+1)x_\ell$. Hence we have

$$d_{j\ell}x_\ell<\max(d_{i\ell}x_\ell,d_{k\ell}x_\ell). \tag{2.8}$$

Similarly (2.8) holds if $\ell\in T_3$. If $\ell\in T_2$, then $d_{i\ell}x_\ell=(d_{j\ell}+1)x_\ell$ and $d_{k\ell}x_\ell=(d_{j\ell}+1)x_\ell$. Hence (2.8) holds as well. Therefore (2.8) holds for any $\ell\in V(T)$. Since g is strictly increasing, (2.7) follows from (2.8).

It follows that the sum of the largest p components in $g(d_{j\ell})x_\ell$, $\ell = 1, \ldots, n$ is less than the sum of the largest p components in $g(d_{i\ell})x_\ell$, $\ell = 1, \ldots, n$ and the largest p components in $g(d_{k\ell})x_\ell$, $\ell = 1, \ldots, n$. Therefore $g(D) \otimes_p x$ is strictly quasiconvex and the proof is complete. □

We note two special cases of Theorem 2.3. Let T be a tree with $V(T) = \{1, \ldots, n\}$. Recall that the *eccentricity* of vertex $\ell \in V(T)$ is defined to be $\mathrm{ecc}(\ell) = \max(d_{\ell 1}, \ldots, d_{\ell n})$, while the transmission index of $\ell \in V(T)$ is given by $Tr(\ell) = \sum_{i=1}^{n} d(\ell i)$. It follows from Theorem 2.3 that both eccentricity and transmission index are strictly quasiconvex functions and hence satisfy (i) and (ii) of Lemma 2.1. The vertex (or vertices) where the eccentricity is minimized constitute the *center* of the tree, whereas the vertex (or vertices) where the transmission index is minimized, constitute the *median* of the tree.

2.2. Centroid

Let T be a tree with $V(T) = \{1, \ldots, n\}$. If $i \in V(T)$, then the weight of i, denoted $\mathrm{wt}(i)$, is defined to be the maximum number of vertices in a component of $T \setminus i$.

Let i, j, k be vertices of T such that $i \sim j$, $j \sim k$. Let T_1, T_2 and T_3 be the components of $T \setminus \{e_1, e_2\}$, containing i, j and k, respectively. Let T_ℓ^i, $\ell = 1, \ldots, u$, be the components of $T_1 \setminus i$, let T_ℓ^j, $\ell = 1, \ldots, v$, be the components of $T_2 \setminus j$, and let T_ℓ^k, $\ell = 1, \ldots, w$, be the components of $T_3 \setminus k$. We denote the number of vertices in the graph G by $|G|$. We have

$$wt(i) = \max\{|T_1^i|, \ldots, |T_u^i|, |T_1^j| + \cdots + |T_v^j| + |T_1^k| + \cdots + |T_w^k| + 2\}, \tag{2.9}$$

and

$$wt(k) = \max\{|T_1^k|, \ldots, |T_w^k|, |T_1^j| + \cdots + |T_v^j| + |T_1^i| + \cdots + |T_u^i| + 2\}. \tag{2.10}$$

It follows from (2.9), (2.10) that

$$\begin{aligned}
&\max(wt(i), wt(k)) \\
&= \max\{|T_1^j|, \ldots, |T_v^j| + |T_1^k| + \cdots + |T_w^k| + 2, |T_1^j| + \cdots + |T_v^j| + |T_1^i| + \cdots + |T_u^i| + 2\} \\
&> \max\{|T_1^i| + \cdots + |T_u^i| + 1, |T_1^j| + \ldots + |T_v^j|, |T_1^k| + \cdots + |T_w^k| + 1\} \\
&= wt(j).
\end{aligned}$$

Thus we have proved the following result.

Theorem 2.4. *Let T be a tree with $V(T) = \{1, \ldots, n\}$. Then $wt(i)$, $i = 1, \ldots, n$ is strictly quasiconvex.*

It follows from Theorem 2.4 that the weight function satisfies (i) and (ii) of Lemma 2.1. The vertices with minimum weight constitute the *centroid* of the tree. It can be shown that the centroid and the median of a tree coincide.

3. Perron vector of distance and distance signless Laplacian matrices

The following result is a reformulation of Theorem 2.2. The proof is omitted.

Theorem 3.1. *Let $f : [0, +\infty) \longrightarrow [0, +\infty)$ be a strictly increasing convex function and $\mathcal{D}$ be the distance matrix of a tree T. Then $f(\mathcal{D})z$ is strictly convex on T for any positive vector z, where $f(\mathcal{D})$ is the matrix with (i, j)-th entry $f(d_{ij})$.*

The following corollary, which follows immediately from Theorem 3.1, says that the Perron vector of $f(\mathcal{D})$ is strictly convex, which implies that the Perron vector satisfies (i) and (ii) of Lemma 2.1.

Corollary 3.2. *Let $\mathcal{D}$ be the distance matrix of a tree T and let $f : [0, +\infty) \longrightarrow [0, +\infty)$ be a strictly increasing convex function. Then the Perron vector of $f(\mathcal{D})$ is strictly convex on T.*

Proof. Let ρ be the spectral radius and $\mathbf{x}$ be the Perron vector of $f(\mathcal{D})$. By the Perron–Frobenius theorem $\mathbf{x}$ is positive. Since $f(\mathcal{D})\mathbf{x} = \rho\mathbf{x}$, then $\mathbf{x} = \frac{1}{\rho} f(\mathcal{D})\mathbf{x}$. By Theorem 3.1 $\mathbf{x}$ is strictly convex. □

It follows from Corollary 3.2 that the Perron vector of $f(\mathcal{D})$ satisfies (i) and (ii) of Lemma 2.1. The next result says that the Perron vector of the distance signless Laplacian matrix of a tree is strictly quasiconvex and hence satisfies (i) and (ii) of Lemma 2.1.

Theorem 3.3. *Let $\mathcal{D}^Q$ be the distance signless Laplacian of a tree T. Then the Perron vector of $\mathcal{D}^Q$ is strictly quasiconvex on T.*

Proof. Let i, j, k be vertices of T such that $i \sim j$, $j \sim k$. Let $e_1 = \{i, j\}$, $e_2 = \{j, k\}$ be edges of T. Let T_1, T_2 and T_3 be the components of $T \setminus \{e_1, e_2\}$, containing i, j and k, respectively. Let $\mathbf{x}$ be the Perron vector of $\mathcal{D}^Q$. By Theorem 2.2 the transmission index is strictly convex and therefore

$$Tr(i) + Tr(k) - 2Tr(j) > 0.$$

Thus

$$2\{\rho^Q - Tr(j)\} > 2\rho^Q - \{Tr(i) + Tr(k)\}. \tag{3.11}$$

By eigenvector equations of $\mathbf{x}$ at the vertices i, j, k we have

$$\{\rho^{\mathcal{Q}} - Tr(i)\}\mathbf{x}_i = \sum_{\ell\in T_1} d_{i\ell}\mathbf{x}_\ell + \sum_{\ell\in T_2}(d_{j\ell}+1)\mathbf{x}_\ell + \sum_{\ell\in T_3}(d_{k\ell}+2)\mathbf{x}_\ell \tag{3.12}$$

$$\{\rho^{\mathcal{Q}} - Tr(j)\}\mathbf{x}_j = \sum_{\ell\in T_1}(d_{i\ell}+1)\mathbf{x}_\ell + \sum_{\ell\in T_2} d_{j\ell}\mathbf{x}_\ell + \sum_{\ell\in T_3}(d_{k\ell}+1)\mathbf{x}_\ell \tag{3.13}$$

$$\{\rho^{\mathcal{Q}} - Tr(k)\}\mathbf{x}_k = \sum_{\ell\in T_1}(d_{i\ell}+2)\mathbf{x}_\ell + \sum_{\ell\in T_2}(d_{j\ell}+1)\mathbf{x}_\ell + \sum_{\ell\in T_3} d_{k\ell}\mathbf{x}_\ell. \tag{3.14}$$

It follows from (3.12)–(3.14) that

$$\{\rho^{\mathcal{Q}} - Tr(i)\}\mathbf{x}_i + \{\rho^{\mathcal{Q}} - Tr(k)\}\mathbf{x}_k - 2\{\rho^{\mathcal{Q}} - Tr(j)\}\mathbf{x}_j > 0. \tag{3.15}$$

From (3.11) and (3.15), it follows that

$$\{\rho^{\mathcal{Q}} - Tr(i)\}\mathbf{x}_i + \{\rho^{\mathcal{Q}} - Tr(k)\}\mathbf{x}_k > \{2\rho^{\mathcal{Q}} - \{Tr(i) + Tr(k)\}\}\mathbf{x}_j.$$

Thus $\beta\mathbf{x}_i + (1-\beta)\mathbf{x}_k > \mathbf{x}_j$, where $\beta = \frac{\rho^{\mathcal{Q}} - Tr(i)}{2\rho^{\mathcal{Q}} - \{Tr(i)+Tr(k)\}}$. Therefore $\mathbf{x}_j < \max\{\mathbf{x}_i, \mathbf{x}_k\}$ and hence the Lemma holds. □

4. Distance spectral radius

For a matrix A we use A_i to denote the i-th column of A. By $\mathbb{1}$ we denote the column vector of all ones and by $\mathbb{J}$ the matrix of all ones of appropriate size. We recall some known results which will be used.

Lemma 4.1 *(Weyl's inequalities). [3] Let $\lambda_i(M)$ denote the i-th largest eigenvalue of a real symmetric matrix M. If A and B are two real symmetric matrices of order n, then*

$$\lambda_1(A) + \lambda_i(B) \geq \lambda_i(A+B) \geq \lambda_n(A) + \lambda_i(B) \text{ for } i = 1, 2, \ldots, n.$$

Lemma 4.2. *[3] If A is a symmetric $n \times n$ matrix with λ_1 as the largest eigenvalue then for any normalized vector $\mathbf{x} \in R^n (\mathbf{x} \neq 0)$,*

$$\mathbf{x}^t A\mathbf{x} \leq \lambda_1.$$

The equality holds if and only if $\mathbf{x}$ is an eigenvector of A corresponding to the eigenvalue λ_1.

Lemma 4.3. *Let G_v be the graph obtained by identifying a vertex v of the graph G with a fixed vertex u of another graph G'. If i, j, k are consecutive vertices of G such that i and k are on different components of $G - j$ and $\rho_v^{\mathcal{D}}$ denotes the distance spectral radius of G_v, then $\rho_i^{\mathcal{D}} + \rho_k^{\mathcal{D}} - 2\rho_j^{\mathcal{D}} > 0$.*

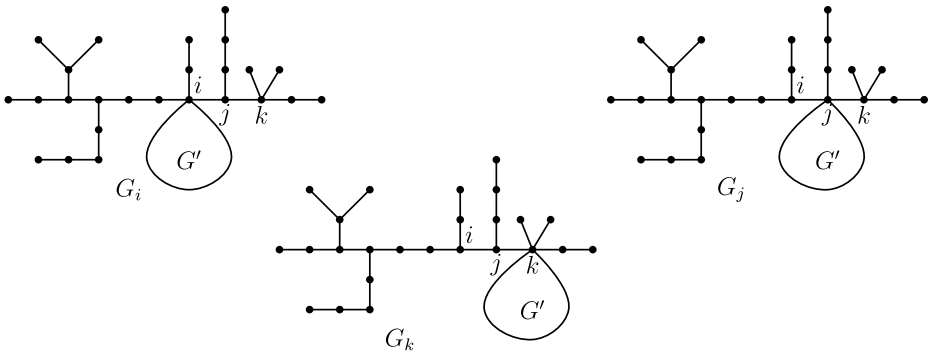


Fig. 1. Graphs G_i, G_j and G_k.

Proof. Let D_j be the jth column of $\mathcal{D}(G)$ and $d_{G'}(u,x)$ be the distance between u and x in G'. By permutation similarity, we have

$$\mathcal{D}(G_j) = \begin{bmatrix} \mathcal{D}(G) & B \\ B^t & \mathcal{D}(G')^u \end{bmatrix},$$

where $B_x = \mathcal{D}_j + d_{G'}(u,x)\mathbb{1}$ for $x \in V(G'-u)$ and $\mathcal{D}(G')^u$ is the submatrix of $\mathcal{D}(G')$ after deleting the row and column corresponding to the vertex u.

Thus

$$\mathcal{D}(G_i) + \mathcal{D}(G_k) - 2\mathcal{D}(G_j) = \begin{bmatrix} 0 & (\mathcal{D}_i + \mathcal{D}_k - 2\mathcal{D}_j)\mathbb{1}^t \\ \mathbb{1}(\mathcal{D}_i + \mathcal{D}_k - 2\mathcal{D}_j)^t & 0 \end{bmatrix}.$$

Let $\mathbf{x}$ be the Perron vector of $\mathcal{D}(G_j)$. Since $\mathbf{x}$ is positive, $\mathcal{D}_i + \mathcal{D}_k - 2\mathcal{D}_j \geq 0$ and $\mathcal{D}_i + \mathcal{D}_k - 2\mathcal{D}_j \neq 0$ we see that

$$\mathbf{x}^t \{\mathcal{D}(G_i) + \mathcal{D}(G_k) - 2\mathcal{D}(G_j)\}\mathbf{x} > 0. \tag{4.16}$$

Also

$$\begin{aligned} \rho_i^{\mathcal{D}} + \rho_k^{\mathcal{D}} - 2\rho_j^{\mathcal{D}} &\geq \rho\{\mathcal{D}(G_i) + \mathcal{D}(G_k)\} - 2\rho_j^{\mathcal{D}} \text{ [by Lemma 4.1]} \\ &\geq \mathbf{x}^t \{\mathcal{D}(G_i) + \mathcal{D}(G_k) - 2\mathcal{D}(G_j)\}\mathbf{x} \text{ [by Lemma 4.2]}. \end{aligned} \tag{4.17}$$

From (4.16) and (4.17) we have $\rho_i^{\mathcal{D}} + \rho_k^{\mathcal{D}} - 2\rho_j^{\mathcal{D}} > 0$. (See Fig. 1.) □

Corollary 4.4. *Let T be a tree and G' be a graph. If T_i and $\rho_i^{\mathcal{D}}$ are as defined in Lemma 4.3, then $\rho_i^{\mathcal{D}}$ is strictly convex on T.*

Let G be a graph with v as one of the vertices. For $k, \ell \geq 1$, let $G(v,k,\ell)$ be the graph obtained by attaching two paths P_k and P_ℓ at v. We use $\rho_{v,k,\ell}^{\mathcal{D}}$ to denote the distance

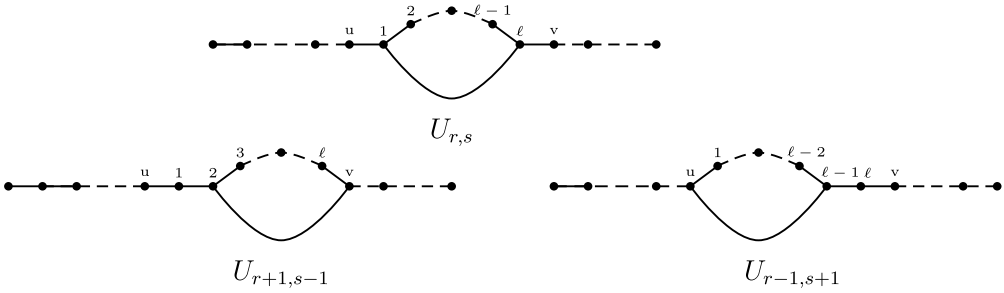


Fig. 2. Graphs $U_{r,s}$, $U_{r+1,s-1}$ and $U_{r-1,s+1}$.

spectral radius of $G(v,k,\ell)$. Stevanović, and Ilić [7] proved that $\rho^{\mathcal{D}}_{v,k,\ell} < \rho^{\mathcal{D}}_{v,k+1,\ell-1}$ for $k \geq \ell \geq 1$, which can now be obtained as an immediate application of Corollary 4.4.

Corollary 4.5. *[7, Lemma 3.3] Let G be a graph and v be one of its vertices. If $k \geq \ell \geq 1$, then $\rho^{\mathcal{D}}_{v,k,\ell} < \rho^{\mathcal{D}}_{v,k+1,\ell-1}$.*

Example 4.6. The graphs $U_{r,s}$, $U_{r+1,s-1}$ and $U_{r-1,s+1}$ with vertex labelling are as shown in Fig. 2.

The following lemma is crucial in proving our main result of this section.

Lemma 4.7. *Let $U_{r,s}$ be a unicyclic graph of girth g with two pendent paths of length r, s $(r, s \geq 1)$ respectively such that the intermediate vertices of the shortest path between the roots of the two paths are of degree two. If $\rho_{r,s}$ denotes the distance spectral radius of $U_{r,s}$ then $\rho_{r+1,s-1} + \rho_{r-1,s+1} > 2\rho_{r,s}$.*

Proof. Let the vertices 1 and ℓ of the cycle be roots of the paths P_1 and P_2 respectively. Let $P := 1 - 2 - \cdots - \ell$ be the shortest path from 1 to ℓ and S be the set of vertices not on these three paths. Partition the vertex set as follows:

$$\begin{aligned}
V_1 &= \{j \in S : d_{1j} > d_{2j} \text{ and } d_{\ell j} < d_{(\ell-1)j}\}\\
V_2 &= \{j \in S : d_{1j} < d_{2j} \text{ and } d_{\ell j} > d_{(\ell-1)j}\}\\
V_3 &= \{j \in S : d_{1j} < d_{2j} \text{ and } d_{\ell j} < d_{(\ell-1)j}\}\\
V_4 &= \{j \in S : d_{1j} = d_{2j}\}\\
V_5 &= \{j \in S : d_{\ell j} = d_{(\ell-1)j}\}\\
V_6 &= V(P_1 \cup P_2) - \{1, \ell\}\\
V_7 &= V(P) - \{1, \ell\}.
\end{aligned}$$

For $i \in V_7$, $j \in \cup_{i=3}^{i=5} V_i$ if the lengths of the two paths in $U_{r,s}$ from i to j differ by k then following changes occur.

(i) For $k = 0$, d_{ij} decreases by 1 in both $U_{r+1,s-1}$ and $U_{r-1,s+1}$.
(ii) For $k = 1$, d_{ij} remains unchanged in one of $U_{r+1,s-1}$ and $U_{r-1,s+1}$ and decreases by 1 in the other.
(iii) For $k \geq 2$, d_{ij} increases by 1 in one of $U_{r+1,s-1}$ and $U_{r-1,s+1}$ and decreases by 1 in the other.

Similarly we can describe the change in distances among other vertices.

For $i \in V_7$, let S_i be the set of vertices j in $\cup_{i=3}^{i=5} V_i$ such the length of the two paths from i to j differ by at most one. Note that if girth g is even then $S_i \subseteq V_3$ for all $i \in V_7$ as $V_4 = V_5 = \emptyset$.

Let $\mathcal{D}_{r,s}$ be the distance matrix of $U_{r,s}$. Then with respect to the ordered partition $V(U_{r,s}) = \{1, \ell\} \cup (V_1 \cup V_2) \cup (V_3 \cup V_4 \cup V_5) \cup V_6 \cup V_7$, we have

$$\mathcal{D}_{r+1,s-1} + \mathcal{D}_{r-1,s+1} - 2\mathcal{D}_{r,s} = \left[\begin{array}{c|c|c|c|c} \{1,\ell\} & V_1 \cup V_2 & V_3 \cup V_4 \cup V_5 & V_6 & V_7 \\ \hline 0 & \begin{matrix} 0 & 2\mathbb{1}^t \\ 2\mathbb{1}^t & 0 \end{matrix} & \begin{matrix} 2\mathbb{1}^t & \mathbb{1}^t & 2\mathbb{1}^t \\ 2\mathbb{1}^t & 2\mathbb{1}^t & \mathbb{1}^t \end{matrix} & 0 & 0 \\ \hline \begin{matrix} 0 & 2\mathbb{1} \\ 2\mathbb{1} & 0 \end{matrix} & 0 & 0 & 0 & 0 \\ \hline \begin{matrix} 2\mathbb{1} & 2\mathbb{1} \\ \mathbb{1} & 2\mathbb{1} \\ 2\mathbb{1} & \mathbb{1} \end{matrix} & 0 & 0 & 0 & A^t \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & A & 0 & 0 \end{array}\right]$$

where $A_{ij} = \begin{cases} -1, & \text{if } j \in S_i \text{ and } g \text{ is odd} \\ -2, & \text{if } j \in S_i \text{ and } g \text{ is even} \\ 0, & \text{otherwise.} \end{cases}$

Let $\mathbf{x}$ be the Perron vector of $\mathcal{D}_{r,s}$, then we have

$$\begin{aligned} & \rho_{r+1,s-1} + \rho_{r-1,s+1} - 2\rho_{r,s} \\ & \geq \rho(\mathcal{D}_{r+1,s-1} + \mathcal{D}_{r-1,s+1}) - 2\rho_{r,s} \quad \text{[by Lemma 4.1]} \\ & \geq \mathbf{x}^t \left(\mathcal{D}_{r+1,s-1} + \mathcal{D}_{r-1,s+1} - 2\mathcal{D}_{r,s}\right) \mathbf{x} \quad \text{[by Lemma 4.2]} \\ & = \mathbf{x}_1 \left(\sum_{j \in V_4} \mathbf{x}_j + 2 \sum_{j \in V_2 \cup V_3 \cup V_5} \mathbf{x}_j \right) + \mathbf{x}_\ell \left(\sum_{j \in V_5} \mathbf{x}_j + 2 \sum_{j \in V_1 \cup V_3 \cup V_4} \mathbf{x}_j \right) - \alpha \sum_{j \in S_i} \sum_{i \in V_7} \mathbf{x}_i \mathbf{x}_j \\ & \quad (\alpha = 1 \text{ or } 2 \text{ according as } g \text{ is odd or even}) \\ & \geq \mathbf{x}_1 \left(\sum_{j \in V_4} \mathbf{x}_j + 2 \sum_{j \in V_3 \cup V_5} \mathbf{x}_j \right) + \mathbf{x}_\ell \left(\sum_{j \in V_5} \mathbf{x}_j + 2 \sum_{j \in \cup V_3 \cup V_4} \mathbf{x}_j \right) - \alpha \sum_{j \in S_i} \sum_{i \in V_7} \mathbf{x}_i \mathbf{x}_j \\ & > \alpha \sum_{j \in S_i} \sum_{i \in V_7} \mathbf{x}_j (\mathbf{x}_1 + \mathbf{x}_\ell - \mathbf{x}_i). \end{aligned} \tag{4.18}$$

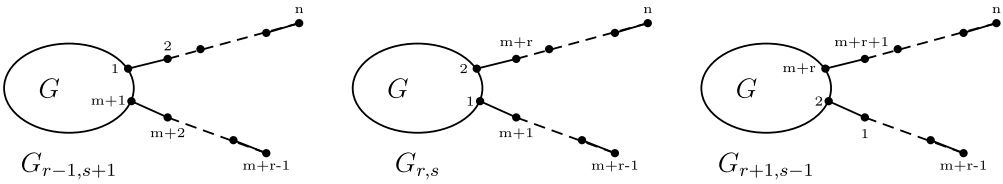


Fig. 3. Graphs $G_{r-1,s+1}, G_{r,s}$ and $G_{r+1,s-1}$.

From eigenequations at the vertices 1, ℓ, i we have

$$\begin{aligned}\rho_{r,s}(\mathbf{x}_1+\mathbf{x}_\ell-\mathbf{x}_i) &= \sum_{j\in V(U_{r,s})} d_{1j}\mathbf{x}_j + \sum_{j\in V(U_{r,s})} d_{\ell j}\mathbf{x}_j - \sum_{j\in V(U_{r,s})} d_{ij}\mathbf{x}_j \\ &= \sum_{j\in V(U_{r,s})} (d_{1j}+d_{\ell j}-d_{ij})\mathbf{x}_j. \qquad (4.19)\end{aligned}$$

Since $d_{1j}+d_{\ell j} > d_{ij}$ for all $i \in V_7$ and any vertex j, the result follows from (4.18) and (4.19). □

We use $\mathcal{U}_{n,g}$ to denote the class of unicyclic graph of girth g on n vertices. U_g^n is the unicyclic graph on n vertices obtained by attaching a g-cycle to one of the end vertices of P_{n-g}. The next result is one of our main results of this section which provides the unique graph that maximizes the distance spectral radius among all unicyclic graph with fixed girth.

Theorem 4.8. *The unique graph that maximizes the distance spectral radius in $\mathcal{U}_{n,g}$ is U_g^n.*

Proof. If $g = n$, there is nothing to prove. Let U be a graph that maximizes the distance spectral radius in $\mathcal{U}_{n,g}$. Let C be the cycle in U and T_u be a tree branch attached at a vertex u of C. If T_u is not a path, then T_u contains a vertex of degree at least 3 except u. Let w be a vertex of degree at least 3 which is farthest from u. Then there exist two pendent paths attached at w. Thus $U \cong G(w,k,\ell)$ for some $k,\ell \geq 1$ and unicyclic graph G of girth g. Observe that $G(w,k-1,\ell+1), G(w,k+1,\ell-1) \in \mathcal{U}_{n,g}$. By Lemma 4.3, the distance spectral radius of $G(w,k+1,\ell-1)$ or $G(w,k-1,\ell+1)$ is bigger than that of U, which is a contradiction. Hence every tree branch of U attached at a vertex of C is a path. Next we prove that at most one vertex of C has degree 3. To the contrary, assume that u and v are two nearest vertices in C of degree 3. Then $U \cong U_{r,s}$ for some r, s $(r \geq s \geq 1)$. By Lemma 4.7, distance spectral radius of $U_{r+1,s-1}$ or $U_{r-1,s+1}$ is bigger than that of U, which is a contradiction. Hence $U \cong U_g^n$. □

Example 4.9. The graphs $G_{r,s}$, $G_{r+1,s-1}$ and $G_{r-1,s+1}$ with vertex labelling are as shown above.

Lemma 4.10. *Let $G_{r,s}$ be the graph obtained by attaching two paths of length r and s at two adjacent vertices u and v of a connected graph G, respectively. If $\rho_{r,s}$ denotes the distance spectral radius of $G_{r,s}$ then $\rho_{r+1,s-1} + \rho_{r-1,s+1} > 2\rho_{r,s}$.*

Proof. Let $V(G) = \{1, 2, \ldots, m\}$ such that $u = 1$, $v = 2$. Let $\mathcal{D}_{r,s}$ be the distance matrix of $G_{r,s}$. Observe that $\mathcal{D}_{r+1,s-1} + \mathcal{D}_{r-1,s+1} - 2\mathcal{D}_{r,s} = [\hat{d}_{ij}]$, where

$$\hat{d}_{ij} = \begin{cases} 1 & \text{if } i = 1, 2 \text{ and } d_{1j} = d_{2j}, j = 3, \ldots, m \\ 2 & \text{if } d_{1j} < d_{2j}, i = 1, j = 3, \ldots, m \text{ or } d_{1j} > d_{2j}, i = 2, j = 3, \ldots, m \\ 0 & \text{otherwise.} \end{cases}$$

Thus we see that $\mathbf{x}^t(\mathcal{D}_{r+1,s-1} + \mathcal{D}_{r-1,s+1} - 2\mathcal{D}_{r,s})\mathbf{x} > 0$, if $\mathbf{x}$ is the Perron vector of $\mathcal{D}_{r,s}$. Hence the lemma holds. □

As an application of Lemma 4.10, we obtain the following result proved by Zhang and Godsil in [9].

Corollary 4.11. *[9, Theorem 1.1] Let $G_{r,s}$ be the graph obtained by attaching two paths of length r and s at two adjacent vertices u and v of a connected graph G, respectively. Let $\rho^{\mathcal{D}}_{r,s}$ be the distance spectral radius of $G_{r,s}$. Then the following holds:*

(i) $\rho^{\mathcal{D}}_{r,s} < \rho^{\mathcal{D}}_{r+1,s-1}$ *if* $r > s \geq 1$.
(ii) $\rho^{\mathcal{D}}_{r,s} < \rho^{\mathcal{D}}_{r+1,s-1}$ *or* $\rho^{\mathcal{D}}_{r,s} < \rho^{\mathcal{D}}_{r-1,s+1}$ *if* $r = s \geq 1$.

5. Distance Laplacian spectral radius

The following Lemma is similar to Lemma 4.3 for distance matrix.

Lemma 5.1. *Let G_v be the graph obtained by identifying a vertex v of the graph G with a fixed vertex of another graph G'. If i, j, k are consecutive vertices of G such that i and k are on different components of $G - j$ and ρ^L_v denotes the distance Laplacian spectral radius of G_v, then $\rho^L_i + \rho^L_k - 2\rho^L_j \geq 0$. Furthermore, either $\rho^L_i > \rho^L_j$ or $\rho^L_k > \rho^L_j$.*

Proof. Let G' be of order $m + 1$ with w as a fixed vertex. Assume that G_i, G_j, G_k are obtained by identifying the vertices i, j, k of G with the vertex w of G', respectively. Let H_i and H_k be the components of $G - j$ containing i and k respectively. Suppose

$$\begin{aligned} V_1 &= \{u \in G : d_{uk} = d_{ui} = d_{uj} + 1\} \\ V_2 &= \{u \in H_k : d_{uk} = d_{jk} \text{ and } d_{ui} = d_{uj} + 1\} \\ V_3 &= \{u \in H_i : d_{ui} = d_{uj} \text{ and } d_{uk} = d_{uj} + 1\} \\ V_4 &= \{u \in H_k : d_{uk} = d_{uj} - 1 \text{ and } d_{ui} = d_{uj} + 1\} \\ V_5 &= \{u \in H_i : d_{ui} = d_{uj} - 1 \text{ and } d_{uk} = d_{uj} + 1\}. \end{aligned}$$

Then $V(G_j) = V(G'-j) \cup \left(\cup_{i=1}^{5} V_i\right)$ is a partition for $V(G_j)$. If $a = |V_1|$ and $b = |V_2 \cup V_3|$, then by permutation similarity we have

$$\mathcal{D}^L(G_i) + \mathcal{D}^L(G_k) - 2\mathcal{D}^L(G_j) = \left[\begin{array}{c|c|c|c} V(G'-j) & V_1 & V_2 \cup V_3 & V_4 \cup V_5 \\ \hline (2a+b)I_m & -2\mathbb{J}_{m\times a} & -\mathbb{J}_{m\times b} & 0 \\ \hline -2\mathbb{J}_{a\times m} & 2mI_a & 0 & 0 \\ \hline -\mathbb{J}_{b\times m} & 0 & mI_b & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right].$$

Let $\mathbf{x}$ be an eigenvector corresponding to spectral radius of $\mathcal{D}^L(G_j)$. Observe that

$$\begin{aligned} \rho_i^L + \rho_k^L - 2\rho_j^L &\geq \rho\{\mathcal{D}^L(G_j) + \mathcal{D}^L(G_k)\} - 2\rho_j^L \text{ [by Lemma 4.1]} \\ &\geq \mathbf{x}^t \left\{\mathcal{D}^L(G_j) + \mathcal{D}^L(G_k) - 2\mathcal{D}^L(G_j)\right\} \mathbf{x} \text{ [by Lemma 4.2]} \\ &= \sum_{p\in V(G'-j)} \left[2\sum_{q\in V_1} (\mathbf{x}_p - \mathbf{x}_q)^2 + \sum_{q\in V_2\cup V_3} (\mathbf{x}_p - \mathbf{x}_q)^2\right] \geq 0. \quad (5.20) \end{aligned}$$

To the contrary, assume that $\rho_i^L = \rho_j^L = \rho_k^L = \rho$ (say). Then above inequalities hold with equality. Therefore by (5.20), $\mathbf{x}$ is also an eigenvector of $\mathcal{D}^L(G_i)$ and $\mathcal{D}^L(G_k)$ corresponding to ρ and $\mathbf{x}_p = \mathbf{x}_q$ for $p \in V(G'-j)$, $q \in V_1 \cup V_2 \cup V_3$. Thus

$$\{\mathcal{D}^L(G_j) - \mathcal{D}^L(G_i)\}\mathbf{x} = 0 \quad (5.21)$$

$$\text{and} \quad \mathbf{x}_p = \mathbf{x}_w \text{ for all } p \in V(G'-j) \cup V_1 \cup V_2 \cup V_3. \quad (5.22)$$

Let $k = |V_4| - |V_5|$, then we have

$$\mathcal{D}^L(G_j) - \mathcal{D}^L(G_i) = \left[\begin{array}{c|c|c|c} V(G'-j) & V_1 \cup V_2 \cup V_3 & V_4 & V_5 \\ \hline kI_m & 0 & \mathbb{J} & -\mathbb{J} \\ \hline 0 & 0 & 0 & 0 \\ \hline \mathbb{J} & 0 & -mI & 0 \\ \hline -\mathbb{J} & 0 & 0 & mI \end{array}\right].$$

Now for $u \in V_4 \cup V_5$, using (5.22) in the u-th equation of (5.21) we have $\mathbf{x}_u = \mathbf{x}_w$ for all $u \in V_4 \cup V_5$. Thus $\mathbf{x} = \mathbf{x}_w \mathbb{1}$, which implies $\rho = 0$, a contradiction. Hence either $\rho_i^L \neq \rho_j^L$ or $\rho_k^L \neq \rho_j^L$. □

Corollary 5.2. *Let T be a tree and G' be a graph. If T_i and ρ_i^L are as defined in Lemma 5.1, then ρ_i^L is strictly quasiconvex on T.*

For the distance Laplacian we obtain the following results similar to that of Corollary 4.5 and that of [7, Lemma 3.3] as an immediate corollary to Corollary 5.2.

Corollary 5.3. *Let G be a simple graph and v be one of its vertices. If $k \geq \ell \geq 1$, then $\rho^L_{v,k,\ell} < \rho^L_{v,k+1,\ell-1}$.*

Theorem 5.4. *Let T be a tree with maximal distance Laplacian spectral radius in $\mathcal{T}_n^m$. Then $T \cong D(n, a, b)$, $a + b = n + 1 - 2m$.*

Proof. If $m = 1$, then the result is trivial. Suppose $m \geq 2$, and let T be a tree in $\mathcal{T}_n^m$ with maximal distance Laplacian spectral radius. Then T has 2 quasipendent vertices. Otherwise, there exists a vertex v in T such that $T - v$ has at least 3 components, and at least two of which are nontrivial. Let T_x, T_y, T_z be three branches at the vertex v of T, and M be an m-matching of T. Then at least two of vx, vy and vz are not in M.

Case 1: Suppose T_x and T_y are non-trivial and vz is not in M. Let x_1, y_1 be quasipendent vertices in T_x, T_y respectively. Then the trees $T' = T - vz + x_1z$ and $T'' = T - vz + y_1z$ are in $\mathcal{T}_n^m$, and by Corollary 5.2, $\rho^L(T') > \rho^L(T)$ or $\rho^L(T'') > \rho^L(T)$, a contradiction to the maximality of T.

Case 2: Suppose vz is in M and T_z is non-trivial. Then both vx and vy are not in M, and say T_x is non-trivial. Reversing the roles of y and z in Case 1, we have a contradiction.

Case 3: Suppose vz is in M and T_z is trivial. Since T_y is nontrivial therefore there exists a vertex u in T_y which is in a matching. Then both the trees $T' = T - vx + zx$ and $T'' = T - vx + ux$ have matching number m, but by Corollary 5.2 at least one has a larger distance Laplacian spectral radius than T, a contradiction to the maximality of T. □

Theorem 5.5. *Let T be a tree with maximal distance Laplacian spectral radius in $\mathcal{T}_{n,k}$. Then $T \cong D(n, a, b)$, $a + b = k$.*

Proof. Let $P := i_1 - i_2 - \cdots - i_m$ be a path of maximum length in T. If $m = 1, 2, 3$, then there is nothing to prove. Suppose $m \geq 4$. Then degrees of $i_3, \ldots, i_{m-2}$ in T are all 2. To the contrary assume that $deg(i_j) \geq 3$. Let T' be the component of $T - i_{j-1} - i_{j+1}$ containing vertex i_j. Let T_1, T_2 be the new trees obtained by shifting T' to i_{j-1} and i_{j+1}, respectively. By Lemma 5.1, the distance Laplacian spectral radius of T_1 or T_2 is bigger than that of T, a contradiction. Hence the Theorem holds. □

Let $\mathcal{B}_{n,\Delta}$ denote the class of all trees with n vertices and maximum vertex degree Δ. The broom, denoted by $B_{n,\Delta}$ is the tree obtained from the star $S_{\Delta+1}$ and path $P_{n-\Delta-1}$ by attaching one end of the path to one of the pendent vertex of the star (see Fig. 4).

Stevanović, and Ilić in [7] proved that among all trees with fixed maximum degree Δ, the broom tree $B_{n,\Delta}$ has maximal distance spectral radius. The next result says that broom is the unique tree that maximize the distance Laplacian spectral radius among all trees with fixed maximum degree.

Theorem 5.6. *Among all trees in $\mathcal{B}_{n,\Delta}$ the unique tree that maximizes the distance Laplacian spectral radius is $B_{n,\Delta}$.*

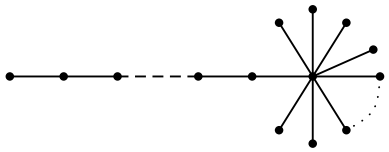


Fig. 4. The broom $B_{n,\Delta}$.

Proof. Let T be tree with maximum distance Laplacian spectral radius among all trees in $\mathcal{B}_{n,\Delta}$. Let v be a vertex of degree Δ and $T_1, \ldots, T_\Delta$ be the trees attached at v. By Corollary 5.3, it follows that each T_i is a path for $i = 1, \ldots, \Delta$. Without loss of generality we assume that the path T_1 has maximum length. By Corollary 5.3, it follows that length of each path $T_2, \ldots, T_\Delta$ is exactly one. Hence $T \cong B_{n,\Delta}$. □

Lemma 5.7. *Let $G_{r,s}$ be the graph obtained by attaching two paths of length r and s at two adjacent vertices u and v of a graph G, respectively. If $\rho^L_{r,s}$ denotes the distance Laplacian spectral radius of $G_{r,s}$ then $\rho^L_{r+1,s-1} + \rho^L_{r-1,s+1} \geq 2\rho^L_{r,s}$. Furthermore, either $\rho^L_{r+1,s-1} > \rho^L_{r,s}$ or $\rho^L_{r-1,s+1} > \rho^L_{r,s}$.*

Proof. Let $V(G) = \{1, 2, \ldots, m\}$ such that $u = 1$, $v = 2$. Let $\mathcal{D}^L_{r,s}$ be the distance Laplacian matrix of $G_{r,s}$. Let $V_1 = \{j \in V(G) - \{1,2\} | d_{1j} = d_{2j}\}$, $V_2 = \{j \in V(G) - \{1,2\} | d_{1j} < d_{2j}\}$, $V_3 = \{j \in V(G) - \{1,2\} | d_{1j} > d_{2j}\}$ and a, b, c be the number of vertices in V_1, V_2 and V_3, respectively. With respect to the ordered partition $V(G_{r,s}) = \{1,2\} \cup V_1 \cup V_2 \cup V_3 \cup (V(G_{r,s}) - V(G))$ we have

$$\mathcal{D}^L_{r+1,s-1} + \mathcal{D}^L_{r-1,s+1} - 2\mathcal{D}^L_{r,s} = \left[\begin{array}{cc|c|c|c|c} 1 & 2 & V_1 & V_2 & V_3 & V(G_{r,s}) - V(G) \\ \hline a+2b & 0 & -\mathbb{1}^t & -2\mathbb{1}^t & 0 & 0 \\ 0 & a+2c & -\mathbb{1}^t & 0 & -2\mathbb{1}^t & 0 \\ \hline -\mathbb{1} & -\mathbb{1} & 2I_a & 0 & 0 & 0 \\ \hline -2\mathbb{1} & 0 & 0 & 2I_b & 0 & 0 \\ 0 & -2\mathbb{1} & 0 & 0 & 2I_c & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Let $\mathbf{x}$ be an eigenvector corresponding to the spectral radius of $\mathcal{D}^L_{r,s}$. Observe that

$$\begin{aligned} \rho^L_{r+1,s-1} + \rho^L_{r-1,s+1} - 2\rho^L_{r,s} &\geq \rho(\mathcal{D}^L_{r+1,s-1} + \mathcal{D}^L_{r-1,s+1}) - 2\rho^L_{r,s} \text{ [by Lemma 4.1]} \\ &\geq \mathbf{x}^t(\mathcal{D}^L_{r+1,s-1} + \mathcal{D}^L_{r-1,s+1} - 2\mathcal{D}^L_{r,s})\mathbf{x} \text{ [by Lemma 4.2]} \\ &= 2\left[\sum_{j\in V_2}(\mathbf{x}_1 - \mathbf{x}_j)^2 + \sum_{j\in V_3}(\mathbf{x}_2 - \mathbf{x}_j)^2\right] \\ &\quad + \sum_{i=1}^{2}\sum_{j\in V_1}(\mathbf{x}_i - \mathbf{x}_j)^2 \geq 0. \end{aligned} \tag{5.23}$$

To the contrary, assume that $\rho_{r-1,s+1} = \rho_{r+1,s-1} = \rho_{r,s} = \rho$ (say). Then $\mathbf{x}$ is also an eigenvector of $\mathcal{D}^L_{r+1,s-1}$ and $\mathcal{D}^L_{r-1,s+1}$ corresponding to ρ. Label the vertices of $G_{r,s}$, $G_{r+1,s-1}$ and $G_{r-1,s+1}$ as in Fig. 3. We see that

$$\mathcal{D}^L_{r+1,s-1} - \mathcal{D}^L_{r,s} = \left[\begin{array}{cc|c|c|c} 1 & 2 & \overbrace{3,\ldots,m} & \overbrace{m+1,\ldots,m+r-1} & \overbrace{m+r,\ldots,n} \\ \hline m-2 & 0 & -\mathbb{1}^t & 0^t & 0^t \\ 0 & 0 & z^t & 0^t & 0^t \\ \hline -\mathbb{1} & z & C & -\mathbb{J} & \mathbb{J} \\ \hline 0 & 0 & -\mathbb{J}^t & (m-2)I & 0 \\ 0 & 0 & \mathbb{J}^t & 0 & -(m-2)I \end{array}\right].$$

Now we have

$$(\mathcal{D}^L_{r+1,s-1} - \mathcal{D}^L_{r,s})\mathbf{x} = 0. \tag{5.24}$$

From the first equation in (5.24), we have

$$(m-2)\mathbf{x}_1 = \sum_{j=3}^{m} \mathbf{x}_j. \tag{5.25}$$

Again from the i-th equation in (5.24), we have

$$(m-2)\mathbf{x}_i = \sum_{j=3}^{m} \mathbf{x}_j, \text{ for } i = m+1, m+2, \ldots, n. \tag{5.26}$$

From the equality in (5.23), we have

$$\mathbf{x}_1 = \mathbf{x}_j \text{ for } d_{2j} \leq d_{1j}, j = 3, \ldots, m. \tag{5.27}$$

From equations (5.25), (5.26) and (5.27), we have

$$\mathbf{x}_1 = \mathbf{x}_{m+1} = \mathbf{x}_{m+2} = \ldots = \mathbf{x}_n = \mathbf{x}_j \text{ for } d_{2j} \leq d_{1j}, j = 3, \ldots, m. \tag{5.28}$$

Similarly, considering the matrix $\mathcal{D}^L_{r-1,s+1} - \mathcal{D}^L_{r,s}$ we can show that

$$\mathbf{x}_2 = \mathbf{x}_{m+1} = \mathbf{x}_{m+2} = \ldots = \mathbf{x}_n = \mathbf{x}_j \text{ for } d_{1j} \leq d_{2j}, j = 3, \ldots, m. \tag{5.29}$$

By equations (5.28) and (5.29), we see that $\mathbf{x} = \alpha\mathbb{1}$ for some $\alpha \neq 0$, which implies that $\rho = 0$, a contradiction. Hence either $\rho^L_{r+1,s-1} > \rho^L_{rs}$ or $\rho^L_{r-1,s+1} > \rho^L_{r,s}$. □

Lemma 5.8. *Let U_0 be a graph obtained from a unicyclic graph by attaching a pendent path P at a vertex p of the cycle. Let $U_1 := U_0 - \{p,r\} + \{q,r\}$ and $U_2 := U_0 - \{p,r\} + \{r,s\}$, where q, r are vertices on the cycle adjacent to p and s on the path adjacent to p. If*

ρ_i^L denotes the Laplacian spectral radius of U_i for $i = 0, 1, 2$ then $\rho_1^L + \rho_2^L - 2\rho_0^L \geq 0$. Furthermore, either $\rho_1^L > \rho_0^L$ or $\rho_2^L > \rho_0^L$.

Proof. Let $V_1 = \{j \in V(U_0 - P) : d_{pj} \leq d_{qj}\}$, and $V_2 = V(U_0) - V_1 - \{p\}$. Let $\mathcal{D}_i^L$ be the distance Laplacian matrix of U_i for $i = 0, 1, 2$. With respect to the ordered partition $\{p\} \cup V_1 \cup V_2$ of $V(U_0)$, we have

$$\mathcal{D}_1^L + \mathcal{D}_2^L - 2\mathcal{D}_0^L = \begin{bmatrix} 2a & -2\mathbb{1}_a^t & 0 \\ -2\mathbb{1}_a & 2I_a & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ where } a = |V_1|.$$

Let $\mathbf{x}$ be an eigenvector corresponding to the spectral radius of $\mathcal{D}_0^L$. We see that

$$\begin{aligned} \rho_1^L + \rho_2^L - 2\rho_0^L &\geq \rho(\mathcal{D}_1^L + \mathcal{D}_2^L) - 2\rho_0^L \\ &\geq \mathbf{x}^t(\mathcal{D}_1^L + \mathcal{D}_2^L - 2\mathcal{D}_0^L)\mathbf{x} \\ &= 2\sum_{j \in V_1} (\mathbf{x}_p - \mathbf{x}_j)^2 \geq 0. \end{aligned}$$

Thus $\rho_1^L + \rho_2^L - 2\rho_0^L \geq 0$. If $\rho_1^L = \rho_2^L = \rho_0^L$, then with similar arguments as in the proof of Lemma 5.1 we can prove that $\mathbf{x} = \mathbf{0}$, which is a contradiction. Hence the Lemma holds. □

Lemma 5.9. *Let $C_n = [1, \ldots, \lceil \frac{n+1}{2} \rceil, n, n-1, \ldots, \lceil \frac{n+1}{2} \rceil + 1]$ be a cycle. Then the following holds.*

(i) The distance Laplacian spectral radius of C_n has multiplicity two.
(ii) If $n = 2k + 1$, then an eigenvector corresponding to spectral radius of $\mathcal{D}^L(C_n)$ is of the form $\begin{bmatrix} 0 \\ x \\ -x \end{bmatrix}$, where x is positive and strictly decreases along the two paths starting at $\lceil \frac{k+3}{2} \rceil$ and $\mathbf{x}_i > \mathbf{x}_{k-i+3}$ for $i = 2, 3, \ldots, \lceil \frac{k+1}{2} \rceil$.
(iii) If $n = 2k$, then an eigenvector corresponding to spectral radius of $\mathcal{D}^L(C_n)$ is of the form $\begin{bmatrix} x \\ -x \end{bmatrix}$, where x is positive and strictly decreases along the two paths starting at $\lceil \frac{k+1}{2} \rceil$ and $\mathbf{x}_i = \mathbf{x}_{k-i+1}$ for $i = 1, 2, \ldots, \lceil \frac{k}{2} \rceil$.

Proof. Let $n = 2k + 1$. Then we observe that

$$\mathcal{D}^L(C_{2k+1}) = \begin{bmatrix} k^2 + k & z^T & z^T \\ z & A & B \\ z & B & A \end{bmatrix},$$

where

$$z^t = [-1 \quad -2 \ldots \quad -k],\ A = \begin{bmatrix} k^2+k & -1 & -2 & \ldots & -(k-1) \\ -1 & k^2+k & -1 & \ldots & -(k-2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -(k-1) & -(k-2) & -(k-3) & \ldots & k^2+k \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} -2 & -3 & -4 & \ldots & -(k-1) & -k & -k \\ -3 & -4 & -5 & \ldots & -k & -k & -(k-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ -k & -k & -(k-1) & \ldots & -4 & -3 & -2 \\ -k & -(k-1) & -(k-2) & \ldots & -3 & -2 & -1 \end{bmatrix}.$$

Also, we see that for some permutation matrix P,

$$\mathcal{D}^L(C_{2k+1}) = \left[\begin{array}{c|c} P^T \begin{pmatrix} A & B \\ B & A \end{pmatrix} P & \begin{matrix} z_1 \\ z_2 \end{matrix} \\ \hline \begin{matrix} z_1^T & z_2^T \end{matrix} & k^2+k \end{array}\right],$$

where $z_1^t = [-k \quad -k \quad -(k-1) \quad \ldots \quad -2]$, $z_2^t = [-1 \quad -(k-1) \quad -(k-2) \quad \ldots \quad -1]$.

Thus it follows that if λ is an eigenvalue of $A-B$ with an eigenvector x, then $\begin{bmatrix} 0 \\ x \\ -x \end{bmatrix}$ and $\begin{bmatrix} Px \\ -Px \\ 0 \end{bmatrix}$ are two linearly independent eigenvectors of $\mathcal{D}^L(C_{2k+1})$ corresponding to the eigenvalue λ. Since $A-B$ has k eigenvalues and 0 is not an eigenvalue of $A-B$, we see that the spectral radius of $A-B$ is the spectral radius of $\mathcal{D}^L(C_{2k+1})$. Moreover, $A-B$ is positive. Thus by Perron–Frobenius theorem x is positive.

For $2 \le i \le \lceil \frac{k+1}{2} \rceil$, from eigen-equations we have

$$\left[\rho^L(C_{2k+1}) - k^2 - k\right](\mathbf{x}_{i+1} - \mathbf{x}_i) = 2\sum_{j=i+1}^{k-i+1} \mathbf{x}_j + \mathbf{x}_{k-i+2} > 0, \tag{5.30}$$

$$\left[\rho^L(C_{2k+1}) - k^2 - k\right](\mathbf{x}_i - \mathbf{x}_{k-i+3}) = \sum_{j=i}^{k-i+2} \mathbf{x}_j > 0 \tag{5.31}$$

$$\text{and } \left[\rho^L(C_{2k+1}) - k^2 - k\right](\mathbf{x}_{k-i+2} - \mathbf{x}_{k-i+3}) = \mathbf{x}_i + 2\sum_{j=i+1}^{k-i+2} \mathbf{x}_j > 0. \tag{5.32}$$

Since $\rho^L(C_{2k+1}) > k^2+k$, (ii) follows from (5.30)–(5.32). Similarly we can prove (iii). □

Lemma 5.10. *Let C_n be a cycle. Then $\rho^L(U_3^n) > \rho^L(C_n)$.*

Proof. Let $C_n = [1, \ldots, \lceil \frac{n+1}{2} \rceil, n, \ldots, \lceil \frac{n+1}{2} \rceil + 1]$. Remove the edge $\{1, \lceil \frac{n+1}{2} \rceil + 1\}$ and add the edge $\{1, 3\}$ to obtain U_3^n and then observe that distances between 1 and i decreases by 1 for $i = 3, \ldots, \lceil \frac{n+1}{2} \rceil$, increases by at least 1 for $i = \lceil \frac{n+1}{2} \rceil, \ldots, n$ and distance among

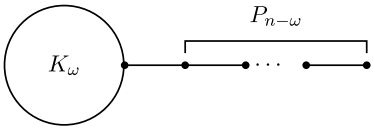


Fig. 5. The graph $PK_{n-\omega,\omega}$.

all other vertices do not decrease. Let $\mathbf{x}$ be an eigenvector of $\mathcal{D}^L(C_n)$ corresponding to the spectral radius. Then we see that

$$\begin{aligned}
\rho^L(U_3^n) - \rho^L(C_n) &\geq \mathbf{x}^t(\mathcal{D}^L(U_3^n) - \mathcal{D}^L(C_n))\mathbf{x} \text{ [by Lemma 4.2]} \\
&\geq \sum_{i=\lceil \frac{n+1}{2} \rceil + 1}^{n-1} (\mathbf{x}_1 - \mathbf{x}_i)^2 - \sum_{i=3}^{\lceil \frac{n+1}{2} \rceil} (\mathbf{x}_1 - \mathbf{x}_i)^2 \\
&\geq \sum_{i=3}^{\lceil \frac{n+1}{2} \rceil - 1} \left[(\mathbf{x}_1 + \mathbf{x}_i)^2 - (\mathbf{x}_1 - \mathbf{x}_i)^2 \right] \\
&+ (\mathbf{x}_1 + \mathbf{x}_2)^2 - (\mathbf{x}_1 - \mathbf{x}_{\lceil \frac{n+1}{2} \rceil})^2, \text{ using Lemma 5.9} \\
&= 4\mathbf{x}_1 \sum_{i=3}^{\lceil \frac{n+1}{2} \rceil - 1} \mathbf{x}_i + 2\mathbf{x}_1(\mathbf{x}_2 + \mathbf{x}_{\lceil \frac{n+1}{2} \rceil}) + {\mathbf{x}_2}^2 - {\mathbf{x}_{\lceil \frac{n+1}{2} \rceil}}^2 \\
&> 0, \text{ as } \mathbf{x}_2 > \mathbf{x}_{\lceil \frac{n+1}{2} \rceil} \text{ by Lemma 5.9.}
\end{aligned}$$

Hence $\rho^L(U_3^n) > \rho^L(C_n)$. □

Theorem 5.11. *The unique graph that maximizes the distance Laplacian spectral radius among all unicyclic graph on n vertices is U_3^n.*

Proof. Let U be the unicyclic graph that maximizes the distance Laplacian spectral radius among all unicyclic graph of order n. By Lemma 5.1, we see that every tree branch of U attached at a vertex of the cycle is a path and by Lemma 5.7, U has no tree branch at two adjacent vertices of cycle. By repeated application of Lemma 5.8, we see that U is either C_n or U_g^n for some g. By Lemma 5.10, we see that $\rho^L(U_3^n) > \rho^L(C_n)$. Thus $U \ncong C_n$. If $U \cong U_g^n$, then by Lemma 5.8, we see that either $\rho^L(U_{g-1}^n) > \rho^L(U)$ or $\rho^L(U_{g+1}^n) > \rho^L(U)$. Hence the Theorem holds. □

A *clique* of a graph is a maximal complete subgraph and *clique number* of a graph is the order of a maximal clique. Let $\mathbf{G}_n^\omega$ be the class of all graphs of order n with clique number ω. Here, we find the graph with maximal distance Laplacian spectral radius in $\mathbf{G}_n^\omega$, where $2 \leq \omega \leq n$. A *kite graph*, denoted by $PK_{n-\omega,\omega}$, is a graph obtained by attaching a path $P_{n-\omega}$ to a vertex of a complete graph K_ω [see Fig. 5]. Clearly, $PK_{n-\omega,\omega} \in \mathbf{G}_n^\omega$.

Theorem 5.12. *If $G \in \mathbf{G}_n^\omega$, then $\rho^L(G) \leq \rho^L(PK_{n-\omega,\omega})$. The equality holds if and only if G is isomorphic to $PK_{n-\omega,\omega}$.*

Proof. If $n = \omega$, then $\mathbf{G}_n^\omega = \{K_n\}$. Since $K_n = PK_{0,n}$, so the result holds in this case. Therefore we consider $n > \omega \geq 2$.

(i) If we delete an edge (if any) from G which is not in K but on a cycle, then the resulting graph is still in $\mathbf{G}_n^\omega$. Let G_1 be a spanning subgraph of G obtained by deleting such edges (if any) in an arbitrary order as long as they exist. Thus, G_1 consists of a clique K with a number of rooted trees attached to clique vertices. Since deletion of an edge from a graph does not decrease its distance Laplacian spectral radius, so $\rho^L(G_1) \geq \rho^L(G)$.

(ii) Let $T_1, T_2, \ldots, T_\ell$ be maximal disjoint trees attached at a vertex v of K, in G_1. We can repeatedly apply the transformation from Lemma 5.1 at any vertex of degree at least three with the largest eccentricity from v in every tree T_i, as long as T_i does not become a path. Each application of this transformation strictly increases the distance Laplacian spectral radius.

Now we can again apply the transformation from Lemma 5.1 at the vertex v as long as there exist at least two paths of length at least one, further increasing the distance Laplacian spectral radius.

(iii) If $\omega = 2$, then $G_2 = P_n = PK_{n-2,2}$, and the result holds in this case. Otherwise, let G_2 consist of clique K and the paths $P_{\ell_1}, P_{\ell_2}, \ldots, P_{\ell_m}$ attached to m distinct vertices of K. With the repeated use of Lemma 5.7 to these paths, we finally arrive at the kite graph $PK_{n-\omega,\omega}$. By Lemma 5.7, the distance Laplacian spectral radius is increased at each step.

Therefore, we may conclude that the kite graph $PK_{n-\omega,\omega}$ has larger distance Laplacian spectral radius than any other graph in $\mathbf{G}_n^\omega$. This completes the proof. □

6. Distance signless Laplacian spectral radius

The following Lemma is similar to Lemma 4.3 and therefore its proof is omitted.

Lemma 6.1. *Let G_v be the graph obtained by identifying a vertex v of the graph G with a fixed vertex u of another graph G'. If i, j, k are consecutive vertices of G such that i and k are on different components of $G - j$ and $\rho_v^{\mathcal{Q}}$ denotes the distance signless Laplacian spectral radius of G_v, then $\rho_i^{\mathcal{Q}} + \rho_k^{\mathcal{Q}} - 2\rho_j^{\mathcal{Q}} > 0$.*

Corollary 6.2. *Let T be a tree and G be a graph. If T_i and $\rho_i^{\mathcal{Q}}$ are as defined in the above Lemma then $\rho_i^{\mathcal{Q}}$ is strictly convex on T.*

The following result similar to that of Corollary 4.5 and Corollary 5.3 follows immediately from Corollary 6.2.

Corollary 6.3. *Let G be a simple graph and v be one of its vertices. If $k \geq \ell \geq 1$, then $\rho_{v,k,\ell}^{\mathcal{Q}} < \rho_{v,k+1,\ell-1}^{\mathcal{Q}}$.*

Theorem 6.4. *Let T be a tree with maximal distance signless Laplacian spectral radius in $\mathcal{T}_n^m$. Then $T \cong D(n, a, b)$, $a + b = n + 1 - 2m$.*

Proof. If $m = 1$, then the result is trivial. Suppose $m \geq 2$, and let T be a tree in $\mathcal{T}_n^m$ with maximal distance signless Laplacian spectral radius. Then T has 2 quasipendent vertices. Otherwise, there exists a vertex v in T such that $T - v$ has at least 3 components, and at least two of which are nontrivial. Let T_x, T_y, T_z be three branches at the vertex v of T, and M be an m-matching of T. Then at least two of vx, vy and vz are not in M.

Case 1: Suppose T_x and T_y are non-trivial and vz is not in M. Let x_1, y_1 be quasipendent vertices in T_x, T_y respectively. Then the trees $T' = T - vz + x_1z$ and $T'' = T - vz + y_1z$ are in $\mathcal{T}_n^m$, and by Corollary 6.2, $\rho^{\mathcal{Q}}(T') > \rho^{\mathcal{Q}}(T)$ or $\rho^{\mathcal{Q}}(T'') > \rho^{\mathcal{Q}}(T)$, a contradiction to the maximality of T.

Case 2: Suppose vz is in M and T_z is non-trivial. Then both vx and vy are not in M, and say T_x is non-trivial. Reversing the roles of y and z in Case 1, we have a contradiction.

Case 3: Suppose vz is in M and T_z is trivial. Since T_y is nontrivial therefore there exists a vertex u in T_y which is in a matching. Then both the trees $T' = T - vx + zx$ and $T'' = T - vx + ux$ have matching number m, but by Corollary 6.2 at least one has a larger distance signless Laplacian spectral radius than T, a contradiction to the maximality of T.

Thus $T \cong D(n, a, b)$, for some a and b. □

Theorem 6.5. *Let T be a tree with maximal distance signless Laplacian spectral radius in $\mathcal{T}_{n,k}$. Then $T \cong D(n, a, b)$, $a + b = k$.*

Proof. Let $P = i_1 - i_2 - \cdots - i_k$ be a path of maximum length in T. If $m = 1, 2, 3$, then there is nothing to prove. Suppose $m \geq 4$. Then degrees of $i_3, \ldots, i_{m-2}$ in T are all 2. To the contrary assume that $deg(i_j) \geq 3$. Let T' be the component of $T - i_{j-1} - i_{j+1}$ containing vertex i_j. Let T_1, T_2 be the new trees obtained by shifting T' to i_{j-1} and i_{j+1}, respectively. By Lemma 6.1, the distance Laplacian spectral radius of T_1 or T_2 is bigger than that of T, a contradiction. Hence the Theorem holds. □

The next result says that broom is the unique tree that maximizes the distance signless Laplacian spectral radius among all trees with fixed maximum degree.

Theorem 6.6. *Among all trees in $\mathcal{B}_{n,\Delta}$ the unique tree that maximizes the distance signless Laplacian spectral radius is $B_{n,\Delta}$.*

Proof. Let T be a tree with maximum distance Laplacian spectral radius among all trees in $\mathcal{B}_{n,\Delta}$. Let v be a vertex of degree Δ and $T_1, \ldots, T_\Delta$ be the trees attached at v. By Corollary 6.3, it follows that each T_i is a path for $i = 1, \ldots, \Delta$. Without loss of generality we assume that the path T_1 has maximum length. By Corollary 6.3, it follows that length of each path $T_2, \ldots, T_\Delta$ is exactly one. Hence $T \cong B_{n,\Delta}$. □

The following Lemma is similar to Lemma 4.10 and Lemma 5.7 whose proof also follows similarly.

Lemma 6.7. *Let $G_{r,s}$ be the graph obtained by attaching two paths of length r and s at two adjacent vertices u and v of a connected graph G, respectively. Let $\rho^{\mathcal{Q}}_{r,s}$ be the distance Laplacian spectral radius of $G_{r,s}$. Then*

$$\rho^{\mathcal{Q}}_{r+1,s-1} + \rho^{\mathcal{Q}}_{r-1,s+1} > 2\rho^{\mathcal{Q}}_{r,s}.$$

Lemma 6.8. *Let U be a unicyclic graph with the cycle $C = [1, 2 \ldots, m]$. Let T_1 and T_2 be two tree branches at the vertices 1 and k of C such that $deg(i) = 2$ for $i = 2, \ldots, k-1$. Let $\mathbf{x}$ be the Perron vector of the distance signless Laplacian of U then $-\mathbf{x}$ induces a quasiconvex function on $\{1, \ldots k\}$.*

Proof. Since $Tr(i) + Tr(i+2) \leq 2Tr(i+1)$, we see that

$$2\rho^{\mathcal{Q}} - Tr(i) - Tr(i+2) \geq 2(\rho^{\mathcal{Q}} - Tr(i+1)) \text{ for } 1 \leq i \leq k-2.$$

By eigenequation of $\mathbf{x}$ at $i, i+1, i+2$ we have

$$\begin{aligned}
&(\rho^{\mathcal{Q}} - Tr(i))\mathbf{x}_i + (\rho^{\mathcal{Q}} - Tr(i+2))\mathbf{x}_{i+2} - 2(\rho^{\mathcal{Q}} - Tr(i+1))\mathbf{x}_{i+1} < 0 \\
\Rightarrow\ &(\rho^{\mathcal{Q}} - Tr(i))\mathbf{x}_i + (\rho^{\mathcal{Q}} - Tr(i+2))\mathbf{x}_{i+2} - (2\rho^{\mathcal{Q}} - Tr(i) - Tr(i+2))\mathbf{x}_{i+1} < 0 \\
\Rightarrow\ &\lambda\mathbf{x}_i + (1-\lambda)\mathbf{x}_{i+2} < \mathbf{x}_{i+1}, \text{ where } \lambda = \frac{\rho^{\mathcal{Q}} - Tr(i)}{2\rho^{\mathcal{Q}} - Tr(i) - Tr(i+2)}.
\end{aligned}$$

Thus $\mathbf{x}_{i+1} > \min\{\mathbf{x}_i, \mathbf{x}_{i+2}\}$ and the proof is complete. □

Lemma 6.9. *Let U be a unicyclic graph obtained by attaching a path $P := 1-2-\cdots-k$ at the vertex k of the cycle $C := [k, \ldots, n]$. Let $\mathbf{x}$ be the Perron vector of the distance signless Laplacian of U. Then $\mathbf{x}$ induces a strictly convex function on $\{1, 2, \ldots, k, \ldots, k + \lceil \frac{g}{2} \rceil\}$.*

Proof. For our convenience let $t = k + \lceil \frac{g}{2} \rceil$. By eigenequation of $\mathbf{x}$ at t and $t-1$, we have

$$\rho^{\mathcal{Q}}(\mathbf{x}_t - \mathbf{x}_{t-1}) = \sum_{j=1}^{t-2} \mathbf{x}_j - \sum_{j=t+1}^{t+\lceil \frac{g}{2} \rceil} \mathbf{x}_j + Tr(t)\mathbf{x}_t - Tr(t-1)\mathbf{x}_{t-1}.$$

Thus we have the following:

$$(\rho^{\mathcal{Q}} - Tr(t) + 1)\mathbf{x}_t - (\rho^{\mathcal{Q}} - Tr(t-1) - 1)\mathbf{x}_{t-1} > 0.$$

Since $\rho^{\mathcal{Q}} - Tr(t) + 1 < \rho^{\mathcal{Q}} - Tr(t-1) - 1$, we see that $\mathbf{x}_t > \mathbf{x}_{t-1}$.

By Lemma 6.8, it follows that $-[\mathbf{x}_k, \ldots, \mathbf{x}_{k+\lceil \frac{g}{2} \rceil}]$ is convex. Thus $\mathbf{x}_{k+\lceil \frac{g}{2} \rceil} > \cdots > \mathbf{x}_k$. Similarly we can show that $[\mathbf{x}_1, \ldots, \mathbf{x}_{k+1}]$ is strictly convex. Using eigenequation we can show that $\mathbf{x}_1 > \mathbf{x}_2$. Hence the lemma holds. □

Lemma 6.10. *Let C_n be a cycle. Then $\rho^{\mathcal{Q}}(U_3^n) > \rho^{\mathcal{Q}}(C_n)$.*

Proof. Observe that $\mathbb{1}$ is a Perron vector of $\mathcal{D}^Q(C_n)$ and

$$\mathbb{1}^t \left\{ \mathcal{D}^Q(U_3^n) - \mathcal{D}^Q(C_n) \right\} \mathbb{1} = 2 \left\{ \sum_{u \in V(U_3^n)} Tr(u) - \sum_{u \in V(C_n)} Tr(u) \right\} > 0.$$

By Lemma 4.2 we have $\rho^{\mathcal{Q}}(U_3^n) > \rho^{\mathcal{Q}}(C_n)$. □

The following Lemma is similar to Lemma 5.8 whose proof also follows similarly.

Lemma 6.11. *Let U_0 be a graph obtained from a unicyclic graph by attaching a pendent path P at a vertex p of the cycle. Let $U_1 := U_0 - \{p, r\} + \{q, r\}$ and $U_2 := U_0 - \{p, r\} + \{r, s\}$, where q, r are vertices on the cycle adjacent to p and s on the path adjacent to p. If $\rho_i^{\mathcal{Q}}$ denotes the distance signless Laplacian spectral radius of U_i for $i = 0, 1, 2$ then $\rho_1^{\mathcal{Q}} + \rho_2^{\mathcal{Q}} - 2\rho_0^{\mathcal{Q}} \geq 0$. Furthermore, either $\rho_1^{\mathcal{Q}} > \rho_0^{\mathcal{Q}}$ or $\rho_2^{\mathcal{Q}} > \rho_0^{\mathcal{Q}}$.*

Theorem 6.12. *The unique graph that maximizes the distance signless Laplacian spectral radius among all unicyclic graph is U_3^n.*

Proof. Let U be a unicyclic graph with maximum distance signless Laplacian spectral radius in the class of all unicyclic graph. By Lemma 6.1, we see that every tree branch of U attached at a vertex of the cycle is a path and by Lemma 6.7, U has no path branch at two adjacent vertices of the cycle. By repeated application of Lemma 6.11 and 6.7, we see that U is either C_n or U_3^n. Hence by Lemma 6.10 the Theorem holds. □

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Convex and quasiconvex functions on trees and their applications

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ABSTRACT

We introduce convex and quasiconvex functions on trees and prove that for a tree the eccentricity, transmission and weight functions are strictly quasiconvex. It is shown that the Perron vector of the distance matrix is strictly convex whereas the Perron vector of the distance signless Laplacian is quasiconvex for a tree. In the class of all trees with a given number of pendent vertices, we prove that the distance Laplacian and distance signless Laplacian spectral radius are both maximized at a dumbbell. Among all trees with fixed maximum degree, we prove that the broom is the unique tree that maximizes the distance Laplacian and distance signless Laplacian spectral radius. We find the unique graph that maximizes the distance spectral radius in the class of all unicyclic graphs of girth g on n vertices. Also we find the unique graph that maximizes the distance signless Laplacian and the distance Laplacian spectral radius in the class of all unicyclic graphs on n vertices.

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1. Introduction

All our graphs are finite, undirected, connected and simple. Let G be a graph on vertices $\{1, 2, \ldots, n\}$. At times, we use $V(G)$ and $E(G)$ to denote the set of vertices and the set of edges of G, respectively. For $i, j \in V(G)$, the *distance* between i and j, denoted by $d_G(i, j)$ or simply d_{ij}, is the length of a shortest path from i to j in G. The *distance matrix* of G, denoted by $\mathcal{D}(G)$ is the $n \times n$ matrix with (i, j)-th entry d_{ij}.

Like the Laplacian and signless Laplacian matrix of a graph, Aouchiche and Hansen[1] first introduced the distance Laplacian and distance signless Laplacian of a graph. The transmission, denoted by $Tr(v)$ of a vertex v is the sum of the distances from v to all other vertices in G. Let $Tr(G)$ be the $n \times n$ diagonal matrix with i-th diagonal entry $Tr(i)$. The distance Laplacian and the distance signless Laplacian of G are defined as the matrix $\mathcal{D}^L(G) = Tr(G) - \mathcal{D}(G)$ and $\mathcal{D}^Q(G) = Tr(G) + \mathcal{D}(G)$, respectively, see [1].

For a column vector $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ we have

$$x^T\mathcal{D}^L(G)x = \sum_{1\le i<j\le n} d_{ij}(x_i - x_j)^2 \text{ and } x^T\mathcal{D}^Q(G)x = \sum_{1\le i<j\le n} d_{ij}(x_i + x_j)^2. \quad (1.1)$$

It follows that $\mathcal{D}^L(G)$ and $\mathcal{D}^Q(G)$ are positive semidefinite. Thus the eigenvalues of distance Laplacian and distance signless Laplacian matrix of a graph are all real. The largest eigenvalue of the distance, distance Laplacian and the distance signless Laplacian matrix are known as the distance spectral radius, distance Laplacian spectral radius and distance signless Laplacian spectral radius, respectively. We use $\rho^{\mathcal{D}}$, ρ^L and $\rho^{\mathcal{Q}}$ to denote the distance, distance Laplacian and distance signless Laplacian spectral radius, respectively.

Note that for a connected graph G, both $\mathcal{D}(G)$ and $\mathcal{D}^Q(G)$ are irreducible nonnegative matrices. Thus by the Perron–Frobenius Theorem, both $\rho^{\mathcal{D}}$ and $\rho^{\mathcal{Q}}$ are simple, and there is a positive eigenvector of $\mathcal{D}(G)$ and $\mathcal{D}^Q(G)$ corresponding to $\rho^{\mathcal{D}}$ and $\rho^{\mathcal{Q}}$, respectively. Such eigenvectors corresponding to $\rho^{\mathcal{D}}$ and $\rho^{\mathcal{Q}}$ are called *Perron vector* of $\mathcal{D}(G)$ and $\mathcal{D}^Q(G)$, respectively. By an eigenvector we mean a unit eigenvector.

Recently, the study of distance, distance Laplacian and distance signless Laplacian spectral radius has attracted many researchers. The problem of finding all graphs with maximal or minimal distance spectral radius among certain class of graphs has been studied extensively. For related results, one may refer to [2,4–6,9] and the references therein.

Subhi and Powers in [8] proved that for $n \ge 3$ the path P_n has the maximum distance spectral radius among trees on n vertices. Stevanović and Ilić in [7] generalized this result, and proved that among trees with fixed maximum degree Δ, the broom graph has maximum distance spectral radius and proved that the star S_n is the unique graph with minimal distance spectral radius among all trees on n vertices.

If vertices i and j are adjacent, we write $i \sim j$. Vertices i, j, k in graph G are said to be consecutive if j is adjacent to both i and k. An edge independent set of a graph G is a set of edges such that any two distinct edges of the set are not incident on a common vertex.

The matching number of G, denoted by $m(G)$, is the maximum of the cardinalities of all edge independent sets. For a connected graph G of order n, its matching number $m(G)$ satisfies $1 \leq m(G) \leq \lfloor \frac{n}{2} \rfloor$. By $\mathcal{T}_n^m$ we denote the class of all trees on n vertices with matching number m.

If u, v are vertices in a graph G, by $G-u$ and $G-\{u,v\}$ we mean the graphs obtained from G by deleting the vertex u and the edge $\{u,v\}$ respectively. Also by $G+\{u,v\}$ we mean the graph obtained from G by adding the edge $\{u,v\}$. A *quasipendent vertex* is a vertex which is adjacent to a pendent vertex. If v is a vertex of a tree T, then the components of $T-v$ are called the *branches* of T at v.

The *dumbbell* $D(n,a,b)$ consists of the path P_{n-a-b} together with a independent vertices adjacent to one pendent vertex of P and b independent vertices adjacent to the other pendent vertex, where $a,b \geq 1$. By $\mathcal{T}_{n,k}$ we denote the class of all trees on n vertices and k pendent vertices.

We now outline the contents of this paper. In section 2 of this article, we introduce some convex and quasiconvex functions on trees and prove that for a tree the eccentricity, transmission and weight functions are strictly quasiconvex. In section 3, we prove that the Perron vector of distance matrix of a tree is strictly convex and the Perron vector of distance signless Laplacian matrix of a tree is strictly quasiconvex. In section 4, we have obtained the unicyclic graph that maximizes the distance spectral radius among all unicyclic graphs of fixed girth and order. In section 5, we prove that the tree that maximizes the distance Laplacian spectral radius among all trees with a given number of pendent vertices is a dumbbell. Moreover, we obtain the tree that maximizes the distance Laplacian spectral radius among all trees with fixed maximum degree. We obtain the unique unicyclic graph that maximizes the distance Laplacian spectral radius among all unicyclic graph. In section 6, we prove that the tree that maximizes the distance signless Laplacian spectral radius among all trees with a given number of pendent vertices is a dumbbell. Moreover, we obtain the tree that maximizes the distance signless Laplacian spectral radius among all trees with fixed maximum degree. We obtain the unique unicyclic graph that maximizes the distance signless Laplacian spectral radius among all unicyclic graphs.

2. Convex and quasiconvex functions on trees

Definition. Let T be a tree with $V(T)=\{1,\ldots,n\}$, $n \geq 3$, and let $f : V(T) \longrightarrow [0,+\infty)$. We say that f is *convex* if for any distinct $i,j,k \in V(T)$ with $i \sim j$, $j \sim k$ we have $2f(j) \leq f(i)+f(k)$ and *strictly convex* if $2f(j) < f(i)+f(k)$. We say that f is *quasiconvex* if for any distinct $i,j,k \in V(T)$ with $i \sim j$, $j \sim k$ we have $f(j) \leq \max\{f(i),f(k)\}$ and *strictly quasiconvex* if $f(j) < \max\{f(i),f(k)\}$.

Lemma 2.1. *Let T be a tree with $V(T)=\{1,\ldots,n\}$ and let $f : V(T) \longrightarrow [0,+\infty)$ be either strictly convex or strictly quasiconvex. Then f attains its minimum, either at a unique vertex, or at two adjacent vertices. Furthermore:*

(i) *If f attains its minimum at the unique vertex i, then for any path $i = i_1 - i_2 - \cdots - i_k$, starting at i, $f(i_1) < f(i_2) < \cdots < f(i_k)$. (We say that f is strictly increasing along any path starting at i.)*

(ii) *If f attains its minimum at the two adjacent vertices i and j, then f is strictly increasing along any path starting at i, and not containing j, or starting at j, and not containing i.*

Proof. First suppose that f is strictly convex. Observe that if $i, j, k \in V(T)$ with $i \sim j$, $j \sim k$, then $f(i) < f(j)$ and the strict convexity of f imply that $f(j) < f(k)$. Let $S \subset V(T)$ be the set of vertices on which f attains its minimum. Let $i, j \in S$ and suppose i and j are not adjacent. Let $i = i_1 - i_2 - \cdots - i_k = j$ be an ij-path. By the observation made earlier, $f(i) = f(i_1) < f(i_2)$ implies that $f(i_2) < \cdots < f(i_k) = f(j)$ and hence $f(i) < f(j)$, which is a contradiction. Thus any two vertices in S must be adjacent. Since T has no cycles, it follows that S has either 1 or 2 vertices. The second part is proved similarly, using the observation made in the beginning. This completes the proof.

If f is strictly quasiconvex, then again if $i, j, k \in V(T)$ with $i \sim j$, $j \sim k$, and $f(i) < f(j)$, then it follows that $f(j) < f(k)$. The rest of the proof is similar. □

Theorem 2.2. *Let T be a tree with $V(T) = \{1, \ldots, n\}$, let $g : [0, +\infty) \longrightarrow [0, +\infty)$ be a strictly increasing convex function. Let $x_1, \ldots, x_n$ be positive numbers and let $f(i) = \sum_{j=1}^{n} g(d_{ij})x_j$, $i = 1, \ldots, n$. Then f is strictly convex.*

Proof. Let i, j, k be vertices of T such that $i \sim j$, $j \sim k$. Let $e_1 = \{i, j\}$, $e_2 = \{j, k\}$ be edges of T. Let T_1, T_2 and T_3 be the components of $T \setminus \{e_1, e_2\}$, containing i, j and k, respectively. We have

$$f(i) = \sum_{\ell \in T_1} g(d_{i\ell})x_\ell + \sum_{\ell \in T_2} g(d_{j\ell} + 1)x_\ell + \sum_{\ell \in T_3} g(d_{k\ell} + 2)x_\ell \tag{2.2}$$

$$f(j) = \sum_{\ell \in T_1} g(d_{i\ell} + 1)x_\ell + \sum_{\ell \in T_2} g(d_{j\ell})x_\ell + \sum_{\ell \in T_3} g(d_{k\ell} + 1)x_\ell \tag{2.3}$$

$$f(k) = \sum_{\ell \in T_1} g(d_{i\ell} + 2)x_\ell + \sum_{\ell \in T_2} g(d_{j\ell} + 1)x_\ell + \sum_{\ell \in T_3} g(d_{k\ell})x_\ell \tag{2.4}$$

Since g is convex, we have

$$g(d_{i\ell}) + g(d_{i\ell} + 2) \geq 2g(d_{i\ell} + 1) \tag{2.5}$$

and

$$g(d_{k\ell} + 2) + g(d_{k\ell}) \geq 2g(d_{k\ell} + 1). \tag{2.6}$$

It follows from (2.2)–(2.6) that

$$f(i)+f(k)\geq \sum_{\ell\in T_1} 2g(d_{i\ell}+1)x_\ell+\sum_{\ell\in T_2} 2g(d_{j\ell}+1)x_\ell+\sum_{\ell\in T_3} 2g(d_{k\ell}+1)x_\ell$$

and since g is strictly increasing we have

$$\begin{aligned} f(i)+f(k)-2f(j) &\geq \sum_{\ell\in T_2} 2(g(d_{j\ell}+1)-g(d_{j\ell}))x_\ell \\ &> 0. \end{aligned}$$

That completes the proof. □

2.1. Sum of the largest p distances

We introduce some notations. Let p be a fixed positive integer, $1\leq p\leq n$. If $x,y\in R^n$, we define $x\otimes_p y$ to be the sum of the p largest components in $\{x_1y_1,\ldots,x_ny_n\}$. If A is an $n\times n$ matrix and if $x\in R^n$, then $A\otimes_p x$ is defined in the natural way, that is, the i-th component of $A\otimes_p x$ is given by $(a_{i1},\ldots,a_{in})\otimes_p x$, $i=1,\ldots,n$. Note that when $p=n$, $x\otimes y=\sum_{i=1}^n x_iy_i$, whereas when $p=1$, then $x\otimes y=\max_i x_iy_i$.

If A is an $n\times n$ matrix with $a_{ij}\geq 0$ for all i, j and if $g:[0,+\infty)\longrightarrow[0,+\infty)$ is a function, then we define $g(A)$ to be the $n\times n$ matrix with (i,j)-entry $g(a_{ij})$.

Let T be a tree with $V(T)=\{1,\ldots,n\}$ and let $y\in R^n$ be a vector with nonnegative coordinates. We say that y is convex (quasiconvex) if $f(i)=y_i$, $i=1,\ldots,n$ is a convex (quasiconvex) function with respect to the tree T. Similar definition is used with respect to strict convexity (quasiconvexity).

Theorem 2.3. *Let T be a tree with $V(T)=\{1,\ldots,n\}$, let $g:[0,+\infty)\longrightarrow[0,+\infty)$ be a strictly increasing function. Let $x\in R^n$ be a vector with positive coordinates. Let p be a positive integer, $1\leq p\leq n$. Then $g(D)\otimes_p x$ is strictly quasiconvex.*

Proof. Let i, j, k be vertices of T such that $i\sim j$, $j\sim k$. We claim that for any $\ell\in V(T)$ we have

$$g(d_{j\ell})x_\ell<\max(g(d_{i\ell})x_\ell,g(d_{k\ell})x_\ell). \tag{2.7}$$

Let $e_1=\{i,j\}$, $e_2=\{j,k\}$ be edges of T. Let T_1, T_2 and T_3 be the components of $T\setminus\{e_1,e_2\}$, containing i, j and k, respectively. If $\ell\in T_1$, then $d_{i\ell}x_\ell=(d_{j\ell}-1)x_\ell$ and $d_{k\ell}x_\ell=(d_{j\ell}+1)x_\ell$. Hence we have

$$d_{j\ell}x_\ell<\max(d_{i\ell}x_\ell,d_{k\ell}x_\ell). \tag{2.8}$$

Similarly (2.8) holds if $\ell\in T_3$. If $\ell\in T_2$, then $d_{i\ell}x_\ell=(d_{j\ell}+1)x_\ell$ and $d_{k\ell}x_\ell=(d_{j\ell}+1)x_\ell$. Hence (2.8) holds as well. Therefore (2.8) holds for any $\ell\in V(T)$. Since g is strictly increasing, (2.7) follows from (2.8).

It follows that the sum of the largest p components in $g(d_{j\ell})x_\ell$, $\ell = 1, \ldots, n$ is less than the sum of the largest p components in $g(d_{i\ell})x_\ell$, $\ell = 1, \ldots, n$ and the largest p components in $g(d_{k\ell})x_\ell$, $\ell = 1, \ldots, n$. Therefore $g(D) \otimes_p x$ is strictly quasiconvex and the proof is complete. □

We note two special cases of Theorem 2.3. Let T be a tree with $V(T) = \{1, \ldots, n\}$. Recall that the *eccentricity* of vertex $\ell \in V(T)$ is defined to be $\mathrm{ecc}(\ell) = \max(d_{\ell 1}, \ldots, d_{\ell n})$, while the transmission index of $\ell \in V(T)$ is given by $Tr(\ell) = \sum_{i=1}^{n} d(\ell i)$. It follows from Theorem 2.3 that both eccentricity and transmission index are strictly quasiconvex functions and hence satisfy (i) and (ii) of Lemma 2.1. The vertex (or vertices) where the eccentricity is minimized constitute the *center* of the tree, whereas the vertex (or vertices) where the transmission index is minimized, constitute the *median* of the tree.

2.2. Centroid

Let T be a tree with $V(T) = \{1, \ldots, n\}$. If $i \in V(T)$, then the weight of i, denoted $\mathrm{wt}(i)$, is defined to be the maximum number of vertices in a component of $T \setminus i$.

Let i, j, k be vertices of T such that $i \sim j$, $j \sim k$. Let T_1, T_2 and T_3 be the components of $T \setminus \{e_1, e_2\}$, containing i, j and k, respectively. Let T_ℓ^i, $\ell = 1, \ldots, u$, be the components of $T_1 \setminus i$, let T_ℓ^j, $\ell = 1, \ldots, v$, be the components of $T_2 \setminus j$, and let T_ℓ^k, $\ell = 1, \ldots, w$, be the components of $T_3 \setminus k$. We denote the number of vertices in the graph G by $|G|$. We have

$$wt(i) = \max\{|T_1^i|, \ldots, |T_u^i|, |T_1^j| + \cdots + |T_v^j| + |T_1^k| + \cdots + |T_w^k| + 2\}, \tag{2.9}$$

and

$$wt(k) = \max\{|T_1^k|, \ldots, |T_w^k|, |T_1^j| + \cdots + |T_v^j| + |T_1^i| + \cdots + |T_u^i| + 2\}. \tag{2.10}$$

It follows from (2.9), (2.10) that

$$\begin{aligned}
&\max(wt(i), wt(k)) \\
&= \max\{|T_1^j|, \ldots, |T_v^j| + |T_1^k| + \cdots + |T_w^k| + 2, |T_1^j| + \cdots + |T_v^j| + |T_1^i| + \cdots + |T_u^i| + 2\} \\
&> \max\{|T_1^i| + \cdots + |T_u^i| + 1, |T_1^j| + \ldots + |T_v^j|, |T_1^k| + \cdots + |T_w^k| + 1\} \\
&= wt(j).
\end{aligned}$$

Thus we have proved the following result.

Theorem 2.4. *Let T be a tree with $V(T) = \{1, \ldots, n\}$. Then $wt(i)$, $i = 1, \ldots, n$ is strictly quasiconvex.*

It follows from Theorem 2.4 that the weight function satisfies (i) and (ii) of Lemma 2.1. The vertices with minimum weight constitute the *centroid* of the tree. It can be shown that the centroid and the median of a tree coincide.

3. Perron vector of distance and distance signless Laplacian matrices

The following result is a reformulation of Theorem 2.2. The proof is omitted.

Theorem 3.1. *Let $f : [0, +\infty) \longrightarrow [0, +\infty)$ be a strictly increasing convex function and $\mathcal{D}$ be the distance matrix of a tree T. Then $f(\mathcal{D})z$ is strictly convex on T for any positive vector z, where $f(\mathcal{D})$ is the matrix with (i, j)-th entry $f(d_{ij})$.*

The following corollary, which follows immediately from Theorem 3.1, says that the Perron vector of $f(\mathcal{D})$ is strictly convex, which implies that the Perron vector satisfies (i) and (ii) of Lemma 2.1.

Corollary 3.2. *Let $\mathcal{D}$ be the distance matrix of a tree T and let $f : [0, +\infty) \longrightarrow [0, +\infty)$ be a strictly increasing convex function. Then the Perron vector of $f(\mathcal{D})$ is strictly convex on T.*

Proof. Let ρ be the spectral radius and $\mathbf{x}$ be the Perron vector of $f(\mathcal{D})$. By the Perron–Frobenius theorem $\mathbf{x}$ is positive. Since $f(\mathcal{D})\mathbf{x} = \rho\mathbf{x}$, then $\mathbf{x} = \frac{1}{\rho} f(\mathcal{D})\mathbf{x}$. By Theorem 3.1 $\mathbf{x}$ is strictly convex. □

It follows from Corollary 3.2 that the Perron vector of $f(\mathcal{D})$ satisfies (i) and (ii) of Lemma 2.1. The next result says that the Perron vector of the distance signless Laplacian matrix of a tree is strictly quasiconvex and hence satisfies (i) and (ii) of Lemma 2.1.

Theorem 3.3. *Let $\mathcal{D}^Q$ be the distance signless Laplacian of a tree T. Then the Perron vector of $\mathcal{D}^Q$ is strictly quasiconvex on T.*

Proof. Let i, j, k be vertices of T such that $i \sim j$, $j \sim k$. Let $e_1 = \{i, j\}$, $e_2 = \{j, k\}$ be edges of T. Let T_1, T_2 and T_3 be the components of $T \setminus \{e_1, e_2\}$, containing i, j and k, respectively. Let $\mathbf{x}$ be the Perron vector of $\mathcal{D}^Q$. By Theorem 2.2 the transmission index is strictly convex and therefore

$$Tr(i) + Tr(k) - 2Tr(j) > 0.$$

Thus

$$2\{\rho^Q - Tr(j)\} > 2\rho^Q - \{Tr(i) + Tr(k)\}. \tag{3.11}$$

By eigenvector equations of $\mathbf{x}$ at the vertices i, j, k we have

$$\{\rho^{\mathcal{Q}} - Tr(i)\}\mathbf{x}_i = \sum_{\ell\in T_1} d_{i\ell}\mathbf{x}_\ell + \sum_{\ell\in T_2}(d_{j\ell}+1)\mathbf{x}_\ell + \sum_{\ell\in T_3}(d_{k\ell}+2)\mathbf{x}_\ell \tag{3.12}$$

$$\{\rho^{\mathcal{Q}} - Tr(j)\}\mathbf{x}_j = \sum_{\ell\in T_1}(d_{i\ell}+1)\mathbf{x}_\ell + \sum_{\ell\in T_2} d_{j\ell}\mathbf{x}_\ell + \sum_{\ell\in T_3}(d_{k\ell}+1)\mathbf{x}_\ell \tag{3.13}$$

$$\{\rho^{\mathcal{Q}} - Tr(k)\}\mathbf{x}_k = \sum_{\ell\in T_1}(d_{i\ell}+2)\mathbf{x}_\ell + \sum_{\ell\in T_2}(d_{j\ell}+1)\mathbf{x}_\ell + \sum_{\ell\in T_3} d_{k\ell}\mathbf{x}_\ell. \tag{3.14}$$

It follows from (3.12)–(3.14) that

$$\{\rho^{\mathcal{Q}} - Tr(i)\}\mathbf{x}_i + \{\rho^{\mathcal{Q}} - Tr(k)\}\mathbf{x}_k - 2\{\rho^{\mathcal{Q}} - Tr(j)\}\mathbf{x}_j > 0. \tag{3.15}$$

From (3.11) and (3.15), it follows that

$$\{\rho^{\mathcal{Q}} - Tr(i)\}\mathbf{x}_i + \{\rho^{\mathcal{Q}} - Tr(k)\}\mathbf{x}_k > \{2\rho^{\mathcal{Q}} - \{Tr(i) + Tr(k)\}\}\mathbf{x}_j.$$

Thus $\beta\mathbf{x}_i + (1-\beta)\mathbf{x}_k > \mathbf{x}_j$, where $\beta = \frac{\rho^{\mathcal{Q}} - Tr(i)}{2\rho^{\mathcal{Q}} - \{Tr(i)+Tr(k)\}}$. Therefore $\mathbf{x}_j < \max\{\mathbf{x}_i, \mathbf{x}_k\}$ and hence the Lemma holds. □

4. Distance spectral radius

For a matrix A we use A_i to denote the i-th column of A. By $\mathbb{1}$ we denote the column vector of all ones and by $\mathbb{J}$ the matrix of all ones of appropriate size. We recall some known results which will be used.

Lemma 4.1 *(Weyl's inequalities). [3] Let $\lambda_i(M)$ denote the i-th largest eigenvalue of a real symmetric matrix M. If A and B are two real symmetric matrices of order n, then*

$$\lambda_1(A) + \lambda_i(B) \geq \lambda_i(A+B) \geq \lambda_n(A) + \lambda_i(B) \text{ for } i = 1, 2, \ldots, n.$$

Lemma 4.2. *[3] If A is a symmetric $n \times n$ matrix with λ_1 as the largest eigenvalue then for any normalized vector $\mathbf{x} \in R^n (\mathbf{x} \neq 0)$,*

$$\mathbf{x}^t A\mathbf{x} \leq \lambda_1.$$

The equality holds if and only if $\mathbf{x}$ is an eigenvector of A corresponding to the eigenvalue λ_1.

Lemma 4.3. *Let G_v be the graph obtained by identifying a vertex v of the graph G with a fixed vertex u of another graph G'. If i, j, k are consecutive vertices of G such that i and k are on different components of $G - j$ and $\rho_v^{\mathcal{D}}$ denotes the distance spectral radius of G_v, then $\rho_i^{\mathcal{D}} + \rho_k^{\mathcal{D}} - 2\rho_j^{\mathcal{D}} > 0$.*

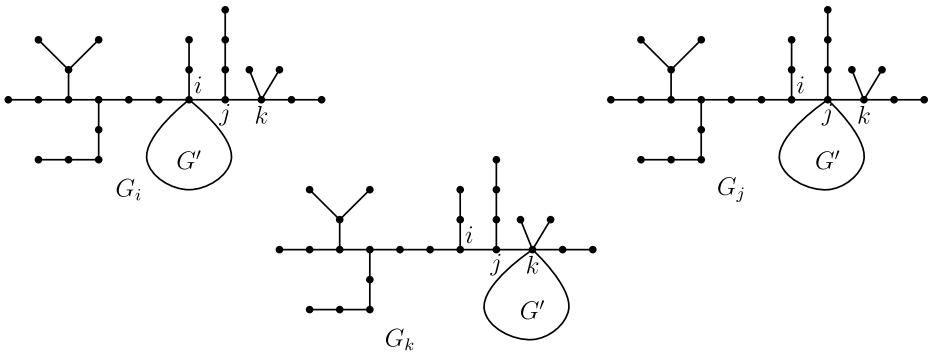


Fig. 1. Graphs G_i, G_j and G_k.

Proof. Let D_j be the jth column of $\mathcal{D}(G)$ and $d_{G'}(u,x)$ be the distance between u and x in G'. By permutation similarity, we have

$$\mathcal{D}(G_j) = \begin{bmatrix} \mathcal{D}(G) & B \\ B^t & \mathcal{D}(G')^u \end{bmatrix},$$

where $B_x = \mathcal{D}_j + d_{G'}(u,x)\mathbb{1}$ for $x \in V(G'-u)$ and $\mathcal{D}(G')^u$ is the submatrix of $\mathcal{D}(G')$ after deleting the row and column corresponding to the vertex u.

Thus

$$\mathcal{D}(G_i) + \mathcal{D}(G_k) - 2\mathcal{D}(G_j) = \begin{bmatrix} 0 & (\mathcal{D}_i + \mathcal{D}_k - 2\mathcal{D}_j)\mathbb{1}^t \\ \mathbb{1}(\mathcal{D}_i + \mathcal{D}_k - 2\mathcal{D}_j)^t & 0 \end{bmatrix}.$$

Let $\mathbf{x}$ be the Perron vector of $\mathcal{D}(G_j)$. Since $\mathbf{x}$ is positive, $\mathcal{D}_i + \mathcal{D}_k - 2\mathcal{D}_j \geq 0$ and $\mathcal{D}_i + \mathcal{D}_k - 2\mathcal{D}_j \neq 0$ we see that

$$\mathbf{x}^t \{\mathcal{D}(G_i) + \mathcal{D}(G_k) - 2\mathcal{D}(G_j)\}\mathbf{x} > 0. \tag{4.16}$$

Also

$$\begin{aligned} \rho_i^{\mathcal{D}} + \rho_k^{\mathcal{D}} - 2\rho_j^{\mathcal{D}} &\geq \rho\{\mathcal{D}(G_i) + \mathcal{D}(G_k)\} - 2\rho_j^{\mathcal{D}} \text{ [by Lemma 4.1]} \\ &\geq \mathbf{x}^t \{\mathcal{D}(G_i) + \mathcal{D}(G_k) - 2\mathcal{D}(G_j)\}\mathbf{x} \text{ [by Lemma 4.2]}. \end{aligned} \tag{4.17}$$

From (4.16) and (4.17) we have $\rho_i^{\mathcal{D}} + \rho_k^{\mathcal{D}} - 2\rho_j^{\mathcal{D}} > 0$. (See Fig. 1.) □

Corollary 4.4. *Let T be a tree and G' be a graph. If T_i and $\rho_i^{\mathcal{D}}$ are as defined in Lemma 4.3, then $\rho_i^{\mathcal{D}}$ is strictly convex on T.*

Let G be a graph with v as one of the vertices. For $k, \ell \geq 1$, let $G(v,k,\ell)$ be the graph obtained by attaching two paths P_k and P_ℓ at v. We use $\rho_{v,k,\ell}^{\mathcal{D}}$ to denote the distance

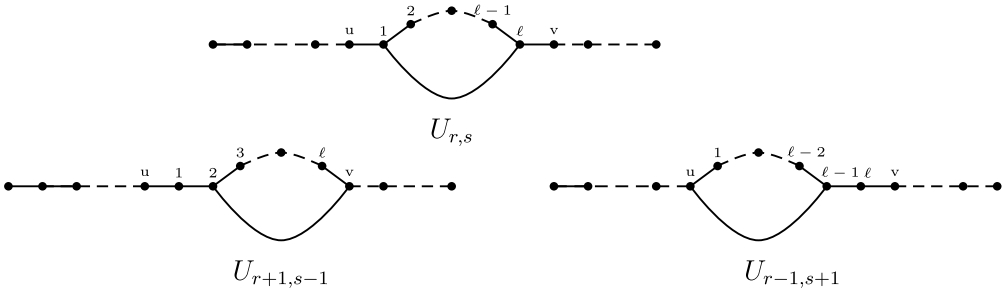


Fig. 2. Graphs $U_{r,s}$, $U_{r+1,s-1}$ and $U_{r-1,s+1}$.

spectral radius of $G(v,k,\ell)$. Stevanović, and Ilić [7] proved that $\rho^{\mathcal{D}}_{v,k,\ell} < \rho^{\mathcal{D}}_{v,k+1,\ell-1}$ for $k \geq \ell \geq 1$, which can now be obtained as an immediate application of Corollary 4.4.

Corollary 4.5. *[7, Lemma 3.3] Let G be a graph and v be one of its vertices. If $k \geq \ell \geq 1$, then $\rho^{\mathcal{D}}_{v,k,\ell} < \rho^{\mathcal{D}}_{v,k+1,\ell-1}$.*

Example 4.6. The graphs $U_{r,s}$, $U_{r+1,s-1}$ and $U_{r-1,s+1}$ with vertex labelling are as shown in Fig. 2.

The following lemma is crucial in proving our main result of this section.

Lemma 4.7. *Let $U_{r,s}$ be a unicyclic graph of girth g with two pendent paths of length r,s $(r,s \geq 1)$ respectively such that the intermediate vertices of the shortest path between the roots of the two paths are of degree two. If $\rho_{r,s}$ denotes the distance spectral radius of $U_{r,s}$ then $\rho_{r+1,s-1} + \rho_{r-1,s+1} > 2\rho_{r,s}$.*

Proof. Let the vertices 1 and ℓ of the cycle be roots of the paths P_1 and P_2 respectively. Let $P := 1 - 2 - \cdots - \ell$ be the shortest path from 1 to ℓ and S be the set of vertices not on these three paths. Partition the vertex set as follows:

$$\begin{aligned}
V_1 &= \{j \in S : d_{1j} > d_{2j} \text{ and } d_{\ell j} < d_{(\ell-1)j}\}\\
V_2 &= \{j \in S : d_{1j} < d_{2j} \text{ and } d_{\ell j} > d_{(\ell-1)j}\}\\
V_3 &= \{j \in S : d_{1j} < d_{2j} \text{ and } d_{\ell j} < d_{(\ell-1)j}\}\\
V_4 &= \{j \in S : d_{1j} = d_{2j}\}\\
V_5 &= \{j \in S : d_{\ell j} = d_{(\ell-1)j}\}\\
V_6 &= V(P_1 \cup P_2) - \{1,\ell\}\\
V_7 &= V(P) - \{1,\ell\}.
\end{aligned}$$

For $i \in V_7$, $j \in \cup_{i=3}^{i=5} V_i$ if the lengths of the two paths in $U_{r,s}$ from i to j differ by k then following changes occur.

(i) For $k = 0$, d_{ij} decreases by 1 in both $U_{r+1,s-1}$ and $U_{r-1,s+1}$.
(ii) For $k = 1$, d_{ij} remains unchanged in one of $U_{r+1,s-1}$ and $U_{r-1,s+1}$ and decreases by 1 in the other.
(iii) For $k \geq 2$, d_{ij} increases by 1 in one of $U_{r+1,s-1}$ and $U_{r-1,s+1}$ and decreases by 1 in the other.

Similarly we can describe the change in distances among other vertices.

For $i \in V_7$, let S_i be the set of vertices j in $\cup_{i=3}^{i=5} V_i$ such the length of the two paths from i to j differ by at most one. Note that if girth g is even then $S_i \subseteq V_3$ for all $i \in V_7$ as $V_4 = V_5 = \emptyset$.

Let $\mathcal{D}_{r,s}$ be the distance matrix of $U_{r,s}$. Then with respect to the ordered partition $V(U_{r,s}) = \{1, \ell\} \cup (V_1 \cup V_2) \cup (V_3 \cup V_4 \cup V_5) \cup V_6 \cup V_7$, we have

$$\mathcal{D}_{r+1,s-1} + \mathcal{D}_{r-1,s+1} - 2\mathcal{D}_{r,s} = \left[\begin{array}{c|c|c|c|c} \{1,\ell\} & V_1 \cup V_2 & V_3 \cup V_4 \cup V_5 & V_6 & V_7 \\ \hline 0 & \begin{matrix} 0 & 2\mathbb{1}^t \\ 2\mathbb{1}^t & 0 \end{matrix} & \begin{matrix} 2\mathbb{1}^t & \mathbb{1}^t & 2\mathbb{1}^t \\ 2\mathbb{1}^t & 2\mathbb{1}^t & \mathbb{1}^t \end{matrix} & 0 & 0 \\ \hline \begin{matrix} 0 & 2\mathbb{1} \\ 2\mathbb{1} & 0 \end{matrix} & 0 & 0 & 0 & 0 \\ \hline \begin{matrix} 2\mathbb{1} & 2\mathbb{1} \\ \mathbb{1} & 2\mathbb{1} \\ 2\mathbb{1} & \mathbb{1} \end{matrix} & 0 & 0 & 0 & A^t \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & A & 0 & 0 \end{array}\right]$$

where $A_{ij} = \begin{cases} -1, & \text{if } j \in S_i \text{ and } g \text{ is odd} \\ -2, & \text{if } j \in S_i \text{ and } g \text{ is even} \\ 0, & \text{otherwise.} \end{cases}$

Let $\mathbf{x}$ be the Perron vector of $\mathcal{D}_{r,s}$, then we have

$$\begin{aligned} & \rho_{r+1,s-1} + \rho_{r-1,s+1} - 2\rho_{r,s} \\ & \geq \rho(\mathcal{D}_{r+1,s-1} + \mathcal{D}_{r-1,s+1}) - 2\rho_{r,s} \quad \text{[by Lemma 4.1]} \\ & \geq \mathbf{x}^t \left(\mathcal{D}_{r+1,s-1} + \mathcal{D}_{r-1,s+1} - 2\mathcal{D}_{r,s}\right) \mathbf{x} \quad \text{[by Lemma 4.2]} \\ & = \mathbf{x}_1 \left(\sum_{j \in V_4} \mathbf{x}_j + 2 \sum_{j \in V_2 \cup V_3 \cup V_5} \mathbf{x}_j \right) + \mathbf{x}_\ell \left(\sum_{j \in V_5} \mathbf{x}_j + 2 \sum_{j \in V_1 \cup V_3 \cup V_4} \mathbf{x}_j \right) - \alpha \sum_{j \in S_i} \sum_{i \in V_7} \mathbf{x}_i \mathbf{x}_j \\ & \quad (\alpha = 1 \text{ or } 2 \text{ according as } g \text{ is odd or even}) \\ & \geq \mathbf{x}_1 \left(\sum_{j \in V_4} \mathbf{x}_j + 2 \sum_{j \in V_3 \cup V_5} \mathbf{x}_j \right) + \mathbf{x}_\ell \left(\sum_{j \in V_5} \mathbf{x}_j + 2 \sum_{j \in \cup V_3 \cup V_4} \mathbf{x}_j \right) - \alpha \sum_{j \in S_i} \sum_{i \in V_7} \mathbf{x}_i \mathbf{x}_j \\ & > \alpha \sum_{j \in S_i} \sum_{i \in V_7} \mathbf{x}_j (\mathbf{x}_1 + \mathbf{x}_\ell - \mathbf{x}_i). \end{aligned} \tag{4.18}$$

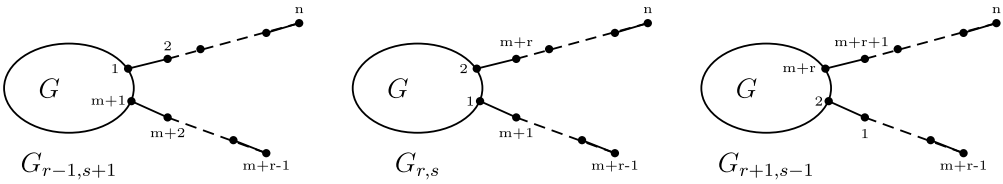


Fig. 3. Graphs $G_{r-1,s+1}, G_{r,s}$ and $G_{r+1,s-1}$.

From eigenequations at the vertices 1, ℓ, i we have

$$\begin{aligned}\rho_{r,s}(\mathbf{x}_1+\mathbf{x}_\ell-\mathbf{x}_i) &= \sum_{j\in V(U_{r,s})} d_{1j}\mathbf{x}_j + \sum_{j\in V(U_{r,s})} d_{\ell j}\mathbf{x}_j - \sum_{j\in V(U_{r,s})} d_{ij}\mathbf{x}_j \\ &= \sum_{j\in V(U_{r,s})} (d_{1j}+d_{\ell j}-d_{ij})\mathbf{x}_j. \qquad (4.19)\end{aligned}$$

Since $d_{1j}+d_{\ell j} > d_{ij}$ for all $i \in V_7$ and any vertex j, the result follows from (4.18) and (4.19). □

We use $\mathcal{U}_{n,g}$ to denote the class of unicyclic graph of girth g on n vertices. U_g^n is the unicyclic graph on n vertices obtained by attaching a g-cycle to one of the end vertices of P_{n-g}. The next result is one of our main results of this section which provides the unique graph that maximizes the distance spectral radius among all unicyclic graph with fixed girth.

Theorem 4.8. *The unique graph that maximizes the distance spectral radius in $\mathcal{U}_{n,g}$ is U_g^n.*

Proof. If $g = n$, there is nothing to prove. Let U be a graph that maximizes the distance spectral radius in $\mathcal{U}_{n,g}$. Let C be the cycle in U and T_u be a tree branch attached at a vertex u of C. If T_u is not a path, then T_u contains a vertex of degree at least 3 except u. Let w be a vertex of degree at least 3 which is farthest from u. Then there exist two pendent paths attached at w. Thus $U \cong G(w,k,\ell)$ for some $k,\ell \geq 1$ and unicyclic graph G of girth g. Observe that $G(w,k-1,\ell+1), G(w,k+1,\ell-1) \in \mathcal{U}_{n,g}$. By Lemma 4.3, the distance spectral radius of $G(w,k+1,\ell-1)$ or $G(w,k-1,\ell+1)$ is bigger than that of U, which is a contradiction. Hence every tree branch of U attached at a vertex of C is a path. Next we prove that at most one vertex of C has degree 3. To the contrary, assume that u and v are two nearest vertices in C of degree 3. Then $U \cong U_{r,s}$ for some r, s $(r \geq s \geq 1)$. By Lemma 4.7, distance spectral radius of $U_{r+1,s-1}$ or $U_{r-1,s+1}$ is bigger than that of U, which is a contradiction. Hence $U \cong U_g^n$. □

Example 4.9. The graphs $G_{r,s}$, $G_{r+1,s-1}$ and $G_{r-1,s+1}$ with vertex labelling are as shown above.

Lemma 4.10. *Let $G_{r,s}$ be the graph obtained by attaching two paths of length r and s at two adjacent vertices u and v of a connected graph G, respectively. If $\rho_{r,s}$ denotes the distance spectral radius of $G_{r,s}$ then $\rho_{r+1,s-1} + \rho_{r-1,s+1} > 2\rho_{r,s}$.*

Proof. Let $V(G) = \{1, 2, \ldots, m\}$ such that $u = 1$, $v = 2$. Let $\mathcal{D}_{r,s}$ be the distance matrix of $G_{r,s}$. Observe that $\mathcal{D}_{r+1,s-1} + \mathcal{D}_{r-1,s+1} - 2\mathcal{D}_{r,s} = [\hat{d}_{ij}]$, where

$$\hat{d}_{ij} = \begin{cases} 1 & \text{if } i = 1, 2 \text{ and } d_{1j} = d_{2j}, j = 3, \ldots, m \\ 2 & \text{if } d_{1j} < d_{2j}, i = 1, j = 3, \ldots, m \text{ or } d_{1j} > d_{2j}, i = 2, j = 3, \ldots, m \\ 0 & \text{otherwise.} \end{cases}$$

Thus we see that $\mathbf{x}^t(\mathcal{D}_{r+1,s-1} + \mathcal{D}_{r-1,s+1} - 2\mathcal{D}_{r,s})\mathbf{x} > 0$, if $\mathbf{x}$ is the Perron vector of $\mathcal{D}_{r,s}$. Hence the lemma holds. □

As an application of Lemma 4.10, we obtain the following result proved by Zhang and Godsil in [9].

Corollary 4.11. *[9, Theorem 1.1] Let $G_{r,s}$ be the graph obtained by attaching two paths of length r and s at two adjacent vertices u and v of a connected graph G, respectively. Let $\rho^{\mathcal{D}}_{r,s}$ be the distance spectral radius of $G_{r,s}$. Then the following holds:*

(i) $\rho^{\mathcal{D}}_{r,s} < \rho^{\mathcal{D}}_{r+1,s-1}$ *if* $r > s \geq 1$.
(ii) $\rho^{\mathcal{D}}_{r,s} < \rho^{\mathcal{D}}_{r+1,s-1}$ *or* $\rho^{\mathcal{D}}_{r,s} < \rho^{\mathcal{D}}_{r-1,s+1}$ *if* $r = s \geq 1$.

5. Distance Laplacian spectral radius

The following Lemma is similar to Lemma 4.3 for distance matrix.

Lemma 5.1. *Let G_v be the graph obtained by identifying a vertex v of the graph G with a fixed vertex of another graph G'. If i, j, k are consecutive vertices of G such that i and k are on different components of $G - j$ and ρ^L_v denotes the distance Laplacian spectral radius of G_v, then $\rho^L_i + \rho^L_k - 2\rho^L_j \geq 0$. Furthermore, either $\rho^L_i > \rho^L_j$ or $\rho^L_k > \rho^L_j$.*

Proof. Let G' be of order $m + 1$ with w as a fixed vertex. Assume that G_i, G_j, G_k are obtained by identifying the vertices i, j, k of G with the vertex w of G', respectively. Let H_i and H_k be the components of $G - j$ containing i and k respectively. Suppose

$$\begin{aligned} V_1 &= \{u \in G : d_{uk} = d_{ui} = d_{uj} + 1\} \\ V_2 &= \{u \in H_k : d_{uk} = d_{jk} \text{ and } d_{ui} = d_{uj} + 1\} \\ V_3 &= \{u \in H_i : d_{ui} = d_{uj} \text{ and } d_{uk} = d_{uj} + 1\} \\ V_4 &= \{u \in H_k : d_{uk} = d_{uj} - 1 \text{ and } d_{ui} = d_{uj} + 1\} \\ V_5 &= \{u \in H_i : d_{ui} = d_{uj} - 1 \text{ and } d_{uk} = d_{uj} + 1\}. \end{aligned}$$

Then $V(G_j) = V(G'-j) \cup \left(\cup_{i=1}^{5} V_i\right)$ is a partition for $V(G_j)$. If $a = |V_1|$ and $b = |V_2 \cup V_3|$, then by permutation similarity we have

$$\mathcal{D}^L(G_i) + \mathcal{D}^L(G_k) - 2\mathcal{D}^L(G_j) = \left[\begin{array}{c|c|c|c} V(G'-j) & V_1 & V_2 \cup V_3 & V_4 \cup V_5 \\ \hline (2a+b)I_m & -2\mathbb{J}_{m\times a} & -\mathbb{J}_{m\times b} & 0 \\ \hline -2\mathbb{J}_{a\times m} & 2mI_a & 0 & 0 \\ \hline -\mathbb{J}_{b\times m} & 0 & mI_b & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right].$$

Let $\mathbf{x}$ be an eigenvector corresponding to spectral radius of $\mathcal{D}^L(G_j)$. Observe that

$$\begin{aligned} \rho_i^L + \rho_k^L - 2\rho_j^L &\geq \rho\{\mathcal{D}^L(G_j) + \mathcal{D}^L(G_k)\} - 2\rho_j^L \text{ [by Lemma 4.1]} \\ &\geq \mathbf{x}^t \left\{\mathcal{D}^L(G_j) + \mathcal{D}^L(G_k) - 2\mathcal{D}^L(G_j)\right\} \mathbf{x} \text{ [by Lemma 4.2]} \\ &= \sum_{p\in V(G'-j)} \left[2\sum_{q\in V_1} (\mathbf{x}_p - \mathbf{x}_q)^2 + \sum_{q\in V_2\cup V_3} (\mathbf{x}_p - \mathbf{x}_q)^2\right] \geq 0. \quad (5.20) \end{aligned}$$

To the contrary, assume that $\rho_i^L = \rho_j^L = \rho_k^L = \rho$ (say). Then above inequalities hold with equality. Therefore by (5.20), $\mathbf{x}$ is also an eigenvector of $\mathcal{D}^L(G_i)$ and $\mathcal{D}^L(G_k)$ corresponding to ρ and $\mathbf{x}_p = \mathbf{x}_q$ for $p \in V(G'-j)$, $q \in V_1 \cup V_2 \cup V_3$. Thus

$$\{\mathcal{D}^L(G_j) - \mathcal{D}^L(G_i)\}\mathbf{x} = 0 \quad (5.21)$$

$$\text{and} \quad \mathbf{x}_p = \mathbf{x}_w \text{ for all } p \in V(G'-j) \cup V_1 \cup V_2 \cup V_3. \quad (5.22)$$

Let $k = |V_4| - |V_5|$, then we have

$$\mathcal{D}^L(G_j) - \mathcal{D}^L(G_i) = \left[\begin{array}{c|c|c|c} V(G'-j) & V_1 \cup V_2 \cup V_3 & V_4 & V_5 \\ \hline kI_m & 0 & \mathbb{J} & -\mathbb{J} \\ \hline 0 & 0 & 0 & 0 \\ \hline \mathbb{J} & 0 & -mI & 0 \\ \hline -\mathbb{J} & 0 & 0 & mI \end{array}\right].$$

Now for $u \in V_4 \cup V_5$, using (5.22) in the u-th equation of (5.21) we have $\mathbf{x}_u = \mathbf{x}_w$ for all $u \in V_4 \cup V_5$. Thus $\mathbf{x} = \mathbf{x}_w \mathbb{1}$, which implies $\rho = 0$, a contradiction. Hence either $\rho_i^L \neq \rho_j^L$ or $\rho_k^L \neq \rho_j^L$. □

Corollary 5.2. *Let T be a tree and G' be a graph. If T_i and ρ_i^L are as defined in Lemma 5.1, then ρ_i^L is strictly quasiconvex on T.*

For the distance Laplacian we obtain the following results similar to that of Corollary 4.5 and that of [7, Lemma 3.3] as an immediate corollary to Corollary 5.2.

Corollary 5.3. *Let G be a simple graph and v be one of its vertices. If $k \geq \ell \geq 1$, then $\rho^L_{v,k,\ell} < \rho^L_{v,k+1,\ell-1}$.*

Theorem 5.4. *Let T be a tree with maximal distance Laplacian spectral radius in $\mathcal{T}_n^m$. Then $T \cong D(n, a, b)$, $a + b = n + 1 - 2m$.*

Proof. If $m = 1$, then the result is trivial. Suppose $m \geq 2$, and let T be a tree in $\mathcal{T}_n^m$ with maximal distance Laplacian spectral radius. Then T has 2 quasipendent vertices. Otherwise, there exists a vertex v in T such that $T - v$ has at least 3 components, and at least two of which are nontrivial. Let T_x, T_y, T_z be three branches at the vertex v of T, and M be an m-matching of T. Then at least two of vx, vy and vz are not in M.

Case 1: Suppose T_x and T_y are non-trivial and vz is not in M. Let x_1, y_1 be quasipendent vertices in T_x, T_y respectively. Then the trees $T' = T - vz + x_1z$ and $T'' = T - vz + y_1z$ are in $\mathcal{T}_n^m$, and by Corollary 5.2, $\rho^L(T') > \rho^L(T)$ or $\rho^L(T'') > \rho^L(T)$, a contradiction to the maximality of T.

Case 2: Suppose vz is in M and T_z is non-trivial. Then both vx and vy are not in M, and say T_x is non-trivial. Reversing the roles of y and z in Case 1, we have a contradiction.

Case 3: Suppose vz is in M and T_z is trivial. Since T_y is nontrivial therefore there exists a vertex u in T_y which is in a matching. Then both the trees $T' = T - vx + zx$ and $T'' = T - vx + ux$ have matching number m, but by Corollary 5.2 at least one has a larger distance Laplacian spectral radius than T, a contradiction to the maximality of T. □

Theorem 5.5. *Let T be a tree with maximal distance Laplacian spectral radius in $\mathcal{T}_{n,k}$. Then $T \cong D(n, a, b)$, $a + b = k$.*

Proof. Let $P := i_1 - i_2 - \cdots - i_m$ be a path of maximum length in T. If $m = 1, 2, 3$, then there is nothing to prove. Suppose $m \geq 4$. Then degrees of $i_3, \ldots, i_{m-2}$ in T are all 2. To the contrary assume that $deg(i_j) \geq 3$. Let T' be the component of $T - i_{j-1} - i_{j+1}$ containing vertex i_j. Let T_1, T_2 be the new trees obtained by shifting T' to i_{j-1} and i_{j+1}, respectively. By Lemma 5.1, the distance Laplacian spectral radius of T_1 or T_2 is bigger than that of T, a contradiction. Hence the Theorem holds. □

Let $\mathcal{B}_{n,\Delta}$ denote the class of all trees with n vertices and maximum vertex degree Δ. The broom, denoted by $B_{n,\Delta}$ is the tree obtained from the star $S_{\Delta+1}$ and path $P_{n-\Delta-1}$ by attaching one end of the path to one of the pendent vertex of the star (see Fig. 4).

Stevanović, and Ilić in [7] proved that among all trees with fixed maximum degree Δ, the broom tree $B_{n,\Delta}$ has maximal distance spectral radius. The next result says that broom is the unique tree that maximize the distance Laplacian spectral radius among all trees with fixed maximum degree.

Theorem 5.6. *Among all trees in $\mathcal{B}_{n,\Delta}$ the unique tree that maximizes the distance Laplacian spectral radius is $B_{n,\Delta}$.*

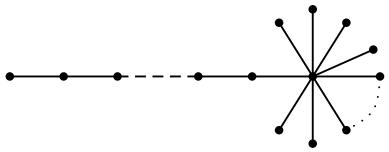


Fig. 4. The broom $B_{n,\Delta}$.

Proof. Let T be tree with maximum distance Laplacian spectral radius among all trees in $\mathcal{B}_{n,\Delta}$. Let v be a vertex of degree Δ and $T_1, \ldots, T_\Delta$ be the trees attached at v. By Corollary 5.3, it follows that each T_i is a path for $i = 1, \ldots, \Delta$. Without loss of generality we assume that the path T_1 has maximum length. By Corollary 5.3, it follows that length of each path $T_2, \ldots, T_\Delta$ is exactly one. Hence $T \cong B_{n,\Delta}$. □

Lemma 5.7. *Let $G_{r,s}$ be the graph obtained by attaching two paths of length r and s at two adjacent vertices u and v of a graph G, respectively. If $\rho^L_{r,s}$ denotes the distance Laplacian spectral radius of $G_{r,s}$ then $\rho^L_{r+1,s-1} + \rho^L_{r-1,s+1} \geq 2\rho^L_{r,s}$. Furthermore, either $\rho^L_{r+1,s-1} > \rho^L_{r,s}$ or $\rho^L_{r-1,s+1} > \rho^L_{r,s}$.*

Proof. Let $V(G) = \{1, 2, \ldots, m\}$ such that $u = 1$, $v = 2$. Let $\mathcal{D}^L_{r,s}$ be the distance Laplacian matrix of $G_{r,s}$. Let $V_1 = \{j \in V(G) - \{1,2\} | d_{1j} = d_{2j}\}$, $V_2 = \{j \in V(G) - \{1,2\} | d_{1j} < d_{2j}\}$, $V_3 = \{j \in V(G) - \{1,2\} | d_{1j} > d_{2j}\}$ and a, b, c be the number of vertices in V_1, V_2 and V_3, respectively. With respect to the ordered partition $V(G_{r,s}) = \{1,2\} \cup V_1 \cup V_2 \cup V_3 \cup (V(G_{r,s}) - V(G))$ we have

$$\mathcal{D}^L_{r+1,s-1} + \mathcal{D}^L_{r-1,s+1} - 2\mathcal{D}^L_{r,s} = \left[\begin{array}{cc|c|c|c|c} 1 & 2 & V_1 & V_2 & V_3 & V(G_{r,s}) - V(G) \\ \hline a+2b & 0 & -\mathbb{1}^t & -2\mathbb{1}^t & 0 & 0 \\ 0 & a+2c & -\mathbb{1}^t & 0 & -2\mathbb{1}^t & 0 \\ \hline -\mathbb{1} & -\mathbb{1} & 2I_a & 0 & 0 & 0 \\ \hline -2\mathbb{1} & 0 & 0 & 2I_b & 0 & 0 \\ 0 & -2\mathbb{1} & 0 & 0 & 2I_c & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Let $\mathbf{x}$ be an eigenvector corresponding to the spectral radius of $\mathcal{D}^L_{r,s}$. Observe that

$$\begin{aligned} \rho^L_{r+1,s-1} + \rho^L_{r-1,s+1} - 2\rho^L_{r,s} &\geq \rho(\mathcal{D}^L_{r+1,s-1} + \mathcal{D}^L_{r-1,s+1}) - 2\rho^L_{r,s} \text{ [by Lemma 4.1]} \\ &\geq \mathbf{x}^t(\mathcal{D}^L_{r+1,s-1} + \mathcal{D}^L_{r-1,s+1} - 2\mathcal{D}^L_{r,s})\mathbf{x} \text{ [by Lemma 4.2]} \\ &= 2\left[\sum_{j\in V_2}(\mathbf{x}_1 - \mathbf{x}_j)^2 + \sum_{j\in V_3}(\mathbf{x}_2 - \mathbf{x}_j)^2\right] \\ &\quad + \sum_{i=1}^{2}\sum_{j\in V_1}(\mathbf{x}_i - \mathbf{x}_j)^2 \geq 0. \end{aligned} \tag{5.23}$$

To the contrary, assume that $\rho_{r-1,s+1} = \rho_{r+1,s-1} = \rho_{r,s} = \rho$ (say). Then $\mathbf{x}$ is also an eigenvector of $\mathcal{D}^L_{r+1,s-1}$ and $\mathcal{D}^L_{r-1,s+1}$ corresponding to ρ. Label the vertices of $G_{r,s}$, $G_{r+1,s-1}$ and $G_{r-1,s+1}$ as in Fig. 3. We see that

$$\mathcal{D}^L_{r+1,s-1} - \mathcal{D}^L_{r,s} = \left[\begin{array}{cc|c|c|c} 1 & 2 & \overbrace{3,\ldots,m} & \overbrace{m+1,\ldots,m+r-1} & \overbrace{m+r,\ldots,n} \\ \hline m-2 & 0 & -\mathbb{1}^t & 0^t & 0^t \\ 0 & 0 & z^t & 0^t & 0^t \\ \hline -\mathbb{1} & z & C & -\mathbb{J} & \mathbb{J} \\ \hline 0 & 0 & -\mathbb{J}^t & (m-2)I & 0 \\ 0 & 0 & \mathbb{J}^t & 0 & -(m-2)I \end{array}\right].$$

Now we have

$$(\mathcal{D}^L_{r+1,s-1} - \mathcal{D}^L_{r,s})\mathbf{x} = 0. \tag{5.24}$$

From the first equation in (5.24), we have

$$(m-2)\mathbf{x}_1 = \sum_{j=3}^{m} \mathbf{x}_j. \tag{5.25}$$

Again from the i-th equation in (5.24), we have

$$(m-2)\mathbf{x}_i = \sum_{j=3}^{m} \mathbf{x}_j, \text{ for } i = m+1, m+2, \ldots, n. \tag{5.26}$$

From the equality in (5.23), we have

$$\mathbf{x}_1 = \mathbf{x}_j \text{ for } d_{2j} \leq d_{1j}, j = 3, \ldots, m. \tag{5.27}$$

From equations (5.25), (5.26) and (5.27), we have

$$\mathbf{x}_1 = \mathbf{x}_{m+1} = \mathbf{x}_{m+2} = \ldots = \mathbf{x}_n = \mathbf{x}_j \text{ for } d_{2j} \leq d_{1j}, j = 3, \ldots, m. \tag{5.28}$$

Similarly, considering the matrix $\mathcal{D}^L_{r-1,s+1} - \mathcal{D}^L_{r,s}$ we can show that

$$\mathbf{x}_2 = \mathbf{x}_{m+1} = \mathbf{x}_{m+2} = \ldots = \mathbf{x}_n = \mathbf{x}_j \text{ for } d_{1j} \leq d_{2j}, j = 3, \ldots, m. \tag{5.29}$$

By equations (5.28) and (5.29), we see that $\mathbf{x} = \alpha\mathbb{1}$ for some $\alpha \neq 0$, which implies that $\rho = 0$, a contradiction. Hence either $\rho^L_{r+1,s-1} > \rho^L_{rs}$ or $\rho^L_{r-1,s+1} > \rho^L_{r,s}$. □

Lemma 5.8. *Let U_0 be a graph obtained from a unicyclic graph by attaching a pendent path P at a vertex p of the cycle. Let $U_1 := U_0 - \{p,r\} + \{q,r\}$ and $U_2 := U_0 - \{p,r\} + \{r,s\}$, where q, r are vertices on the cycle adjacent to p and s on the path adjacent to p. If*

ρ_i^L denotes the Laplacian spectral radius of U_i for $i = 0, 1, 2$ then $\rho_1^L + \rho_2^L - 2\rho_0^L \geq 0$. Furthermore, either $\rho_1^L > \rho_0^L$ or $\rho_2^L > \rho_0^L$.

Proof. Let $V_1 = \{j \in V(U_0 - P) : d_{pj} \leq d_{qj}\}$, and $V_2 = V(U_0) - V_1 - \{p\}$. Let $\mathcal{D}_i^L$ be the distance Laplacian matrix of U_i for $i = 0, 1, 2$. With respect to the ordered partition $\{p\} \cup V_1 \cup V_2$ of $V(U_0)$, we have

$$\mathcal{D}_1^L + \mathcal{D}_2^L - 2\mathcal{D}_0^L = \begin{bmatrix} 2a & -2\mathbb{1}_a^t & 0 \\ -2\mathbb{1}_a & 2I_a & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ where } a = |V_1|.$$

Let $\mathbf{x}$ be an eigenvector corresponding to the spectral radius of $\mathcal{D}_0^L$. We see that

$$\begin{aligned} \rho_1^L + \rho_2^L - 2\rho_0^L &\geq \rho(\mathcal{D}_1^L + \mathcal{D}_2^L) - 2\rho_0^L \\ &\geq \mathbf{x}^t(\mathcal{D}_1^L + \mathcal{D}_2^L - 2\mathcal{D}_0^L)\mathbf{x} \\ &= 2\sum_{j \in V_1} (\mathbf{x}_p - \mathbf{x}_j)^2 \geq 0. \end{aligned}$$

Thus $\rho_1^L + \rho_2^L - 2\rho_0^L \geq 0$. If $\rho_1^L = \rho_2^L = \rho_0^L$, then with similar arguments as in the proof of Lemma 5.1 we can prove that $\mathbf{x} = \mathbf{0}$, which is a contradiction. Hence the Lemma holds. □

Lemma 5.9. *Let $C_n = [1, \ldots, \lceil \frac{n+1}{2} \rceil, n, n-1, \ldots, \lceil \frac{n+1}{2} \rceil + 1]$ be a cycle. Then the following holds.*

(i) The distance Laplacian spectral radius of C_n has multiplicity two.
(ii) If $n = 2k + 1$, then an eigenvector corresponding to spectral radius of $\mathcal{D}^L(C_n)$ is of the form $\begin{bmatrix} 0 \\ x \\ -x \end{bmatrix}$, where x is positive and strictly decreases along the two paths starting at $\lceil \frac{k+3}{2} \rceil$ and $\mathbf{x}_i > \mathbf{x}_{k-i+3}$ for $i = 2, 3, \ldots, \lceil \frac{k+1}{2} \rceil$.
(iii) If $n = 2k$, then an eigenvector corresponding to spectral radius of $\mathcal{D}^L(C_n)$ is of the form $\begin{bmatrix} x \\ -x \end{bmatrix}$, where x is positive and strictly decreases along the two paths starting at $\lceil \frac{k+1}{2} \rceil$ and $\mathbf{x}_i = \mathbf{x}_{k-i+1}$ for $i = 1, 2, \ldots, \lceil \frac{k}{2} \rceil$.

Proof. Let $n = 2k + 1$. Then we observe that

$$\mathcal{D}^L(C_{2k+1}) = \begin{bmatrix} k^2 + k & z^T & z^T \\ z & A & B \\ z & B & A \end{bmatrix},$$

where

$$z^t = [-1 \quad -2 \ldots \quad -k],\ A = \begin{bmatrix} k^2+k & -1 & -2 & \ldots & -(k-1) \\ -1 & k^2+k & -1 & \ldots & -(k-2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -(k-1) & -(k-2) & -(k-3) & \ldots & k^2+k \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} -2 & -3 & -4 & \ldots & -(k-1) & -k & -k \\ -3 & -4 & -5 & \ldots & -k & -k & -(k-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ -k & -k & -(k-1) & \ldots & -4 & -3 & -2 \\ -k & -(k-1) & -(k-2) & \ldots & -3 & -2 & -1 \end{bmatrix}.$$

Also, we see that for some permutation matrix P,

$$\mathcal{D}^L(C_{2k+1}) = \left[\begin{array}{c|c} P^T \begin{pmatrix} A & B \\ B & A \end{pmatrix} P & \begin{matrix} z_1 \\ z_2 \end{matrix} \\ \hline \begin{matrix} z_1^T & z_2^T \end{matrix} & k^2+k \end{array}\right],$$

where $z_1^t = [-k \quad -k \quad -(k-1) \quad \ldots \quad -2]$, $z_2^t = [-1 \quad -(k-1) \quad -(k-2) \quad \ldots \quad -1]$.

Thus it follows that if λ is an eigenvalue of $A-B$ with an eigenvector x, then $\begin{bmatrix} 0 \\ x \\ -x \end{bmatrix}$ and $\begin{bmatrix} Px \\ -Px \\ 0 \end{bmatrix}$ are two linearly independent eigenvectors of $\mathcal{D}^L(C_{2k+1})$ corresponding to the eigenvalue λ. Since $A-B$ has k eigenvalues and 0 is not an eigenvalue of $A-B$, we see that the spectral radius of $A-B$ is the spectral radius of $\mathcal{D}^L(C_{2k+1})$. Moreover, $A-B$ is positive. Thus by Perron–Frobenius theorem x is positive.

For $2 \le i \le \lceil \frac{k+1}{2} \rceil$, from eigen-equations we have

$$\left[\rho^L(C_{2k+1}) - k^2 - k\right](\mathbf{x}_{i+1} - \mathbf{x}_i) = 2\sum_{j=i+1}^{k-i+1} \mathbf{x}_j + \mathbf{x}_{k-i+2} > 0, \tag{5.30}$$

$$\left[\rho^L(C_{2k+1}) - k^2 - k\right](\mathbf{x}_i - \mathbf{x}_{k-i+3}) = \sum_{j=i}^{k-i+2} \mathbf{x}_j > 0 \tag{5.31}$$

$$\text{and } \left[\rho^L(C_{2k+1}) - k^2 - k\right](\mathbf{x}_{k-i+2} - \mathbf{x}_{k-i+3}) = \mathbf{x}_i + 2\sum_{j=i+1}^{k-i+2} \mathbf{x}_j > 0. \tag{5.32}$$

Since $\rho^L(C_{2k+1}) > k^2+k$, (ii) follows from (5.30)–(5.32). Similarly we can prove (iii). □

Lemma 5.10. *Let C_n be a cycle. Then $\rho^L(U_3^n) > \rho^L(C_n)$.*

Proof. Let $C_n = [1, \ldots, \lceil \frac{n+1}{2} \rceil, n, \ldots, \lceil \frac{n+1}{2} \rceil + 1]$. Remove the edge $\{1, \lceil \frac{n+1}{2} \rceil + 1\}$ and add the edge $\{1, 3\}$ to obtain U_3^n and then observe that distances between 1 and i decreases by 1 for $i = 3, \ldots, \lceil \frac{n+1}{2} \rceil$, increases by at least 1 for $i = \lceil \frac{n+1}{2} \rceil, \ldots, n$ and distance among

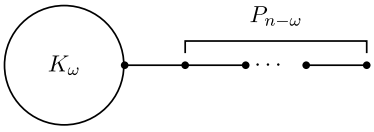


Fig. 5. The graph $PK_{n-\omega,\omega}$.

all other vertices do not decrease. Let $\mathbf{x}$ be an eigenvector of $\mathcal{D}^L(C_n)$ corresponding to the spectral radius. Then we see that

$$\begin{aligned}
\rho^L(U_3^n) - \rho^L(C_n) &\geq \mathbf{x}^t(\mathcal{D}^L(U_3^n) - \mathcal{D}^L(C_n))\mathbf{x} \text{ [by Lemma 4.2]} \\
&\geq \sum_{i=\lceil \frac{n+1}{2} \rceil + 1}^{n-1} (\mathbf{x}_1 - \mathbf{x}_i)^2 - \sum_{i=3}^{\lceil \frac{n+1}{2} \rceil} (\mathbf{x}_1 - \mathbf{x}_i)^2 \\
&\geq \sum_{i=3}^{\lceil \frac{n+1}{2} \rceil - 1} \left[(\mathbf{x}_1 + \mathbf{x}_i)^2 - (\mathbf{x}_1 - \mathbf{x}_i)^2 \right] \\
&+ (\mathbf{x}_1 + \mathbf{x}_2)^2 - (\mathbf{x}_1 - \mathbf{x}_{\lceil \frac{n+1}{2} \rceil})^2, \text{ using Lemma 5.9} \\
&= 4\mathbf{x}_1 \sum_{i=3}^{\lceil \frac{n+1}{2} \rceil - 1} \mathbf{x}_i + 2\mathbf{x}_1(\mathbf{x}_2 + \mathbf{x}_{\lceil \frac{n+1}{2} \rceil}) + {\mathbf{x}_2}^2 - {\mathbf{x}_{\lceil \frac{n+1}{2} \rceil}}^2 \\
&> 0, \text{ as } \mathbf{x}_2 > \mathbf{x}_{\lceil \frac{n+1}{2} \rceil} \text{ by Lemma 5.9.}
\end{aligned}$$

Hence $\rho^L(U_3^n) > \rho^L(C_n)$. □

Theorem 5.11. *The unique graph that maximizes the distance Laplacian spectral radius among all unicyclic graph on n vertices is U_3^n.*

Proof. Let U be the unicyclic graph that maximizes the distance Laplacian spectral radius among all unicyclic graph of order n. By Lemma 5.1, we see that every tree branch of U attached at a vertex of the cycle is a path and by Lemma 5.7, U has no tree branch at two adjacent vertices of cycle. By repeated application of Lemma 5.8, we see that U is either C_n or U_g^n for some g. By Lemma 5.10, we see that $\rho^L(U_3^n) > \rho^L(C_n)$. Thus $U \ncong C_n$. If $U \cong U_g^n$, then by Lemma 5.8, we see that either $\rho^L(U_{g-1}^n) > \rho^L(U)$ or $\rho^L(U_{g+1}^n) > \rho^L(U)$. Hence the Theorem holds. □

A *clique* of a graph is a maximal complete subgraph and *clique number* of a graph is the order of a maximal clique. Let $\mathbf{G}_n^\omega$ be the class of all graphs of order n with clique number ω. Here, we find the graph with maximal distance Laplacian spectral radius in $\mathbf{G}_n^\omega$, where $2 \leq \omega \leq n$. A *kite graph*, denoted by $PK_{n-\omega,\omega}$, is a graph obtained by attaching a path $P_{n-\omega}$ to a vertex of a complete graph K_ω [see Fig. 5]. Clearly, $PK_{n-\omega,\omega} \in \mathbf{G}_n^\omega$.

Theorem 5.12. *If $G \in \mathbf{G}_n^\omega$, then $\rho^L(G) \leq \rho^L(PK_{n-\omega,\omega})$. The equality holds if and only if G is isomorphic to $PK_{n-\omega,\omega}$.*

Proof. If $n = \omega$, then $\mathbf{G}_n^\omega = \{K_n\}$. Since $K_n = PK_{0,n}$, so the result holds in this case. Therefore we consider $n > \omega \geq 2$.

(i) If we delete an edge (if any) from G which is not in K but on a cycle, then the resulting graph is still in $\mathbf{G}_n^\omega$. Let G_1 be a spanning subgraph of G obtained by deleting such edges (if any) in an arbitrary order as long as they exist. Thus, G_1 consists of a clique K with a number of rooted trees attached to clique vertices. Since deletion of an edge from a graph does not decrease its distance Laplacian spectral radius, so $\rho^L(G_1) \geq \rho^L(G)$.

(ii) Let $T_1, T_2, \ldots, T_\ell$ be maximal disjoint trees attached at a vertex v of K, in G_1. We can repeatedly apply the transformation from Lemma 5.1 at any vertex of degree at least three with the largest eccentricity from v in every tree T_i, as long as T_i does not become a path. Each application of this transformation strictly increases the distance Laplacian spectral radius.

Now we can again apply the transformation from Lemma 5.1 at the vertex v as long as there exist at least two paths of length at least one, further increasing the distance Laplacian spectral radius.

(iii) If $\omega = 2$, then $G_2 = P_n = PK_{n-2,2}$, and the result holds in this case. Otherwise, let G_2 consist of clique K and the paths $P_{\ell_1}, P_{\ell_2}, \ldots, P_{\ell_m}$ attached to m distinct vertices of K. With the repeated use of Lemma 5.7 to these paths, we finally arrive at the kite graph $PK_{n-\omega,\omega}$. By Lemma 5.7, the distance Laplacian spectral radius is increased at each step.

Therefore, we may conclude that the kite graph $PK_{n-\omega,\omega}$ has larger distance Laplacian spectral radius than any other graph in $\mathbf{G}_n^\omega$. This completes the proof. □

6. Distance signless Laplacian spectral radius

The following Lemma is similar to Lemma 4.3 and therefore its proof is omitted.

Lemma 6.1. *Let G_v be the graph obtained by identifying a vertex v of the graph G with a fixed vertex u of another graph G'. If i, j, k are consecutive vertices of G such that i and k are on different components of $G - j$ and $\rho_v^{\mathcal{Q}}$ denotes the distance signless Laplacian spectral radius of G_v, then $\rho_i^{\mathcal{Q}} + \rho_k^{\mathcal{Q}} - 2\rho_j^{\mathcal{Q}} > 0$.*

Corollary 6.2. *Let T be a tree and G be a graph. If T_i and $\rho_i^{\mathcal{Q}}$ are as defined in the above Lemma then $\rho_i^{\mathcal{Q}}$ is strictly convex on T.*

The following result similar to that of Corollary 4.5 and Corollary 5.3 follows immediately from Corollary 6.2.

Corollary 6.3. *Let G be a simple graph and v be one of its vertices. If $k \geq \ell \geq 1$, then $\rho_{v,k,\ell}^{\mathcal{Q}} < \rho_{v,k+1,\ell-1}^{\mathcal{Q}}$.*

Theorem 6.4. *Let T be a tree with maximal distance signless Laplacian spectral radius in $\mathcal{T}_n^m$. Then $T \cong D(n, a, b)$, $a + b = n + 1 - 2m$.*

Proof. If $m = 1$, then the result is trivial. Suppose $m \geq 2$, and let T be a tree in $\mathcal{T}_n^m$ with maximal distance signless Laplacian spectral radius. Then T has 2 quasipendent vertices. Otherwise, there exists a vertex v in T such that $T - v$ has at least 3 components, and at least two of which are nontrivial. Let T_x, T_y, T_z be three branches at the vertex v of T, and M be an m-matching of T. Then at least two of vx, vy and vz are not in M.

Case 1: Suppose T_x and T_y are non-trivial and vz is not in M. Let x_1, y_1 be quasipendent vertices in T_x, T_y respectively. Then the trees $T' = T - vz + x_1z$ and $T'' = T - vz + y_1z$ are in $\mathcal{T}_n^m$, and by Corollary 6.2, $\rho^{\mathcal{Q}}(T') > \rho^{\mathcal{Q}}(T)$ or $\rho^{\mathcal{Q}}(T'') > \rho^{\mathcal{Q}}(T)$, a contradiction to the maximality of T.

Case 2: Suppose vz is in M and T_z is non-trivial. Then both vx and vy are not in M, and say T_x is non-trivial. Reversing the roles of y and z in Case 1, we have a contradiction.

Case 3: Suppose vz is in M and T_z is trivial. Since T_y is nontrivial therefore there exists a vertex u in T_y which is in a matching. Then both the trees $T' = T - vx + zx$ and $T'' = T - vx + ux$ have matching number m, but by Corollary 6.2 at least one has a larger distance signless Laplacian spectral radius than T, a contradiction to the maximality of T.

Thus $T \cong D(n, a, b)$, for some a and b. □

Theorem 6.5. *Let T be a tree with maximal distance signless Laplacian spectral radius in $\mathcal{T}_{n,k}$. Then $T \cong D(n, a, b)$, $a + b = k$.*

Proof. Let $P = i_1 - i_2 - \cdots - i_k$ be a path of maximum length in T. If $m = 1, 2, 3$, then there is nothing to prove. Suppose $m \geq 4$. Then degrees of $i_3, \ldots, i_{m-2}$ in T are all 2. To the contrary assume that $deg(i_j) \geq 3$. Let T' be the component of $T - i_{j-1} - i_{j+1}$ containing vertex i_j. Let T_1, T_2 be the new trees obtained by shifting T' to i_{j-1} and i_{j+1}, respectively. By Lemma 6.1, the distance Laplacian spectral radius of T_1 or T_2 is bigger than that of T, a contradiction. Hence the Theorem holds. □

The next result says that broom is the unique tree that maximizes the distance signless Laplacian spectral radius among all trees with fixed maximum degree.

Theorem 6.6. *Among all trees in $\mathcal{B}_{n,\Delta}$ the unique tree that maximizes the distance signless Laplacian spectral radius is $B_{n,\Delta}$.*

Proof. Let T be a tree with maximum distance Laplacian spectral radius among all trees in $\mathcal{B}_{n,\Delta}$. Let v be a vertex of degree Δ and $T_1, \ldots, T_\Delta$ be the trees attached at v. By Corollary 6.3, it follows that each T_i is a path for $i = 1, \ldots, \Delta$. Without loss of generality we assume that the path T_1 has maximum length. By Corollary 6.3, it follows that length of each path $T_2, \ldots, T_\Delta$ is exactly one. Hence $T \cong B_{n,\Delta}$. □

The following Lemma is similar to Lemma 4.10 and Lemma 5.7 whose proof also follows similarly.

Lemma 6.7. *Let $G_{r,s}$ be the graph obtained by attaching two paths of length r and s at two adjacent vertices u and v of a connected graph G, respectively. Let $\rho^{\mathcal{Q}}_{r,s}$ be the distance Laplacian spectral radius of $G_{r,s}$. Then*

$$\rho^{\mathcal{Q}}_{r+1,s-1} + \rho^{\mathcal{Q}}_{r-1,s+1} > 2\rho^{\mathcal{Q}}_{r,s}.$$

Lemma 6.8. *Let U be a unicyclic graph with the cycle $C = [1, 2 \ldots, m]$. Let T_1 and T_2 be two tree branches at the vertices 1 and k of C such that $deg(i) = 2$ for $i = 2, \ldots, k-1$. Let $\mathbf{x}$ be the Perron vector of the distance signless Laplacian of U then $-\mathbf{x}$ induces a quasiconvex function on $\{1, \ldots k\}$.*

Proof. Since $Tr(i) + Tr(i+2) \leq 2Tr(i+1)$, we see that

$$2\rho^{\mathcal{Q}} - Tr(i) - Tr(i+2) \geq 2(\rho^{\mathcal{Q}} - Tr(i+1)) \text{ for } 1 \leq i \leq k-2.$$

By eigenequation of $\mathbf{x}$ at $i, i+1, i+2$ we have

$$\begin{aligned}
&(\rho^{\mathcal{Q}} - Tr(i))\mathbf{x}_i + (\rho^{\mathcal{Q}} - Tr(i+2))\mathbf{x}_{i+2} - 2(\rho^{\mathcal{Q}} - Tr(i+1))\mathbf{x}_{i+1} < 0 \\
\Rightarrow\ &(\rho^{\mathcal{Q}} - Tr(i))\mathbf{x}_i + (\rho^{\mathcal{Q}} - Tr(i+2))\mathbf{x}_{i+2} - (2\rho^{\mathcal{Q}} - Tr(i) - Tr(i+2))\mathbf{x}_{i+1} < 0 \\
\Rightarrow\ &\lambda\mathbf{x}_i + (1-\lambda)\mathbf{x}_{i+2} < \mathbf{x}_{i+1}, \text{ where } \lambda = \frac{\rho^{\mathcal{Q}} - Tr(i)}{2\rho^{\mathcal{Q}} - Tr(i) - Tr(i+2)}.
\end{aligned}$$

Thus $\mathbf{x}_{i+1} > \min\{\mathbf{x}_i, \mathbf{x}_{i+2}\}$ and the proof is complete. □

Lemma 6.9. *Let U be a unicyclic graph obtained by attaching a path $P := 1-2-\cdots-k$ at the vertex k of the cycle $C := [k, \ldots, n]$. Let $\mathbf{x}$ be the Perron vector of the distance signless Laplacian of U. Then $\mathbf{x}$ induces a strictly convex function on $\{1, 2, \ldots, k, \ldots, k + \lceil \frac{g}{2} \rceil\}$.*

Proof. For our convenience let $t = k + \lceil \frac{g}{2} \rceil$. By eigenequation of $\mathbf{x}$ at t and $t-1$, we have

$$\rho^{\mathcal{Q}}(\mathbf{x}_t - \mathbf{x}_{t-1}) = \sum_{j=1}^{t-2} \mathbf{x}_j - \sum_{j=t+1}^{t+\lceil \frac{g}{2} \rceil} \mathbf{x}_j + Tr(t)\mathbf{x}_t - Tr(t-1)\mathbf{x}_{t-1}.$$

Thus we have the following:

$$(\rho^{\mathcal{Q}} - Tr(t) + 1)\mathbf{x}_t - (\rho^{\mathcal{Q}} - Tr(t-1) - 1)\mathbf{x}_{t-1} > 0.$$

Since $\rho^{\mathcal{Q}} - Tr(t) + 1 < \rho^{\mathcal{Q}} - Tr(t-1) - 1$, we see that $\mathbf{x}_t > \mathbf{x}_{t-1}$.

By Lemma 6.8, it follows that $-[\mathbf{x}_k, \ldots, \mathbf{x}_{k+\lceil \frac{g}{2} \rceil}]$ is convex. Thus $\mathbf{x}_{k+\lceil \frac{g}{2} \rceil} > \cdots > \mathbf{x}_k$. Similarly we can show that $[\mathbf{x}_1, \ldots, \mathbf{x}_{k+1}]$ is strictly convex. Using eigenequation we can show that $\mathbf{x}_1 > \mathbf{x}_2$. Hence the lemma holds. □

Lemma 6.10. *Let C_n be a cycle. Then $\rho^{\mathcal{Q}}(U_3^n) > \rho^{\mathcal{Q}}(C_n)$.*

Proof. Observe that $\mathbb{1}$ is a Perron vector of $\mathcal{D}^Q(C_n)$ and

$$\mathbb{1}^t \left\{ \mathcal{D}^Q(U_3^n) - \mathcal{D}^Q(C_n) \right\} \mathbb{1} = 2 \left\{ \sum_{u \in V(U_3^n)} Tr(u) - \sum_{u \in V(C_n)} Tr(u) \right\} > 0.$$

By Lemma 4.2 we have $\rho^{\mathcal{Q}}(U_3^n) > \rho^{\mathcal{Q}}(C_n)$. □

The following Lemma is similar to Lemma 5.8 whose proof also follows similarly.

Lemma 6.11. *Let U_0 be a graph obtained from a unicyclic graph by attaching a pendent path P at a vertex p of the cycle. Let $U_1 := U_0 - \{p, r\} + \{q, r\}$ and $U_2 := U_0 - \{p, r\} + \{r, s\}$, where q, r are vertices on the cycle adjacent to p and s on the path adjacent to p. If $\rho_i^{\mathcal{Q}}$ denotes the distance signless Laplacian spectral radius of U_i for $i = 0, 1, 2$ then $\rho_1^{\mathcal{Q}} + \rho_2^{\mathcal{Q}} - 2\rho_0^{\mathcal{Q}} \geq 0$. Furthermore, either $\rho_1^{\mathcal{Q}} > \rho_0^{\mathcal{Q}}$ or $\rho_2^{\mathcal{Q}} > \rho_0^{\mathcal{Q}}$.*

Theorem 6.12. *The unique graph that maximizes the distance signless Laplacian spectral radius among all unicyclic graph is U_3^n.*

Proof. Let U be a unicyclic graph with maximum distance signless Laplacian spectral radius in the class of all unicyclic graph. By Lemma 6.1, we see that every tree branch of U attached at a vertex of the cycle is a path and by Lemma 6.7, U has no path branch at two adjacent vertices of the cycle. By repeated application of Lemma 6.11 and 6.7, we see that U is either C_n or U_3^n. Hence by Lemma 6.10 the Theorem holds. □

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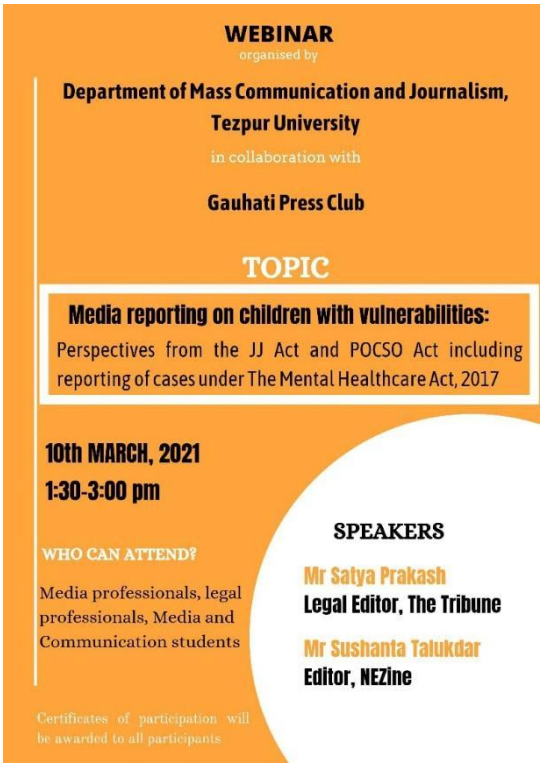
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Webinar Report

Media reporting on children with vulnerabilities: Perspectives from the JJ Act and POCSO Act including reporting of cases under The Mental Healthcare Act, 2017

10 March 2021

The Department of Mass Communication and Journalism, Tezpur University organised a webinar titled '**Media reporting on children with vulnerabilities: Perspectives from the JJ Act and POCSO Act including reporting of cases under The Mental Healthcare Act, 2017'** in collaboration with the Judicial Academy, Assam and Gauhati Press Club on 10th March, 2021.



Mr. Justice N. Kotiswar Singh, honourable judge, Gauhati High Court delivered the inaugural speech. The resource persons for the webinar were- Mr Satya Prakash, legal editor of The Tribune and Mr Sushanta Talukdar, the Editor of Nezine. Over 300 participants including media and law students and faculty, media professionals and legal professionals from across the country participated in the webinar.

Dr. Anjuman Borah, Assistant Professor, Department of MCJ, Tezpur University invited the resource persons and introduced them to the virtual house comprising of more than 200 participants.

Mr. Satya Prakash spoke on three important acts: The Juvenile Justice Act, the POCSO Act and the Mental Healthcare Act, which came into effect from May 29, 2018. He spoke that the journalists need to take care of the differences between Activism and Journalism. He also laid down focus on the balance between media freedom and the responsibility of the journalists. Adding on the JJ Act, he spoke that the ambit of maintaining the anonymity of the victims needs to be extended. He was very optimist of this joint collaboration made between Judicial Academy, Gauhati Press Club and the MCJ Department at Tezpur University.

Mr Sushanta Talukdar started his part of the discussion accepting the realities of current scenario of media, where the outlets are taking part in a mad-rush of headlines and deadlines .This in turn doesn't allow the reporting to be sensible enough and leads to

breakage of many rules of the JJ Act or POCSO Act. He added the importance of proper gate-keeping in the organisations so that a sensitive reporting can be done, unlike the current ways where the media houses intensifies the trauma of the victims. He spoke that the journalists need to be crucial on reporting the geographical location of the victims. If a crime like that happens in Guwahati, they should report it as Guwahati, rather than mentioning the exact location, which makes it easier for the people to identify the victim. Mr. Talukdar laid down many important guidelines that media organisations need to adopt as self-regulatory measures.

The speeches were followed by a question-answer session where the resource person addressed the queries of the students. The session was followed by a brief deliberation from Mr. Sanjoy Ray, the General Secretary of Gauhati Press Club. The session came to an end with Professor Joya Chakraborty, Head of the Department of MCJ, giving a vote of thanks to all the resource persons, collaborators and participants attending the webinar.

Workshop on Social Media and Marketing

By Pulak Sharma

Date: 9-12th February

Time: 12:00-1:00 pm

Duration- 4 days

Platform: Zoom Meeting

DAY 1

First day of Digi-Dialogues, an advertising workshop hosted by a well experienced advertising expert named Pulak Sharma started with the introduction of digital marketing or say social media marketing.It began with a brief detail about digital evolution I.e. how rapidly usage of smartphones grewfrom the launch of first cellphone. Digital media has transformed in such a massive way that passing a new checkpoint didn't get time to celebrate. Launch of Linked-in in 2003, Facebook in 2004, YouTube in 2005, Instagram in 2010 and more such creative internet application were observed as the production of one time. Such remarkable developments in digital field are also backed by another study which further differentiate among the years taken by companies' to reach the consumer base of 50 millions. In this research, it has been founded that airline companies took almost 50 years to attain the consumer base of 50 millions whereas PokemonGo (an online gaming application) took only 19 days to attain such height. In addition to this information, today more than 59% of world's population uses internet and more than 53% is active on social media. The numbers are still growing around the globe with a faster rate of increase. After establishing more such facts, the workshop went into how this huge user base is getting converted into a hub of marketers. Further, Mr. Pulak made awarded about the transition towards 3Vs (Vernacular, Voice, Video) from the traditional advertising approach of 4Ps. On the basis of some growing demands of consumers i.e. comfort, connection and pursuance these 3Vs are being proven the most useful catalyst to win the users or to convert the users into customers in the world of social media. Reaching towards end, he discussed about the major growth drivers of social media like mobile phones, vernacular users,social conditions etc. He also briefly explained the aspects on which digital marketing is based i.e. Performance, branding, social and content. The first day concluded with the study of the role of Consumer Funnel in Digital Marketing.

Digi Dialogues

Day 2

Date: 10/02/2021

Report:

The second day of lecture series on the topic Digi Dialogues by Pulak Sharma, started around 12:15 pm. The lecture started with the introduction of Performance Marketing. It is one of the aspects of digital marketing and it is number driven. Performance marketing is all about sales, leads which are the potential prospects, traffic which is the number of visitors who come to a website, and app downloads. It is about paying for results that can be directly attributed. Then the discussion moves forward to the topic of Performance Marketing Areas. Performance marketing ensures three things tracking, monitoring and controlling. Affiliate marketing, social media, display media, SEA, SEO, couponing and E-mail marketing, these are the areas of performance marketing. Display media are the banner ads. SEA/SEM stands for search engine marketing, the information which is shown to us when we search for a particular product/service keyword. SEO is search engine optimization. Couponing means giving away of coupons containing discounts. E-mail marketing is a cost-effective way of doing performance marketing. Social media including FB, WhatsApp etc.

Affiliate marketing is similar to E-mail marketing based on data collection of the users or potential users. Majority of the performance marketing happens through search which is controlled by Google and Bing. A lot of performance marketing happens on FB, it is the second largest advertising platform. The largest is Google. Then there is display which is largely controlled by Google. Important terminologies are; Traffic – website visitors, CPC – cost per click, CPL – cost per load, CPV – cost per view, CPD – cost per download, CPA - cost per acquisition.

Then the discussion solely focuses on SEM – search engine marketing. It works on a Keyboard based bidding model. Google and Bing are the largest search engines. Bing is owned by Microsoft. Google owns about 90% of the searches that happens across the world followed by Bing at roughly 8%. The advertisers will pay Google to show their ads. Whatever we type in a search box are known as search terms. The words that we type are called keywords. Whichever results show up if ‘ad’ is mentioned at the top of any result that means that the advertisement is paid. Other results can be called as the organic results which are usually non paid form ads. The advertisers have to win the bidding for the top spots while searched for those keywords related to the advertisements. The lecture continues to further discuss SEO, which is search engine optimization. SEO is smart compared to SEM and doesn’t pay Google. For SEO a lot of content optimization has to be done. The best content optimized result will be selected and recognized by Google and be given the top spot while searched upon. For understanding SEO, three R’s are important, namely Reputation, Relevance and Recognition. Google has a software called crawler/spider, every hour this program crawls and reads through all the websites in the world. Based on the readings, crawler ranks those websites and that is how Google shows us the result. If a result is on the top spot that means the crawler has found the website best in terms of the 3R’s. Reputation is built over the years. People writing reviews also affect the

reputation. The content in the website have to relevant. The website has to be recognized. Recognition happens through reviews and the mention of the website in other websites. After this the lecture moves on to the topic of display marketing. Any websites have smaller ads known as the display ads. Google has bonded with around 80% of the websites in the world. Google has bought the ad spaces from them and pays them in terms of the visitors it sends to each website. This is the Google display network. The discussion then shifts to the topic of remarketing. When an individual moves from one website to another website, the same ads are again present on the latter websites. This is called remarketing. Most of the websites asks for the individual's permission. There is a privacy policy that needs to be accepted. Accepting the privacy policy, the individual is identified in terms of his/her choices and gets tagged. Hence the website now contains and stores the behavioral information of the individual. The session finally comes to an end at around 1:25 pm.

Day 3: Digi Drive

Topic: Social media marketing

Time : 12-1:00 pm

Platform: Zoom drive

The workshop was started with a question by Pulak Sharma, who compared the popularity of different

forms of social media, taking YouTube and Facebook as the primary examples. He asked the students

about their opinion on the comparative popularity of these media platforms. He then described the

scope of YouTube as a broadcast medium, which was followed by stats. Focus was given on the

measuring impact and exposure measuring aspects of these medium on the audience which revolved

around topics such as reach and engagements.

Facebook being the largest media platform for engagement has infinite scope. The emergence of local

platforms such as KOO as an alternative of Twitter gives rise to regionalism. These platforms gives in

scope for various local language which results in more participation. Native languages and regionalism is

fuelling such local apps and as communicators it is our role to understand it's importance.

The workshop then proceeded to understand platforms based on time spent and perspectives.

Facebook and WhatsApp leading in this scenario, WhatsApp had introduced a new payment form and

brands are adapting to WhatsApp as a new medium of one to one communication. This is because of its

effectiveness. Moreover WhatsApp is quick and provides a personalized touch. Brands such as make my

trip have started to venture in this platform. WhatsApp was bought by Facebook around six seven years

back for 19 billion USD.

All apps are constantly improved first. The aspect of monetization comes in late. The age group ranging

from 16-54 years are usually active on social media. India dominates this scene. Answering to the

question if a student, Pulak Sharma introduced the monetization of WhatsApp. He stated that

WhatsApp is monetizing through brands and charging license fee.

Brands are incredibly using Facebook and other social media to reach out it's customers. One can now

search for the brand in any social media especially Facebook before purchasing a product. Thus, brand

research on social media is emerging and flourishing. Businesses too use social media for brand

research. The demographic of Facebook allows more flexibility to the brand's.

Social media can also be used as a method of brand discovery. One can be exposed to new products

through these ads. As communicators one must have a content strategy, as to what to produce and when. This growth is due to various factors, the first side being the consumer side. Social networking

allows connectivity, it lends us a strong platform of voice and expression. It provides a sense of self

worth and entertainment.

An advertiser spends time on social media due to factors such as size of audience, the sticky

characteristics of audience and the time spending nature on social media. It also helps in the precise

targeting opportunities and the virality factor is content. Consumer needs and brand needs are perfectly

intertwined here.

Some social networking platforms such as YouTube, Facebook, Twitter, Instagram have now become an

integral part of the consumer mind-set. Social media helps spark a conversation through campaigns and

spark discussion and debated. It can create awareness about a brand, product or service; leading to the

consideration of such brands, products and services. This can be done deploying influencers etc. An

example of a case study was given #DRIVELIKEALADY by Kodak for car insurance. This campaign comes

with an embedded social message and is backed by statistics. This campaign consisted of real life human

stories of achievement of ordinary people. The campaign was divided into three parts Head (emotional

aspect), Heart (connectivity with audience aspect) and Hand (incentives for target consumers). Examples

of Twitter campaign which can be used for the advantage of brands were given. Leverage of each

medium for the brand's was emphasized upon.

In conclusion focus was given on the creation and maintenance of brand images. A brand image must

always be free from scandals and must not do anything that would hurt the sentiments of it's

consumers. The rise of internet connectivity can also result in trolling of the brand, this is where Online

Reputation Management branch of brands come in. They communicate with the leads and consumers.

ORM branch should be creative and constantly analysing and monitoring to identify new leads.

The workshop then ended with a question round by the students to Pulak Sharma, who addressed their

doors and queries.

Day 4 – Digi Dialogue

Date - 12th January

Time – 12am to 1pm

Workshop on Social media marketing

By Pulak Sharmah

On 12th January at 12 am, students of MCJ and C4D participated in the 4th day of Digi Dialogue by Pulak Sharmah, Media Manager at Kotak Mahindra. On virtual platform Zoom, Mrs Anjuman welcomed the guest speaker to the department of mass communication and journalism at Tezpur university.

Event brief by the guest Speaker:-

Social media platforms details

Advantages and disadvantages of social media platforms

Demo for setting up page

Analyzing a web page's growth and data

The workshop started by showing the cover page of 'The Guardian Weekly' a magazine which is a leading publication. Mr. Pulak asked the participants 'what's special about the cover page? The cover contained the picture of Jeff bezoz which the logo of Amazon painted with his smile, many participants replied with enthusiasm.

After the kick start of the workshop, he started with the topic social media management, which is the key to find the right platform for instant in the movie 'Dear Zindagi' the question arises, Have you ever bought a chair? Finding the right chair is very important similarly finding the right platform for message/advertisement is Important.

Facebook: one should have an active content strategy regarding what he /she is going to talk about? How frequently will he/she talk? The strategy must be clear before setting up any page

on social media. Small businesses are coming to Facebook, almost all brands are there, Food, Travel, E-commerce, are some of the biggest categories. Facebook has:-

Good Audience Reach

Better Engagement

Visual Appeal

Storytelling

Precise Targeting

Shareable

Low Investment

Example - On Television you have 30 second to tell your message which cost you around 30 lakh but on social media you will get more time with less money.

Instagram: it is a visual experiential platform where attractive filters catch the attention of users. Instagram provides good exposure for personal branding and Influencers. It is extremely popular among home businesses and visual bloggers.

Further the workshop proceeded with the Case study on 'Facebook Thumbstopper' a new property in vertical form in which a 10 sec video will be uploaded by brands and directors like Prakash Verma, Kiran rao, Kotak and many more contested from across India. The idea was to promote a shorter vertical format of video as the user time span is decreasing day by day. The thumb stopper stories were highly creative and engaging; the participants interacted while explaining each one of them and discussed the idea of it.

Youtube : one of the biggest advertising platforms in terms of money earner. Brands are increasingly using it as a broadcast medium and a good replacement of television. The platform could be used as a content repository for all the users and creators. Youtube promotes their key influencers by annual events. It has:-

High Audience Reach

High Visual Appeal

Engagement

Precise Targeting

Low Investment For Entries

The information aroused the curiosity of the participants, Bishal asked the Question about the money advertisers get on uploading their advertisement as youtube works on views, like and

subscribe. The participants get to know that the channel is under the brand category where youtube don't pay the advertisers.

Twitter – Is more of ORM Hygiene of brands, it is good for announcement, thought leadership of individual and personal branding. It is also being used by brands for customer service purpose and reputation management. The platform provides:-

Intellectual appeal

Conversational

Engagement

Linkden: is a platform with good targeting opportunity for corporates, employers branding and highlighting employers in good light is been done of Linkden. Recruitment demands and freelancers are major content creators here. Mostly used by good employers, universities, college, courses, etc to built in their image around professionals.

Professional imagery

Intellectual appeal

Precise targeting

The session further proceeded with the live facebook page of Mr. Pulak which is his own travel and food page 'tummies on tour'. He had given the live example of how to promote a post, select the target audience, the location and interest of the TA, and their age group. He stated that the broader the location will be the more people will get to target. More text on the image will be disapproved by FB to advertise as they have their own protocols for the advertisers.

He further took the participants into the insight growth of his page, which shows how to check audience, it shows how many people visited your page, liked it, and reached to it and other post engagement. After all It is all about engagement on the page not the followers, he explained with the example of a singer Badsha, he had many followers but later exposed that they were fake. The key learning was the more you interact the more your post will be shown to the people. This is how social media earns money. Without paying, a brand could only reach only 2% of its followers but after paying, it could reach 29 to 30%.

'Create good content not followers' he stated.

The workshop ended with some questions asked by the participants, which is 'If on posting something one of your friends does not get your notification, is it a glitch or FB purposefully does it? It's all about engagements, to whom you are interacting frequently will get notified, others won't. Mr. Pulak answered.

The session concluded with gratitude and Thanks from all the participants for organizing such a workshop.

Workshop on Women and Violence

29-31 August 2016

BACKGROUND:

Tezpur University is a central university established in 1994 by an act of parliament. Since then the university has emerged as a premier institute of science and technology with a strong focus on socio – cultural development within the region.

INTRODUCTION:

Earlier the department of mass communication and journalism has already organized many such workshops. But this is the first workshop for the communication for development programme which was officially inaugurated this very year on the 26th August 2016.This workshop was organized under the guidance of Mrs.Anjuman Borah .Through this workshop we had the opportunity to interact with Dr. Monisha Behal who is the executive director and founder of the North East Network and Mrs.Sheetal Sharma works as project manager in North East Network which is a women's organization based in the northeast region.

About NEN- North east network is a women's organization working in the northeast region of India with a focus on women's human rights. Ever since its inception in 1995 their role has been that of a facilitator to empower women of northeast around issues of livelihood, health, conflict and governance through capacity building, awareness raising, networking, research and advocacy. Their work has been aimed at bringing out gendered understandings of human right situations in the region whether it's in the context of violence, livelihood, conflict, natural resource management or health.

AIMS AND OBJECTIVES OF THE WORKSHOP:

1. To know more about 'feminism'.
2. Existing gender division of labour.

3. To know the deep impacts of Patriarchy.

4. To make us aware of the acts/laws protecting the rights of women.

DAY ONE (29th AUGUST 2016)

Day one of the workshop started the small inaugural session. Our guide Maam Mrs. Anjuman Bora introduced us to Dr. Monisha Behal and Maam Sheetal. They have been associated with NEN since a long time. After our introduction Dr. Monisha gave us an insight about NEN and how it works. They started off the workshop discussing how gender and sex works. They explained us the exact difference between this two terms that is gender is socially constructed and sex is biologically constructed. How everything is based on gender and it is difficult to change these rules which have been continuing in the society since a long time. On the first day they explained us about three important topics that is:

Gender/Sex
Patriarchy
CEDAW

The second most important topic that our mentors dwelt upon was Patriarchy. Patriarchy is a social system in which adult males hold primary position in the society. There are different views regarding patriarchy and they are –

Tradionalistic view –
Pseudo-Scientist view
Radical Feminist view
Marxist Feminist View
Socialist Feminist view

- We also learnt about the Hindu Succession Act which was passed in 2005. Through this act Hindu woman were entitled to equal property rights.
- We were screened a small animated video made by UNICEF called “The Impossible Dream” which showcases a woman’s endless struggles and her shattered dreams.

The third most important topic was CEDAW that is “The Convention on the Elimination of all Forms of Discrimination against Women”. Article 1 of the convention on the elimination of all forms of discrimination against women (1979) defines the discrimination against women as ‘any distinction, exclusion or restriction made on the basis of sex which has the effect or purpose of impairing or nullifying the reorganization, enjoyment or exercise by women, irrespective of their marital status on a basis of equality of men and women, of human rights and fundamental freedom in the political, economic, social, cultural, civil or any other field. It is often described as an international bill of rights for women.

Our mentors also discussed with us about the concept of equality and how equality should be in terms of access, opportunity and outcome. Access in case of how both the sexes should have equal access to resources and rights. Opportunity in terms of how both the sexes should have equal opportunities in every sphere of life for example in job sectors etc. Outcomes in terms of how every outcome should be equal.

We also learnt about the various principles of equality which are-

SAMENESS APPROACH – Assuming that boy and girl have the same strength and they can do same type of work physically.

PROTECTIONIST APPROACH – Women are denied equality of opportunity on the ground that denial is in their best interest. This approach recognizes the problem as a weakness or inferiority in women and not in the environment which needs to be corrected.

SUBSTANTIVE APPROACH – This approach recognizes that women are in unequal position and may have to be treated differently in order to benefit equally. It ensures equality of results, and not only of means.

ACTIVITIES:

We were assigned various activities that day. During the first half of the workshop we were divided into two different groups under the category of rural men and rural women. On different charts we had to showcase their working hours and define the amount of work and rest hours they have the entire day. Through this

activity our mentors wanted us to know about the working patterns of an average rural men and women and how much rest they get during the entire day. This activity actually showed us the difference in the amount of work between a man and a woman and the difference in the rest hours between them.

DAY TWO (30th AUGUST 2016)

It was 30th August 2016, the second day of our workshop with North East Network (NEN) under the mentors of madam Monisha Behal and the program assistant of NEN, madam Sheetal Sharma.

- The day started with the introduction of various acts and laws for protecting the rights of women. Acts like Criminal Law Amendment Act 2013, ICRW and CEDAW. And we were also asked to present our views on how the rights of women can be violated.
- The Criminal Law (Amendment) Act, 2013 is an Indian legislation passed by the Lok Sabha on 19 March 2013, and by the Rajya Sabha on 21 March 2013, which provides for amendment of Indian Penal Code, Indian Evidence Act, and Code of Criminal Procedure, 1973 on laws related to sexual offences.
- ICRW has been the world's premier applied research institute focused on women and girls for almost four decades. Their evidence-based insight seeks to optimize programs, influence policies and identify scalable solutions that help them lead safer, healthier and more empowered lives.
- The Convention on the Elimination of all Forms of Discrimination Against Women (CEDAW) is an international treaty adopted in 1979 by the United Nations General Assembly.
- And apart from that we had discussions on many other social problems in which woman are preyed by the society. The Spiral of Violence, womb to tomb were many such terms through which we came across and the impacts and outcomings were explained.

- Spiral of violence was discussed in which we came across the violation of women rights since their womb to tomb. The rights of women are violated not even before their birth, even the foetus has to go through stages of violence, (female foeticide, discrimination in providing proper nutrients during pregnancy of the mother), infancy (female infanticide, sexual violence), childhood (sexual harassment, incest, child marriage, objectification of female body), teenage (restricted mobility, voyeurism, violation of reproductive health rights), adulthood (stereotyping female, discrimination in jobs, dowry, physical/verbal/emotional violence, marital rape, sexual violence in conflict, acid attacks, forced disrobing) and also in old age.
- The Awareness Videos like "It's my Fault", "Bell Bajao - got milk?", "boys don't cry" were screened made by some leading organisations like Vogue India, Drishti and ICRW. These videos are made in order to make people aware and make them learn to raise their voice against violence that are existing in our society against women.
- We also came across the story of Birubala Rabha, who has been tirelessly fighting the cause of witch hunting since 1980's. Witch-hunting is a process of tracking down people deemed as witches, which often involves episodes of public frenzy. Although it seems like an outdated idea and concept, remotely rural Assam, witch-hunting is not uncommon.

DAY THREE (31st August 2016)

MORNING SESSION (*10:30am onwards*)

1. The session started with a recap of the previous day. One by one, the participants shared their individual experiences. The participants spoke about their learning from the interactive sessions of the previous day.

2. After this, the participants were introduced to the ***Protection of Women against Domestic Violence Act, 2005*** *(PWDVA)* which is a civil and secular law that addresses the needs of women in situations of violence and applies

to women of all communities. It also categorically recognizes the right of a woman to live a life free from violence and also to reside in a shared household. This act also expands the definition of violence to include mental, verbal, physical and economic abuse and threats and demands for dowry. It is applicable not only to married women but all women in a family as well as women in relationships akin to marriage. The resource person then discussed about the ***Domestic Incidence Report*** *(DIR)*. DIR is similar to FIR but this is filed only in case of domestic violence.

3. The NEN team then showed a video titled ***AIB: Rape-It's your Fault*** by ***All India Bakchod*** which responds to the string of abusive and hateful remarks made by people against women.

4. After the video, they discussed about the ***Sexual Harassment of Women at the Work Place Act, 2013*** *(SHWWP)* and showed a video on ***Laws against Sexual Harassment of Women*** so that the participants get a better idea of the act. This acts says that the sexual harassment against women can be categorized into two segments: one 'quid pro quo' which means 'this for that' underlining sexual harassment caused by those in authority/power in workplace resulting in noticeable or tangible employment related action. The second one is 'hostile work environment' in which the abuse does not result in tangible employment action but creates an unfriendly working environment making a women employee uncomfortable or threatened. In both cases, the women can make a complain to the following ***Internal Complaints Committee*** *(ICC)* for organizations with more than 10 employees or ***Local Complaints Committee*** *(LCC)* for organizations with less than 10 employees.

OUTCOMES OF THE SESSION:

- The participants were able to understand the issues concerning women in a more detailed manner from a broader perspective of law and justice.

- The session made the participants aware of the two acts namely;

    Protection of Women against Domestic Violence Act, 2005 (PWDVA) and;

    Sexual Harassment of Women at the Work Place Act, 2013 (SHWWP)

CONCLUDING SESSION:

During the concluding session, Ms. Anjuman Borah, the coordinator of the workshop asked for experiences, feedback, queries and suggestions from the participants in context of the three-day workshop by the North-east Network, held in the Department of Mas Communication and Journalism, Tezpur University. Ms. Chayanika Baruah and Mr. Sujit Muktan shared their experiences on behalf of all the participants of the workshop. After that Ms. Anjuman Borah, the coordinator of the workshop offered the vote of thanks.

ANNEXURE:

I. **Protection of Women against Domestic Violence Act, 2005 (PWDVA) and;**

II. **Sexual Harassment of Women at the Work Place Act, 2013 (SHWWP)**



Students of C4D stand together with Dr. Monisha Behal and Sheetal Sharma after the workshop

About the Department of Mass Communication and Journalism, Tezpur University

Established in 2001, the Department of Mass Communication & Journalism, Tezpur University provides state-of-the-art facilities for the students with an objective of producing skilled manpower in the ever-changing dynamic media world. It offers an intensive two-year M A program in Mass Communication & Journalism that includes print journalism, television production, advertising and public relations, new media, development communication, communication research methods, etc.; and M A in Communication for Development focusing on development communication, content production and social reconstruction.

The department owns some of the most modern infrastructure, allowing a rich exposure and hands on experience on the equipment(s) of the media such as Audio-edit Suite, Non-Linear Video Editing Suite, Television studio, and Exclusive computer workstations with 24 hours net connectivity, Screening Hall, New Media Lab and many more.

There is a balanced emphasis between theory and practice in the syllabus designed keeping in mind the global standards. The curriculum of the various courses offered by the department has a strong focus on communication for social change. Courses on Media Law and Ethics, Development Communication, Specialized reporting on conflict, environment etc. included in the pedagogy aims at creating responsible media professionals with aptitude for technical skills combined with sense of ethical and social commitment. The department has also been ranked No. 1 in the entire country in the recent Times Higher Education Survey 2018.

About the Event

A Seminar and Workshop on “Gandhian Philosophy and Peace Journalism” was organized by Department of Mass Communication and Journalism, Tezpur University, Assam in collaboration with Gandhi Smriti and Darshan Samiti, New Delhi on the 21st and 22nd February 2018. The two day event saw a rich amalgamation of industry and academia where experts in the field of media and journalism shared their thoughts and experiences about Mahatma Gandhi and his pioneering work in advocating peace journalism. The invited speakers and the student participants reflected on the theme and brought forth their knowledge and opinions as to how journalism can be used as a powerful tool to spread peace in the world.

Media plays a very important role in shaping social reality. Mass media are often people's only source of information about events happening outside of one's locality. Particularly during conflict situations, media reportage on a story can be the sole information available to the audience. The slant of public opinion, the society's construction of reality and the role of the media therein can well be decisive factors in the possible escalation or resolution of the conflict. While professionally practiced journalism of constructive agenda-setting in a conflict environment can open up channels for communication, educate people, build confidence, humanize disputants, provide emotional outlets and note solutions elsewhere; poor or deliberately manipulated conflict coverage can further fuel the conflict. A responsible media, which is 'conflict-sensitive' can not only set in motion the process of construction of a general 'culture of tolerance', but can also help dispel the sense of 'singular affiliation' by emphasizing 'multicultural knowledge'.

In this context the concept of peace journalism assumes significance. Peace journalism identifies itself as socially responsible journalism and interprets the events to resolve the conflicts. According to Jake Lynch, Peace journalism is when editors and reporters make choices – about what to report, and how to report it – that create opportunities for society at large to consider and to value non-violent responses to conflict. Peace journalism, is a remedial strategy and an attempt to supplement the news conventions to give peace a chance. It explores the backgrounds and contexts of conflict formation, presenting causes and options on every side (not just 'both sides'); gives voice to the views of all rival parties, from all levels; offers creative ideas for conflict resolution, development, peace-making and peacekeeping; exposes lies, cover-up attempts and culprits on all sides, and reveals excesses committed by, and suffering inflicted on, peoples of all parties; pays attention to peace stories and post-war developments.

Mahatma Gandhi, the father of our nation who led India to freedom through his non-violent struggle was also a great journalist. Gandhiji published Navajivan, Young India, Harijan, Harijansevak and Harijanbandhu during the freedom struggle in India. He advocated peace as an important news value for journalists and practiced peace journalism in his role as a journalist, a columnist, an editor, and a publisher. Ethical journalism in general and peace journalism in particular has a lot to draw from Gandhiji and his ideology of non-violence. It is high time for journalism practitioners, educators and students to reflect upon and learn from Gandhiji's thoughts on non-violent communication as

well as the journalistic values that he espoused. This could go a long way in making the practice of journalism much more ethical, responsible and socially relevant.

Under the aforementioned backdrop, the two day programme on Gandhian Philosophy and Peace Journalism was aimed towards reflecting upon the need for peace Journalism in the present day context and to discuss about the relevance of Gandhiji's philosophy of non-violence and peace communication in this context. The purpose was to mainly orient the MCJ students who are media professionals of the future towards using journalism for peace.

Objectives of the Seminar cum Workshop

To reflect upon the need for peace Journalism in the present day context;

To discuss about the relevance of Gandhiji's philosophy of non-violence and peace communication;

To orient the MCJ students who are media professionals of the future towards using journalism for peace.

List of Participants

Resource Persons:

Sl. No	Name	Designation	Affiliation
1	**Dr. Syed Murtaza Alfarid Hussain**	**Assistant Professor**	**Department of Mass Communication, Assam University**
2	**Mr. P. J. Baruah**	**Executive Editor**	**The Assam Tribune**
3	**Mr. Sushanta Talukdar**	**Editor**	**Nezine**
4	**Mr. Rajeev Bhattacharyya**	**Freelance Journalist and Author**	

Representatives from Local Media:

Sl. No	Name	Designation	Affiliation

1	Mr. Shambhu Boro	Dist. H.Q. Correspondent	The Assam Tribune
2	Mr. Samir Kar	Photo Journalist	UB Photos
3	Mr. Raj Kumar Mahanta	Correspondent	Asomiya Pratidin
4	Mr. Aswini Bora	Correspondent	Assam Talks
5	Mr. Amarjeet Prasad	Video Journalist	Assam Talks

Representatives from Tezpur University

Sl. No	Name	Designation	Affiliation
1	Dr. Biren Das	Registrar	Tezpur University
2	Prof. P.K. Das	Dean, School of Humanities and Social Sciences	Tezpur University
3	Mr. Samaresh Barman	Public Relations & Information Officer	Tezpur University
4	Prof. Abhijit Bora	Associate Professor and Head	Dept. of MCJ, Tezpur University
5	Dr. P Anbarasan	Associate Professor	Dept. of MCJ, Tezpur University
6	Dr. Uttam Kr Pegu	Associate Professor	Dept. of MCJ, Tezpur University
7	Dr. Joya Chakraborty	Associate Professor	Dept. of MCJ, Tezpur University
8	Dr. Anjuman Borah	Assistant Professor	Dept. of MCJ, Tezpur University
9	Dr. Junali Deka	Assistant Professor	Dept. of MCJ, Tezpur University
10	Mr. Manoj Deori	Assistant Professor	Dept. of MCJ, Tezpur University
11	Ms. Veda Yumnam	Assistant Professor	Dept. of Social Work, Tezpur University
12	Mr. Debajit. K. Sarmah	Assistant Professor	Dept. of Law, Tezpur University

Student Participants

Sl. No	Name	Programme	Semester
1.	Ritu Raj Choudhury	M.A. Communication for Development	II
2.	Ankita Dutta	M.A. Communication for Development	II
3.	Shaheed Ahmed Alomgir	M.A. Communication for Development	II
4.	Abhishree Borah	M.A. Communication for Development	II
5.	Rajat Kamal Hazarika	M.A. Communication for Development	II
6.	Pratyush Deep Kotoky	M.A. Communication for Development	II
7.	Arnab Jyoti Arya	M.A. Communication for Development	II

8.	Sujit Bhattacharya	M.A. Communication for Development	II
9.	Pranamee Chutia	M.A. Communication for Development	II
10.	Bhaswati Saharia	M.A. Communication for Development	II
11.	Anamika Mazumder	M.A. Communication for Development	II
12.	Mr. Debajit. K. Sarmah	M.A. Communication for Development	II
13.	Shruti Sharma	M.A. Mass Communication and Journalism	II
14.	Nidarshana Mahanta	M.A. Mass Communication and Journalism	II
15.	Seauj Sajib Nath	M.A. Mass Communication and Journalism	II
16.	Indrani Choudhury	M.A. Mass Communication and Journalism	II
17.	Srishti Saikia	M.A. Mass Communication and Journalism	II
18.	Ruma Kalita	M.A. Mass Communication and Journalism	II
19.	Amlan Jyoti Das	M.A. Mass Communication and Journalism	II
20.	Khushbu Bin	M.A. Mass Communication and Journalism	II
21.	Jiri Swargiary	M.A. Mass Communication and Journalism	II
22.	Pubali Das	M.A. Mass Communication and Journalism	II
23.	Tirtharaj Gohain	M.A. Mass Communication and Journalism	II
24.	Shreejana Rai	M.A. Mass Communication and Journalism	II
25.	Abhilasha Pathak	M.A. Mass Communication and Journalism	II
26.	Barsaindi Naiding	M.A. Mass Communication and Journalism	II
27.	Syed Towhid Momin	M.A. Mass Communication and Journalism	II
28.	Kaushik Das	M.A. Mass Communication and Journalism	II
29.	Nibirjyoti Mriduta	M.A. Mass Communication and Journalism	II
30.	Tanuj Nath	M.A. Mass Communication and Journalism	II
31.	Shyam Gogoi	M.A. Mass Communication and Journalism	II
32.	Komal Doley	M.A. Mass Communication and Journalism	II
33.	Bina Sharma	M.A. Mass Communication and Journalism	II
34.	Pankhi Das	M.A. Mass Communication and Journalism	II
35.	Ayan Goswami	M.A. Mass Communication and Journalism	II
36.	Bishal Sharma	M.A. Mass Communication and Journalism	II
37.	Atlanta Baruah	M.A. Mass Communication and Journalism	II
38.	Partha Protim Das	M.A. Mass Communication and Journalism	II
39.	Anurupa Sonowal	M.A. Mass Communication and Journalism	II
40.	Debojit Das	M.A. Mass Communication and Journalism	II
41.	Debabrat Gogoi	M.A. Mass Communication and Journalism	II
42.	Anindita Kashyap	M.A. Mass Communication and Journalism	II
43.	Ananya Dewgharia	M.A. Mass Communication and Journalism	II
44.	Lipika Chutia	M.A. Mass Communication and Journalism	II

45.	Bedanan Borah	M.A. Mass Communication and Journalism	II
46.	Arindam Goswami	M.A. Mass Communication and Journalism	IV
47.	Upasana Swargiary	M.A. Mass Communication and Journalism	IV
48.	Rinkumani Pathak	M.A. Mass Communication and Journalism	IV
49.	Debashis Pratim Sarma	M.A. Mass Communication and Journalism	IV
50.	Jayasmita Deka	M.A. Mass Communication and Journalism	IV
51.	Pranati Sinha	M.A. Mass Communication and Journalism	IV
52.	R K Yaibiren Sana	M.A. Mass Communication and Journalism	IV
53.	Kuwoli Hazarika	M.A. Mass Communication and Journalism	IV
54.	Tapas Mali	M.A. Mass Communication and Journalism	IV
55.	Deepak Kumar	M.A. Mass Communication and Journalism	IV
56.	Leonard Kanime	M.A. Mass Communication and Journalism	IV
57.	Rahul Sinha	M.A. Mass Communication and Journalism	IV
58.	Abhilekh Gogoi	M.A. Mass Communication and Journalism	IV
59.	Saibal Krishna Sharma	M.A. Mass Communication and Journalism	IV
60.	Pritesh Kumar	M.A. Mass Communication and Journalism	IV
61.	Sudarkhina Basumatari	M.A. Mass Communication and Journalism	IV
62.	Priyadarshini Saikia	M.A. Mass Communication and Journalism	IV
63.	Pulakesh Das	M.A. Mass Communication and Journalism	IV
64.	Maumita Mazumdar	M.A. Mass Communication and Journalism	IV
65.	Sneha Kumari	M.A. Mass Communication and Journalism	IV
66.	Gangutri Borgohain	M.A. Mass Communication and Journalism	IV
67.	Upasana Gogoi	M.A. Mass Communication and Journalism	IV
68.	Mridusmita Chakraborty	M.A. Mass Communication and Journalism	IV
69.	Rimiana Baruah	M.A. Mass Communication and Journalism	IV
70.	Sonal Sidhu K. Sharma	M.A. Mass Communication and Journalism	IV
71.	Gaurav Verma	M.A. Mass Communication and Journalism	IV
72.	Sushmita Dey	M.A. Mass Communication and Journalism	IV
73.	Kakoli Phukan	M.A. Mass Communication and Journalism	IV
74.	Toka V	M.A. Mass Communication and Journalism	IV
75.	Polash Patangia	M.A. Mass Communication and Journalism	IV
76.	Angita Mahanta	M.A. Mass Communication and Journalism	IV
77.	Mahamaya Ghosh	M.A. Mass Communication and Journalism	IV
78.	Kuntala Das	M.A. Mass Communication and Journalism	IV
79.	Saket Kumar Bhardwaj	Ph.D. Research Scholar	
80.	Kumari Pallavi	Ph.D. Research Scholar	
81.	Eishita Das	Ph.D. Research Scholar	

82.	Madhusmita Boruah	Ph.D. Research Scholar
83.	Muktikam Hazarika	Ph.D. Research Scholar
84.	Vansanglura	Ph.D. Research Scholar
85.	Richa Chattapadhyay	Ph.D. Research Scholar
86.	Subhajit Paul	Ph.D. Research Scholar
87.	Harshwardhani Sharma	Ph.D. Research Scholar
88.	Manaswinee Mahanta	Ph.D. Research Scholar

The profiles of the invited speakers

Mr. Prasanta J Baruah

Mr. Prasanta J Baruah is a senior journalist with over thirty years of working experience in Assam and the North East. He is presently the Executive Editor of The Assam Tribune, the oldest and premier English daily of the region. He started his journalistic career with the English daily The Sentinel in Guwahati after completing his postgraduate studies in English from Dibrugarh University In 1984.He joined as the Principal Correspondent of the Assamese newspaper Dainik Janambhumi at their Guwahati office in 1986.In 1987, he became the North East correspondent of the English daily Newstime (Eenadu Group) of Hyderabad. He later worked for a year at the paper's Hyderabad office before returning to Guwahati in 1989 to join The Assam Tribune. He has been associated with the Tribune ever since and took over as the Executive Editor the paper in 1998.

Mr. Baruah has been closely associated with journalism education in the region. He has been a Guest lecturer of the Journalism department of Gauhati University, Dibrugarh University, Assam University, KK Handique State Open University, and Cotton College among others. Widely travelled, Mr. Baruah has been a member of the Prime Minister's Media delegation to South Africa, USA, Philippines, Japan, Malaysia, Vietnam, Bangladesh, Bhutan, Switzerland and Trinidad & Tobago. He has visited the USA and UK on invitation from the respective governments.

He is presently doing a PhD research work on the issues of identity and humanity in the songs of the internationally reputed Assamese singer, composer, film maker and journalist Dr. Bhupen Hazarika.

Mr. Rajeev Bhattacharyya

Mr. Rajeev Bhattacharyya, is currently working as a Freelance Journalist and Author. He has more than two decades of experience in the media world and has worked as a correspondent for various newspapers including The Times of India, The Telegraph, The Indian Express, Times Now, The Bengal Post, and Seven Sisters Post and now with The Wire, First Post and the DNA. In his career he has covered North east India, Bhutan, Jharkhand, New Delhi, and adjacent regions in Myanmar and Bangladesh.

His book “Rendezvous with Rebels: Journey to Meet India's Most Wanted Men” has been inspiring journalist across the country. Rajeev Bhattacharyya walked nearly 800 kilometers, over some of the most hostile terrain and through no man's land. His journey, which took three months and twenty days to complete, is unprecedented in Indian journalism. He visited the rebel bases in Eastern Nagaland, which covers a part of Myanmar's Sagaing Division, stayed in the ULFA camp and interviewed its chief of staff Paresh Baruah, as also chairman of the NSCN (Khaplang), S.S. Khaplang himself. He interacted with rebels from banned outfits like the NDFB, UPPK, PLA and other groups - for many of them, this was their first conversation with an Indian journalist. Bhattacharyya is one of very few journalists in the world to have made this journey, and among the fewer still to have had such intimate access.

He actively participates in seminars and workshops throughout the country on issues like insurgency, ethnic conflicts, border relations, peace struggles and many more.

Mr. Sushanta Talukdar

Mr. Sushanta Talukdar, Editor, Nezine, also has an experience of over 25 years in the field of journalism and has worked for leading newspapers like The Hindu, Frontline, The Telegraph, The North East Daily, and The North East Times. He was awarded with the prestigious Parag Kumar Das journalism award in 2016. He has reported on varied issues like national politics, ethnic conflicts, communal riots, natural disasters, tribes & traditions, environment, trade & commerce,

migration issues, development & government and many more. He is one of the celebrated names in Assamese journalism.

His recent venture, Nezine is an online magazine focusing on India's Northeast region, published by Nezine Media. It makes efforts to bridge the information gap on the region – a repository of traditional knowledge, a bio-diversity hotspot and a living museum of cultural diversities. We generate and publish content that add to a state-of-the-art digital archive of quality information about the region.

Nezine provides a platform for publication of news, features and articles on wide-ranging subjects and issues which will help its visitors derive their own notions about the region instead of looking at it through fixed frames of perception about its geography, history, society, culture and tradition.

Dr. Syed Murtaza Alfarid Hussain

Dr. Syed Murtaza Alfarid Hussain is a senior Assistant Professor at the Department of Mass Communication, Assam University at Silchar. A postgraduate in English Literature from Delhi University, he also has a master's degree in Mass Communication and Journalism from Tezpur Central University. In 2006, he was awarded with a British Council fellowship and attended a training course on Conflict Reporting in the UK, conducted by Thomson Foundation at Cardiff University in Wales and Belfast in Northern Ireland. In 2008, he was awarded a Rotary scholarship to attend a six-month course on Peace and Conflict Studies at Chulalongkorn University in Thailand. He did his PhD on 'Media Framing of Intra-State Conflicts in Assam' from Berhampur University, Odisha. His primary research interest is Media Framing, and is currently supervising doctoral research in the areas of Social Media and Political Communication, Human-Computer Interaction for Development (HCI4D), Mediatisation and Media Literacy. He has presented his research findings at the IVSA 2017 conference held at Concordia University in Montreal, Canada and also at the AMIC conference in Manila, Philippines, besides presenting papers at several national and international seminars and conferences in different parts of the country. He has also directed a few documentary films on issues of culture, health, ethnicity and marginality in northeast India. He was awarded the best postgraduate gold medal in Tezpur University in 2004.

Details of the two day event

DAY 1

Inauguration

[10:30 am – 11:15 am]

The seminar began with a formal inauguration ceremony where the students presented a classical song “Raghupati Raghav Raja Ram” in memory of *Father of the Nation*, whose hard work and determination has always been a source of inspiration. The same took us back to the perennial efforts of Mahatma Gandhi who tried to unite the citizens across all communities and identities by his communication skills, diction, writings and even songs.

Prof Abhijit Bora, Head of the Department, Mass Communication and Journalism, Tezpur University welcomed all the distinguished guests and dignitaries who spared their time to share their knowledge with the students and other participants. He said that Mahatma Gandhi possessed an unimaginable vision that he visualized the use and potential of mass media much ahead of anyone else in the country. The same was evident in Gandhiji’s speech when he used the phrase “Broadcasting is power” in his message on All India Radio, 1947 while addressing the refugees of partition in Kurukshetra. Prof Abhijit Bora also mentioned that it is extremely relevant to hold the seminar amidst budding media aspirants, i.e. the students as this is an opportunity for all of us to contribute towards the noble cause of peace journalism. He invited the distinguished guests to share their expertise and enlighten the audience, while he remembered the wonderful work of John Lennon, “Give Peace a Chance”.

Prof Bora announced with great delight that Tezpur University is ranked 100th in Times Higher Education Asia University Rankings 2018 and that its Department of Mass Communication & Journalism was ranked 1st in the country by the same survey. He gave the credit of this milestone achieved by the department to faculties and students who work ceaselessly and add value to it. He said that maintaining this reputation is always a challenge for the university and the same cannot be

fulfilled without the support of the media and resource personalities who visit the department and enrich and nurture it with a positive flow of information as well as ordeals.



Prof P K Das felicitating Mr. P J Baruah

After the welcome address by Prof Abhijit Bora, a 'Felicitation Ceremony' followed, where the invited guests Mr. P J Baruah, Executive Editor, The Assam Tribune, Mr. Rajeev Bhattacharyya, Freelance Journalist and Author, Mr. Sushanta Talukdar, Editor, Nezine and Dr. Syed Murtaza Alfarid Hussain, Assistant Professor, Department of Mass Communication and Journalism, Assam University were honored by the traditional Assamese 'Gamusa', flowers and small tokens of respect from Tezpur University.

Further, **Dr. Anjuman Borah**, Assistant Professor, Department of Mass Communication and Journalism, Tezpur University, introduced the theme of the event and why it is crucial to deliberate on the topic for the media fraternity.

She emphasized that there was a felt need to discuss the relevance of Peace Journalism as we witness that the whole world is conflict struck. Mass Media these days shapes social realities and it can go miles in resolving warlike situations and spread peace across the globe. She further elaborated that the panel of the speakers were meaningfully selected in order to make the most of the deliberations that the entire audience would engage in and gave a brief introduction about each of the members.

Dr. Anjuman anticipated that post the seminar and the workshop that would take place on Day 2, the students shall acquire the ability to construct media messages that highlight peace and not conflict.

Prof P K Das, Dean, School of Humanities and Social Sciences, Tezpur University delivered the inaugural address and declared the event open. He expressed that he honored to share the dice with such eminent personalities and extended his whole hearted gratitude to them for taking out

time to enlighten the students and other participants. He said that north east itself is a conflict sensitive region and prone to violence and other forms of distress. It is high time that media inclines itself to peaceful narratives and bring about the required change in its communication strategies for a larger social benefit. He recollected the fact that Mahatma Gandhi was a journalist all his life, who knew that journalism can be used for good and bad purposes alike. He envisaged that if Gandhiji was alive in the present world, he would also made use of social networking sites like Twitter and Facebook to spread peace and positivity, just as he electrified the nation using print media in a very effective way.

Prof P K Das declared the event open by announcing a brief lamp lighting ritual where all the dignitaries participated in loving memory of the Mahatma.



A glimpse of the inauguration ceremony

Dr. Biren Das, Registrar, Tezpur University was also present to grace the occasion and he conveyed sincere gratitude to all the guests and participants who assembled to hold such a significant discussion. He remembered the famous quote while addressing the audience, “There is no way to peace, but peace is the only way.” Dr. Biren Das wished that the mandates of the event be fulfilled and it comes out in flying colors of success.

Mr. P J Baruah, Executive Editor, The Assam Tribune delivered his introductory remarks where he conceded that the theme selected for the seminar & workshop is quite different from usual ones and probably cardinal to the profession in the recent times. He said that the 20th century was a game changing century for entire human civilization as it saw the rise and fall of empires, wars, racial conflicts, bloodshed and subsequent attempts of proliferating peace amongst the affected. He mentioned that Gandhiji’s concept of truth, non-violence (Ahinsa) and peace were capable to convince the opponents, i.e. the British rule, that they are wrong and must be fair in their conduct. He asked them to provide the Indian nation, justice, as they deserved to be free and independent in their own native land.

Mr. P J Baruah felt that a lot of circumstances must have affected Gandhiji's philosophy in the Indian freedom struggle. Since he studied in England, he was already familiar with the English way of thinking and also must have built up a connection with the local communal. He avoided any direct conflict and sought for peaceful communication to end the disharmony and bring about social justice in the Indian subcontinent. He further adds in the present times, one is not that familiar with the opponents and hence resort to violent defensive measures in order to safeguard one's own interests. Under these situations, we must look back to the historical accounts of peaceful struggles and their long lasting impact in the achievement of intended goals. It shall also be practical to study the findings of Johan Galtung in the arena of peace and conflict resolution. Mass Media as a profession should be free from any bias and must have the confidence of both the opposing parties in order to practice peace journalism successfully. He said that the eminent panel would further discuss the intricacies of the matter and appraise the gathering on the same.

Giving a formal end to the inaugural ceremony, **Dr. Junali Deka**, Assistant Professor, Department of Mass Communication and Journalism, Tezpur University conveyed the vote of thanks. She conveyed sincere gratitude to Gandhi Smriti and darshan Samiti, New Delhi for agreeing to collaborate with the Department of MCJ, TU to organize the seminar and workshop. She also thanked Mr. P K Das, Dean, Humanities and Social Sciences, Tezpur University and Dr. Biren Das, Registrar, Tezpur University for their presence and support. Also, the guests, speakers, media personnel, faculties and students were appreciated for taking the effort to participate and organize the seminar cum workshop. Dr. Junali said that it is expected that a productive discussion will take place between the experts and the budding journalist, i.e. the students that will in turn help them to grow in the right direction.

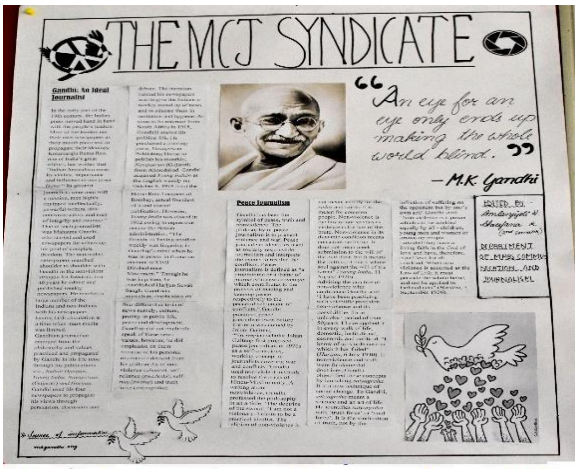
Poster Exhibition

It must be mentioned that the students also presented a **"Poster Exhibition"** that added vibrant colors to the cause. They created artistic pieces on topics like Gandhian Way of Life, Pulitzer Award Winners, Do's and Don'ts of Peace Journalism, Assassination of Journalists, Conflicts in North east and many more. All the invited guests and dignitaries took out time to walk through the poster exhibition to praise

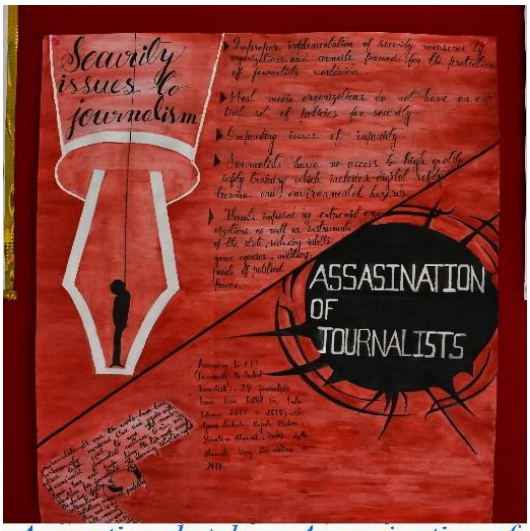


Some Do's and Don'ts of Peace Journalism showcased on a poster

the hard work of the students and encourage them for future endeavors.



A poster representing work and life of Mahatma Gandhi



A creative sketch on Assassination of Journalists

Academic Session 1

[11:30 am – 1 pm]

The first session of the seminar included two student presentations and a detailed lecture by Mr. Rajeev Bhattacharyya, Freelance Journalist and Author who spoke upon Peace Journalism in Theory and Practice: Observations in North East. The session brought forth important insights of the expert as well as the students who spoke about their views on the topic under scrutiny.

The first student presenter was **Mr. Amlan Jyoti Das,** Semester II, Department of Mass Communication and Journalism, Tezpur University and his topic was *Gandhi and Peace Journalism*.

He touched upon Gandhiji's way of life, his philosophy and his journalistic competencies that enriched the Indian society and paved the path of peace journalism that influenced the century.

Mr. Amlan gave a brief account of Gandhiji's early life and how he worked to transform the social structure of the nation. His ideas and practices of non-violence and truth can serve as the best content for the discipline of peace education.

One of the greatest attributes of Mahatma Gandhi was the simplicity with which he spoke, wrote and communicated. Quoting the Mahatma, "Each one has to find his peace from within. And peace to be real must be unaffected by outside circumstances." Today, when the contemporary media scenario bristles with turmoil and over-riding role of market forces in the media, it would be worthwhile to revisit Mahatma Gandhi's philosophy and his contribution as a journalist.

Gandhiji started his journalistic foray in South Africa with Indian Opinion in 1903. It featured articles and editorials that highlighted the discrimination and harsh conditions under which indentured laborers worked. Through the 19th century Indians were brought to South Africa as laborers by the authorities of the British Empire, which governess both South Africa and India. Alongside various multi-ethnic communities, the Indian community suffered from significant political, economic and social discrimination, administered by the system of apartheid. Gandhiji's Indian Opinion became a crucial vehicle for challenging state laws and urging defiance of unjust regulations. His views evolved into the concept of Satyagraha, the philosophy and practice of non-violent resistance, which was adopted by other freedom movements throughout the latter half of the 20th-century.



Student Presenter Mr. Amlan Jyoti Das talking about Gandhian philosophy

Young India was another weekly paper or journal - in English edited by Gandhiji from 1919 to 1931. He used Young India to spread his unique ideology and thoughts regarding the use of non-violence in organizing movements in India's struggle to freedom. It helped masses to understand the concept of 'Satyagraha' and it's far reaching consequences. In 1933, Gandhiji started another weekly named *Harijan* meaning 'People of God' with an intention to eradicate

untouchability, discrimination and poverty. Quoting him in the Harijan, “My faith in truth and non-violence is ever growing, and as I am ever trying to follow them in my life I too am growing every moment. I see new implications about them. I see them in a newer light every day and read in them a newer meaning.”

According to his book, My Experiments with Truth, the cornerstone of Gandhian journalism were to understand popular opinion and give expression to it, to arouse desirable sentiments among the people and to fearlessly expose popular defects. The objectives of his newspapers included provision of news to all the sections of the society in vernacular languages, to advocate their cause, provision of information regarding the various happening in India and to garner contributions from competent writers, Indian as well as Europeans.

Further Mr. Amlan elaborated on his understanding of Peace Journalism, and said that it is a programme or a frame of journalistic news coverage which contributes to the process of making and keeping peace respectively to the peaceful settlement of conflicts. He also illustrated upon five core principles of peaceful and ethical journalism, viz. Truth & Accuracy, Independence, Fairness & Impartiality, Humanity and Accountability.

He further added that ‘Peaceful Journalism’ was one of the key factors that transformed Gandhi into “Mahatma Gandhi’. Even a wooden stick, which we often associate with violence and force was used by Gandhiji as a symbol of support. There were many leaders and freedom fighters who struggled for Indian Freedom and many who remained alongside him, but it is Gandhi that we remember the most because no one could have caused an upheaval in the colonial rule like him. He ended his presentation by the famous words of Mahatma Gandhi, *“Freedom of the press is a precious privilege that no country can forgo.”*

The second student presenter was **Ms. Nidarshana Mahanta**, Semester II, Department of Mass Communication and Journalism, Tezpur University and her topic was *Role of Media in Conflict*. She gave real time examples on how media can arouse or mitigate conflict and crisis situations.



She began with the classic example of William Hearst and his hate journalism which instigated war between the Americans and

the Spanish citizens. He use to tell his out stationed correspondents that “You furnish the pictures, I’ll furnish the war.” His paper vigorously published stories that helped create and foster the American sentiment reflected in the popular phrases, “Remember the Maine, to Hell with Spain!”, “Spanish Treachery”, and many more that finally led to the Spanish-American war. It also signified the growing influence of mass media, which had never before proven to be as immediate and far-reaching.

Another example of was the pictures of Alan Kurdi, a 3 year old Syrian boy drowned to death. The heartbreaking photograph of Alan, whose body washed up in Turkey, had a great impact on policy, with the U.K. agreeing to take thousands more refugees. Thus, the use of mass media can lead to good or bad outcome; it’s the power that needs to be handled with care.

Ms. Nidarshana added that *words can become swords* if used inappropriately. When ULFA attacked Hindi speaking people in the districts of Tinsukia and Dibrugarh, Assam and it was reported as an attack on ‘Biharis’ by the national media; it further led to retaliation by the people in Bihar who counter attacked the north eastern people leading to prolonged crisis. In another case, the assassination of Smt. Indira Gandhi was reported as carried out by her ‘Sikh’ bodyguards which gave communal colors to the conflict. Contemporary media seems to use conflict as a tool to rise up the popularity ladder. In case of war and conflict, sensationalism becomes common and there is a mutual competition among the media houses, as to who will publish or broadcast the news in the most provocative manner. The Indo-Pak conflict sees a number of news channels blaming Pakistan and using hate statements very often. Here, the example of Anchor Gaurav Sawant from India Today was given, who created a war room in the studio and is dressed up like a warrior all set to attack the enemies any moment.

In fact, in a very interesting survey published in The Hoot, the researcher, compared the coverage of India in Dawn (Islamabad edition) and Pakistan in The Hindu (Delhi edition) during the period May 01 - May 31, 2016. Both these newspapers are known to be liberal and, as such, are not representative of the news media. Dawn published more news items, editorials, and opinion pieces on India than The Hindu on Pakistan, which means it better voiced India’s side than the Hindu did for Pakistan.

Another observation made in the presentation was that China's aggressive state-backed media has accused Western media of being biased in favor of India simply because it is a democracy. It emphasized that India "arouses sympathy" because it is "in a weaker position compared with China", elaborated in an opinion piece in China's Global Times.

At times, governments also use conflict as a tool for propaganda as seen in the case of *embedded journalism*. National media journalists are taken into war zones under full security who usually ends up publishing what the military wants them to; while the local media of such conflict ridden states have no voice. This in turn we may refer to as planted stories which are biased in nature as the other side of the conflict remains untouched.

Also, the flare up tension in the race to break news can cause a lot of damage to the safety and security of the country. One such example is the 26/11 Mumbai Attacks that shook the nation; media constantly reported the movement of NSG commanders and soldiers that in turn helped the enemies to cause more damage as they got firsthand information about where the soldiers are located. In the age of social media and digital networking platforms, the circulation of news and information has become easier and faster. This calls for more caution to be exercised in the rush to break the news as once published the message reaches thousand and lakhs of people in a few hours and any mistake cannot be rectified with ease thereafter.

Ms. Nidarshana ended her presentation with the term CSR of mass media; not Corporate Social Responsibility but Cultural and Social Responsibility that the media fraternity should put into practice. Since, media is deemed as the forth pillar of democracy, it should fulfill its larger responsibility of positive social construction and pave the path of peace and prosperity.

After the two presentations by the students, next **Mr. Rajeev Bhattacharyya,** Freelance Journalist and Author enlightened the gathering on the theory and practice of peace journalism in north east India and the various conflict situations he has reported.

He began his speech by recalling his visit to rebel camps of Myanmar in 2011 where it was mandatory to move around in camouflage, else it could risk a journalist’s life. Such situations makes one understand the importance of peace and futility of war and conflict.

Mr. Rajeev said that Mahatma Gandhi was a great journalist and edited three newspapers which had a lasting impact on the society. Once can read truth and non-violence in all his writings. Very few people know that he wrote a book on Srimad Bhagwat Geeta. He was aware that the British rule in India was a technologically strong one and it was impossible to dodge them with aggression and rebellious revolts. It was only through peace and persistent efforts that the country received the rightful independence. We must make India what Gandhiji envisioned it to be so that his divine soul can rest in peace.

He said that North east as a region is not the perfect spot for modernization and industrial activities. It a biodiversity hotspot and not a place to establish smoke emitting factories. The beauty of the region lies in its greenery and agricultural opulence.



Mr. Rajeev Bhattacharyya sharing his experiences

A journalist's role is to report with objectivity and truthfulness; he or she should not and cannot play the role of a facilitator between the government/authorities and the parties in conflict. This is against the modus operandi of peace journalism, as the famous phrase depicts, "media cannot put anyone on trial". Again sometimes, media personnel and journalists should certainly raise issues of social interest again and again for the betterment of the community in large, as sometimes it acts as an awakening message and brings about long awaited transformations.

North eastern media has no culture of follow ups and there is an acute necessity to be more methodical in their approach. Sometimes crucial news items are covered but what happens to them in near future is not even attempted to be reported. For example, there was less emphasis in reporting Tibet earthquakes. There are artificial dams that might affect the flow of water in the Brahmaputra River; might as well cause unexpected floods in Assam and adjoining areas. There is culture to let go off some such climacteric incidents and more focus is on Global Summits and networking conclaves. Mr. Rajeev re-emphasizes that north east is not a place for establishment of jumbo industries; rather its distinguishing feature lies in its natural beauty and rich flora and fauna.

Coming to peace journalism and ethical reporting, he says that a journalist should consult all sources equally and not just rely on one single source. There can be varied sources of information

that may include intelligence agencies, government agents, correspondents and many more; one should try to take in view both sides of the conflict and not just be publishing what one view informs him. At the same time, it's equally important to sort and filter news and use caution in disseminating news related to conflict.

There is always some degree of risk involved in the life of a war journalist and one must be courageous and prepared to face such consternations. For example, Mr. Rajeev said he was detained with a pistol on his head in a case of mistaken identity in the region of Dima Hasao while he was reporting there. He said that north east India is a goldmine of information and usual sight of communal conflicts; but peace journalism was never practiced in the region before. The Anti-Dam project in Dhemaji area gave him the idea that what wonders can peace journalism bring about to conflict resolution where the construction of dam was disputed in the Arunachal Pradesh as it directly affects the district. One should check all the facts and should not publish wrong stories intentionally just for the sake of viewership or readership.

He further gave the example of Karbi Anglong and Dima Hasao conflicts that occurred in October 2005 in Assam where there was a lot of violence that took place. There was attacks on buses, arsons, beating and so on that triggered from one single incident of violence. He consulted both the parties and tried to gather information on what was the root cause of such happenings which was later disclosed to him as a race or competition for collection of taxes in the boundaries. He published his findings in good light so as to resolve the conflict as soon as possible and put an end to the instances of bloodshed. While he was researching in the region, he also came across facts as to who was the miscreant who initiated violence, but preferred not to talk against one party in order to maintain peace. But he also mentioned that sometimes access to leaders and members of both the parties of conflict becomes difficult as they might recognize the journalist as belonging to or representing one side of the conflicting parties.

The next example he gave was of the 2012 Bodo vs. illegal migrants' conflict who were popularly understood as the illegally migrated Muslims. In this scenario, access to Bodo leaders and members was easily available but it was difficult to gather any information or talk to the so called illegal migrants without which the story remains incomplete for a peace journalist as he or she should never consult only one angle. In the north east, terrain and geography at times is a huge

barrier, as there are places where land and road connectivity is even absent. A journalist should be physically and mentally prepare to surpass such barriers in information gathering process.

Further, Mr. Rajeev spoke about the Rohingya refugees of Myanmar and Bangladesh and how the western media have covered their tragic lives. Also, many Rohingya refugees in Bangladesh's Cox's Bazar are doubtful of returning to their homes in Myanmar's Rakhine state engulfed by violence, riots and a brutal crackdown by the army since 2016. Around six lakh Rohingya are estimated to have crossed over to Bangladesh and they join two lakh of earlier Rohingya settlers in the coastal district in camps and at other locations. Most of the refugees, who had arrived in 1978 and 1992, had settled in Chittagong, some granted citizenship, while others migrated to the Middle East and Southeast Asia. We usually see reports of massacre in the refugee camps of the Rohingyas, extreme food shortage, beating, killing and arsons and the whole world has developed a sympathy for the tribe. But the unfortunate fact that they have indulged into drug pedaling is unknown to most of us. About 30 lacs pills of Ya ba drug is smuggled in the adjoining areas. Mr. Rajeev further adds that sometimes a journalist may get some stories that are of importance but even if there is a slight doubt on the authenticity, one should clearly structure the report like, "It is speculated that…." So that in future, if there are any corrections, they can be easily accommodated.

Lastly, Mr. Rajeev said that technology is very democratic and with the advent of social media and citizen journalism, more and more people are getting access to first hand access to information. He believes that news should be filtered and verified before its mass circulation so that the credibility of journalistic profession is maintained.

Academic Session 2

[2:30pm – 3:30pm]

The post lunch second session saw one student presentation, by Mr. Tirtharaj Gohain, Semester II, Department of Mass Communication and Journalism, Tezpur University and one expert lecture by Mr. Sushanta Talukdar, Editor, Nezine, Guwahati.

Mr. Tirtharaj Gohain elaborated on his topic Peace Journalism vs. War Journalism and what are the radical differences between the two. He began with the famous words of Jake Lynch "Peace journalism is when editors and reporters make choices – about what to report, and how to report it – that create opportunities for society at large to consider and to value non-violent responses to conflict." He adds that when peace is combined with journalism, the motive clearly becomes conflict resolution. Peace journalism allows one to explore the background and context of conflict formation, on every side, not just 'both sides', gives voice to the views of all rival parties, offers ideas for conflict resolution, peacemaking and peacekeeping, exposes lies, reveals suffering inflicted on peoples, and pays attention to peace stories & post-war developments.



Student Presenter Mr. Tirtharaj Gohain establishing distinction between war & peace

Mr. Tirtharaj said that in Indian mythology, Narad Muni is perfect personification of war journalism, as he instigated the 'devas' and 'asuras' alike and leaked information that provoked each of the parties to violent consequences. While if we see the epic Mahabharatha, Sanjay, the blind man who narrated the war proceedings to Dhritarashtra, requested him to stop the war as it will lead to killing of brothers amongst themselves; the same we can relate to peace journalism.

War journalism is about conflict that has a value bias towards violence; it usually leads audiences to overvalue violent responses to conflict and ignore non-violent alternative. He further gave examples of the Syrian bombing by American nation and how heart moving pictures of Syria's devastated people have had attracted international aids to the innocent localities. Also, the details of the agony of people of Kashmir and the death of Burhan Wani have given a clear picture of what is happening in the valleys and how local people suffer due to the ongoing wars and military violence. Such a literature is necessary to establish peace and mutual cooperation of the communities in the region.
He also believes that war and war related news sells in news media as due to the interest garnered by violence and fear among the masses that anything can happen to them anytime. So they prefer reading and viewing war related content in order to be prepared for any mishap.

In the era of new media and digital platforms, it becomes the responsibility of each and every journalist and even the common man to only publish peaceful messages and not incite violent and hatred through our words and depictions.

Next was the expert speech by **Mr. Sushanta Talukdar**, Editor, Nezine, who elaborated on the topic: Looking for Stories of Rebuilding Lives and untold stories of Rehabilitation Fault lines in conflict-hit areas. He began with a note of thanks to the earlier speakers who did a lot of context setting and expanded the theme of the seminar and forthcoming workshop.



Mr. Sushanta Talukdar talking about rebuilding lives in conflict zones

He initiated his presentation by reminiscing the Joypur Relief Camp in Kokrajhar district of Assam. A series of violent ethnic clashes between the Bodos and the adivasis in lower Assam's Kokrajhar and Bongaigaon district in 1996 and 1998 and between Bodos and Muslims in 1993 uprooted 48,556 families from their homes. Some of them have been rehabilitated. However, for past 16 years, thousands of refugees, mostly adivasis, are still languishing in congested makeshift relief camps in sub-human conditions without proper food, drinking water, sanitation, healthcare and livelihoods. Children born in the camps do not have any idea what a proper home is. For more than 13,000 occupants of the Joypur relief camp at Karigaon by the side of the National Highway there are only five tube wells as the only source of water for drinking and other household purposes. The refugees have not heard about the Total Sanitation Campaign and open defecation is the normal practice for children and elders alike. Among such strife what impressed Mr. Talukdar the most was school for the refugee children which was set up by the camp members who saved the relief meal and ration that they got, sold it in the market and made arrangements to educate their children in such adversity. This rehabilitation story is truly inspiring and giving people hope that they can have a better future and need not live an entire life in vain.

It is evident that whenever conflict happens, it leads to displacement of innocent people and civilians being victimized. There is always a narrative and a counter narrative to any conflict situation and the challenge for a journalist is not to get trapped between the true and produce only truthful narratives. Access to both the sides of a conflict is usually very difficult and one finds himself

biased or overtly sympathetic to one side that is accessible and visible. Hence, it goes without saying that one must attempt to gather information from both and if possible from all sides of the conflict.

Mr. Sushanta Talukdar further spoke about the role of social scientists in conflict and war like situations. He says that they play a major role in post conflict rehabilitation and rebuilding of lives; but the initial role of communication and information dissemination should be carried on by a professional communicator and not a social scientist. The content that is created for mass circulation at an early stage of any conflict impacts it course. Usually communities are not at war and prefers to remain in peace and interdependence. Hence while reporting on violence or any similar activity, one must use terms like “Miscreants among the community….” And should refrain from accusing any community directly or indirectly as it further aggravates the situation.

He furnished examples of Barpeta riots of 1994 which he investigated as a journalist and told that every party had its own story and it was extremely difficult to judge whose was correct. At times shots were fired in abandoned village to ensure that fear prevails in the region, which some journalists mistook and reported as acts of violence; so a fact finding exercise was missing and resolution of the conflict could not be achieved amidst massacre and bloodshed. Survivors usually loose hope to rebuild their life under such circumstances and prefer to keep quiet and not to give any media statements.

What is needed in such conflict sensitive regions is confidence among the victims and a hope to survive. One such way to evoke confidence is the relief materials and food that is provided to them that brings life to their already wounded and dead minds. He also recalled that in the Joypur Relief camp, children made balls out of waste plastic bags and were playing with the same; what is remarkable here is that they are not even aware about the brutality of war and the pitiable living conditions they are in, joyous and want to live life up to the fullest. He even attended rituals and subsequent religious ceremonies taking place in the camp which streaked in a ray of positive energy and strong desire to lead a normal life and not to be tagged as refugees.
Another story was of a lady named Vimala, in Saron Basti, Assam who was an inhabitant of a relief camp; when all other establishments of the camp were destroyed in violence, she decided to convert her own house into a school for children in a hope that their will to live and rehabilitate should remain undaunted. Such stories should be entertained more and more by the mass media

rather than focusing only on conflict and violence. Also, there is dire need to reframe the rehabilitation policies if we really have to hope for an independent life for the victims. Media should also work in this angle aggressively. At times political forces do not allow two communities to be interdependent for their own selfish motives; such flaws in political procedures need to be checked by the media professionals. Mr. Talukdar ends his lecture by saying that we should analyze history and saints like the Mahatma Gandhi to become a just journalist and offer balance in conflict reporting.

Academic Session 3

[3:45pm – 5:30pm]

The last session of the day one saw a student presentation by Mr. R K Yaibiren Sana, Semester IV, Department of Mass communication and Journalism, Tezpur University and expert talk by Mr. P J Baruah, Executive Editor, The Assam Tribune. Lastly, the concluding remarks were delivered by Dr. Syed Murtaza Alfarid Hussain, Assistant Professor, Department of Mass Communication and Journalism, Assam University summing up the entire learnings of the day.

Mr. R K Yaibiren Sana, presented on his topic Peace Journalism in North East India with special reference to Naga Peace Accord. He began by saying that the objectives of his presentation includes understanding media's reaction to Naga Peace Framework Agreement and to analyze whether the national and local media has made any attempts to resolve conflict and avoid the prevailing violence.



Student Presenter Mr. R K Yaibiren Sana throwing light on the Naga Accord

Mr. R K Yaibiren Sana defines conflict as a process through which two or more actors try to pursue incompatible aims or goals while trying to stop the other(s) from pursuing their own goals. Conflict may sometimes lead to violence, but not always and mass media has a major role to play in information distribution in a conflict situation. It can be a tool for peace building and development of mutual trust and harmony; yet a common journalist's principle is "If it bleeds, it leads." Violence and

conflict often happens to make the headlines while peace building takes a backseat in most of the print and broadcast media channels. This tendency to cover conflict and violence distorts reality and leads many people to think that conflict is pervasive and peace is abnormal. Now the question to be discussed is that is the same fact true with the media vis a vis the Naga Peace Accord?

On the other hand, Peace Journalism is when the editors and the reporters make choices of what to report and how to report in conflict situation. It uses conflict analysis and transformation to update the concept of balance, fairness and accuracy in reporting. It calls for positive representation of facts and appropriate balance between what is to be said and what is not to be said.

Post the signing of the agreement, On the 4th of August 2015, the Huyen Lanpao, a daily vernacular newspaper published from Imphal put its headline as, “The Indian Government and the IM (NSCN) signed Peace accord: Manipur Government left un-consulted; contents of the agreement left secret”. It is understandable in the situation that the vernacular media communalized the whole situation without even any detailing of the draft as seen evident from various Assamese and Manipuri newspapers.

The National Media including the NDTV, Lok Sabha TV, RJ TV, Times Now held panel discussions on the issue--the Peace agreement. These talks were arranged to focus on the objectivity of the issue rather than making assumptions. However, many local channels of Assam exaggerated the issue and brought in a communal element in their dissemination of information. Many things were spoken on mere assumptions. Further on, the local news channel in Manipur- Impact TV and ISTV held a series of panel discussions and talks to bring about objective knowledge on the agreement. Further the impact TV also had a one to one talk with RN Ravi, the interlocutor to have a clear opinion on the framework. It seemed that there is lack of detailed research and the Dima Hasao violence was a result of misinformation.

He ended his presentation by pointing out that much depends on the socialization of the journalist, esp. in context of communal reporting. There are always a few people who instigate violence and media should not be one of them or aid them in any way for the overall benefit of the communities in conflict.

Mr. P J Baruah, Executive Editor, The Assam Tribune initiated his lecture on Issues and Challenges of Covering Conflict: An Editor's Perspective by highlighting the fact that there is a proliferation in the media business with cut throat competition. He recalled that he joined the media business in 1984 in The Sentinel. In those days, The Assam Tribune was the 1st English Daily of Assam and a few vernacular newspapers such as Dainik Asom, Dainik Janambhumi and so on cropped up in the information world. In the same year, Television was developing as a medium in India and the country also won its 1st Cricket world cup. In contrast, the present day Guwahati alone sees 15 plus local language newspapers and 6-7 news channels broadcasting news; all aspiring to top the charts.



Mr. P J Baruah elaborating on the Editor's Perspective

He further recalls an incident of bomb explosion in Guwahati in the 80s, where a leading Kolkata newspaper gave headlines with names of the culprits inciting rage in almost all parts of Assam. This made Mr. P J Baruah to write to the editor not to publish such provocative news items in the larger good of the society. The Editor reacted by publishing his letter under the headline 'Hide the Truth'; whereas his intention was only to instill peace among the common people and assure them of their safety by avoiding communal colors to media reporting. He further sighted the example of Godhara train blasts that was repeatedly projected in media and became an important ingredient to the rising of the Gujrat riots that killed hundreds of innocent souls. He says that if journalists break the lines of ethical reporting, conflict will certainly follow without exception.

He specifically feels that TV news channels exaggerate information and presents a dramatic picture of petty issues that can be resorted with peace. Just like we have guidelines for Press functioning, similar regulations are needed to control the TV news channels so that they do not broadcast misleading or sensationalized news. In a pluralistic country like ours, we need more rules and regulatory bodies due to the diversity we accommodate. Such regulations might not have been anticipated by the makers of the Indian constitution on account of limited technology available in those times. Electronic media should put a check on those news and features that go against the

sovereignty, safety and security of its people. This in turn calls for a greater responsibility on the part of the editor or the 'gatekeepers' as they are the one who filter news.

Mr. Baruah recalled an incident in a Guwahati-Adivasi rally where there arose a conflict and a minor girl was stripped naked and the footage was captured by local television. The same was repeatedly shown in news channels to make the public more furious and thus resulted in larger damage. It was the right to dignity, privacy and respect of the girl which was violated by the news media in lieu of their own profits. Hence, certain professional attributes need to be inculcated and seeking permission before publishing any controversial content is very important and falls under media ethics. An Editor is supposed to practice restraint in the times when he finds a news piece too alluring for the popularity of his channel or newspaper.

He further mentions that Gandhiji and his philosophy has deeply influenced Assam and its people during the freedom struggle. He says that Dr. Bhupen Hazarika, himself was affected by Gandhian thoughts on peace and truthfulness that he composed many verses propagating his messages to the Assamese people. Assam was a land of peace and tranquility in the ancient times owing to its natural heritage; but communal violence has broken this peace into pieces. Media has a critical role to play to mitigate such adversity and restore peace in the land; one should put the interest of people foremost during reporting and gatekeeping.

The last speech of the day was delivered by **Dr. Syed Murtaza Alfarid Hussain**, Assistant Professor, Department of Mass Communication and Journalism, Assam University in his concluding remarks.



Dr. Syed Murtaza Alfarid Hussain sharing the concluding remarks

He said that that there can be no peace without conflict and vice-versa and there is a lot of sensitivity & maturity that goes in construction of peaceful journalistic messages. There can be different categories of conflict, with or without bloodshed and violence that require fruitful resolutions.

He gave the example of ‘Hotel Europa’ in Belfast which has gained popularity to be world’s most bombed hotel as it has been attacked more than 36 times. Many a time terrorist groups informed BBC before attacking in order to gain media coverage. This is exploiting the reach and power of media by the defaulters. The then UK Prime Minister Ms. Margaret Thatcher has requested many a times not to give coverage to insurgent group like Irish Republican Army and cut down their “oxygen of publicity”.

He said that there are deep fissures and faultiness in Northeast too, that require immediate attention. Seminars and events of this nature certainly provide us with a platform to judge the existing media framework in the region and allow us to work to improve upon the drawbacks. Sometimes media professionals deem that they are striking debates on important social issues but the same might not fulfil the desired purpose and end up in adding fuel to the fire.

Dr. Syed also touched upon the importance of sources of gathering information in a journalistic journey. Normally, one attaches a lot of importance to government agents and military personnel as reliable conduits of information, but it is equally important for us to give context to a story and ensure careful construction of media messages with objectivity. At the same time media should give a platform to common people to vent out their grievances. Newsroom diversity can make a huge impact to the purpose of peace journalism and there should be unity, not segregation of journalists based on the communities they hail from. Another important factor is gender sensitivity as people make moral judgements, especially in the contemporary times. There should be equal number of men and women in the newsroom too.

He ended by saying that the seminar was indeed an eye opening experience and gave food for thought in media’s role in covering conflict.



A pledge for peace - Participants putting their thumb impressions

The first day ended with a **‘pledge taking ceremony’** where the respected industry experts, guests, faculties, students and all other participants placed their thumb impressions in a silhouette outline of Mahatma Gandhi and vowed to stand for peace throughout their lives. A simple promise would certainly serve as a beacon

light to show the righteous path to justice and peace.

DAY 2

Full Day workshop on "Peace Journalism"

By

Dr. Syed Murtaza Alfarid Hussain

[10am – 5pm]

Morning Session

Dr. Syed Murtaza Alfarid Hussain, Assistant Professor, Department of Mass Communication and Journalism, Assam University, took an extensive workshop for the second and forth semester students of Tezpur University on the theme of 'Peace Journalism'. He has conducted in depth research on the same in the region of northeast and its conflict areas.



Dr. Syed Murtaza Alfarid Hussain explaining the concept of Peace

He began by clarifying on the conceptual understanding of peace journalism among the students and sculpted their know how on the theme. Conflict reporting constitutes the elements that restore order in crisis conditions. Violence is not only the physical manifestation of it, but lots of trigger points that leads to such physical violence. Though mass media focusses on this form alone and need not give much coverage to the others. As peace journalists, one must tether weightage to all forms of violence alike and endeavor to resolve disputes in any form.

It was felt that the concept of peace journalism is less deliberated among the other topics related to media and mass communication, both in practice as well as academia. One should take small steps to inculcate an attitude towards the discipline and become 'peace warriors', i.e. aggressively acting towards restoring and propagating peace.

Dr. Syed gave a practical situation to the students to ponder about. He asked them that if you know that one of your hostel boarders have stolen INR 500 and some books from your room, what shall be your immediate reaction to the situation? The room was filled with varied responses ranging from beating the culprit, expelling him/her out of the hostel, complaining against him or her to the warden/authorities and so on. But among these answers, a few voices called for a peaceful solution. They said that it was better if we interrogated the boy or the girl on what made him or her commit such an unlawful activity and try to help rather than punish the culprit. At times it is the circumstances that lead to breach of law and moral ethics, it should be dealt with maturity and sensibility rather than outrage and volatile reactions.

The same maturity is needed in a journalistic career and peaceful deliberations can make a lot of impact on a person's or a group's way of thinking. What is needed is 'empathy' and not 'sympathy' under these situations where complex issues can be accommodated with simplest steps.

Another situation presented before the students was when a journalist was told that someone shall be murdered in a particular area well in advanced by a trustworthy informer; he put up cameras and captured the entire brawl without the knowledge of the attackers who actually murdered an individual on a street. As a media worker, what is one's social role in these circumstances? Should they show footages of a man being murdered on national news television with audiences of all age groups watching it? These incidents pose question marks on ethics of a journalist and he or she must look for practical solutions as well as perform their duties as a responsible citizens of the world. Ideally, the journalist should share the evidences generated to the local police who should immediately arrest the attackers and then give a peaceful, non-violent message on their news channel or newspaper about the sad event without arousing emotions of fury and insecurity in the public. Also, the discretion and human ideology of the Editor/Gatekeeper is also very crucial in such significant occasions as to what to show and what to hide from the viewers.

A BBC correspondent while covering the Kosovo bombing in Eastern Europe, came across British troops entering the sight of conflict. He reported the same incident as "Our boys have arrived….." which was criticized immensely by fellow colleagues as he clearly displayed his personal affiliation towards his country and projected an image of being biased with his own people. A peace journalist trying to disseminate information for conflict resolution would have never used such phrases and would always prefer to remain neutral in the times of distress. "We should make patriotic journalists"

is a wrong statement by any nation for the cause of peace in the world as only one side of the conflict shall be looked into by them. Rather, one should attempt to understand the brad paradigm that leads to peace movements across the globe.

The Anti-war movement in Vietnam in the 1960s-1970s furnishes a great example of a peace movement given due importance by the mass media. The movement against U.S. involvement in the Vietnam War began small–among peace activists and leftist intellectuals on college campuses–but gained national prominence in 1965, after the United States began bombing North Vietnam in earnest. Anti-war marches and other protests, such as the ones organized by Students for a Democratic Society (SDS), attracted a widening base of support over the next three years, peaking in early 1968 after the successful Tet Offensive by North Vietnamese troops proved that war's end was nowhere in sight.



The students brainstorming during the workshop

Dr. Syed said that the 'mainstream media' is inclined towards using the logic of 3Ps, viz. Power, Politics and Profit. While the 'alternative media' such as news websites, applications, social networking sites, volunteer movements, blogs and so on (example of The Wire, The Hoot, Firstpost, etc.) are also occupying prominent space in the business. The mainstream media is living in a "delusion of grandeur" that they are setting the agenda; on the contrary they pick up agenda from the alternative media based on citizen interests. It is noteworthy to read the *Agenda Setting Theory* and the *Framing Theory* to a deeper understanding of the role played by different mass media.

Earlier, mass communication was understood to be the one initiated by professional journalists over a specified platform meant for the purpose. But in the contemporary era of citizen journalism, every man or woman is a source of information that can be accessed by the common masses. For example, Indian Prime Minister Shri Narendra Modi has 37 million plus followers on Twitter. If he sends out a message on the Twitter rostrum, and the same gets retweeted by his followers, it automatically becomes a mass communication as it reached millions of people within hours. We can add that 'objectivity' is very 'subjective' in journalism as it depends on the numerous

interpretations of the content circulated, by different groups and individuals exposed to it. The role of academia is to analyze empirically the different works and stories produced by different media houses and making a comparative study of their performances. Stories can be assessed as peace frames and conflict frames. He further suggested that we all live in the age of 'spectacle', i.e. dramatic events affect us more than the peaceful ones.

In the context of north eastern society, the pre industrial era was full of amity where peace had a larger part in our lives and there was interdependence between communities. But the post-industrial times saw that man became more self-centered and stopped interfering in each other's life. These days, people's sense of reality is determined by screen images and they are more depended on simulations, moving away from the authenticity of social reality. The big question here was whether media is fulfilling its duty or making us feel more vulnerable?

Dr. Syed presented a VUCA model (Volatility/Vulnerability, Uncertainty, Complexity and Ambiguity). This implies that mass media wants us to believe that the world is in a bad state with no hope for improvement and that anything can happen any moment. This they furnish by extreme escalation of violence and proliferation of fake news sent with a motive of incitement.

Talking about citizen journalism, he said that apart from the positive faith that it has earned, there is a darker side to it as well. It is making traditional media weak; as they have stopped employing correspondents and rely greatly on the information generated by general public and showcase it on air without any fact checking or verification. Many reporters are losing their jobs in the due course. These days consumers have transformed into "Prosumers = Consumers + Producers". The catch here is one can easily produce content but not in the discipline and ethics of a journalist. This affects the social structure to a great extent.

Coming to the question, why do we need peace journalism? The answer is to add a forth P in the 3Ps policy of mass media conglomerates and motivate them to work for peace in their quest to earn power and profit with political support. We find that people are amused with war scenes on news television and become glued to the screen in case of any violent conflict being broadcasted. News media uses the terms collateral damage, friendly fire, smart bomb, peace process, terrorism, etc. with ease not realizing its true nature and that the innocent soul such activities victimizes. We must

remember that we all are in great danger of losing democracy if journalists act as a conduit to state government and military of a nation.

In the 5W and 1H of the traditional journalism (Who, What, When, Where, What & How) we can add 1C and 1S of peace journalism, i.e. Common ground of agreement and Solution to the problem. Peace journalists should bring about some kind of resolution in the way they structure the content. One should leave his or her ethnic identity and prejudice behind and never allow them to sweep to the content of the story.

Dr. Syed then discussed the 11 Framing Tools of James Tankard that must be looked into while reporting and writing media messages. They are headlines, subheads, photos, photo captions, leads, source selection, quotes selection, pull quotes, logos, statistics and charts, concluding statements and paragraphs. All these aspects must be taken into account while framing a message for the masses and must ensure that they adhere to the peace phenomena.

Afternoon Session

The post lunch session began with Dr. Syed explaining the students some important concepts related to peace and conflict journalism, such as embedded and unilateral war journalism, episodic and thematic frames, culture of impunity, difference between intractable and normal conflict and so on. He furnish India vs. Pakistan, North vs South Korea and Israel vs. Palestine as examples of intractable conflicts where all possible efforts of peace and conflict resolution have failed and there is no chance of mutual agreement. He adds that the worst sufferers of such a conflict are the women and children of any nation who are often exposed to inhuman brutality.

As peace journalists, we must be familiar with International Humanitarian Laws (IHL) commonly known as the Geneva Convention that specifically deal with rules governing the execution of war and treatment ought to be given to the refugees. We must also be capable of distinguishing between IHL and Human Right Laws in detail.

It is worth knowing that peace can be both positive and negative; when peace is forced by external peace keeping forces without the inner will of the parties involved, it is a negative form of peace. Positive peace allows freedom of choice and when people are actually free to pursue their goals without any external interference. Examples of negative peace keeping forces are the digital peace keeping panoptic surveillance systems like the Unique Identification Number.

Apart from physical/direct violence, which is most popular amongst the media fraternity, violence can also be categorized as Cultural violence or Structural violence. Direct violence includes individuals or groups intending to hurt/kill people; example - Mob Violence/Lynching/Stabbing/Shooting/Bombing/Raping/Stone Pelting. Cultural violence takes place when images and stories which justify or glorify violence. Example - hate speech/xenophobia/persecution complex/myths and legends of war heroes/religious justifications for war/gender violence. Structural violence is built in to customs, practices, and organizations like Poverty/systems based on exploitation/excessive material inequality/racism/ sexism/colonialism/corruption, collusion, nepotism/gender inequality/ economic exploitation/ political repression/cultural alienation/climate change, etc. A peace journalist is supposed to look into all aspects of violence in view of restoring peace and not just focus on brutality of bloodshed.

Next, Dr. Syed opened the forum for a healthy debate among the students to interact.

Ali-Aye-Ligang is a spring festival associated with agriculture, especially with the beginning of the Ahu paddy cultivation. It is celebrated by the Mising or Mishing ethnic community of Assam, India. The festival marks onset of sowing seeds. The name of the festival is made up of three terms, 'Ali', legumes, 'Aye', seed and 'Ligang', to sow. The debate was about a tweet by a famous personality, who said that we should call it a festival of whole of the Assam state rather than calling it a Mising culture. Now how far is it correct to generalize this important ritual of a particular tribe? The students presented various views on the statement and in the process what came up was understanding how deep the communal colors can act to disrupt peace. Another important aspect of the debate was to realize that diversity and plurality of a society should be appreciated and the cultures should be preserved in peace.

Next in line, followed the discussion about the ‘Semantics of Peace Journalism’. It included points such as avoiding use of opinions as established facts, avoiding elite bias, single sourced reporting,

avoiding of rumor-mongering, discouraging provocative language, avoiding sensational headlines, curbing inflammatory content- text and visuals, and avoiding unverified information. These points can make an enormous difference in day to day sentence construction of a reporter.

The last session of the workshop saw an open interaction amongst the students and their mentor Dr. Syed Murtaza Alfarid Hussain. He divided the batch into teams if two, each team comprising of a junior (Semester II) and a senior (Semester IV) student. The junior was supposed to narrate a conflict incident that he or she came across and the senior one was supposed to re-narrate it to the entire gathering with a probable solution to the said issue. The conflict could be in any form, direct, cultural or structural. The students after mutual discussions touch based on a variety of issues such as the National Register of Citizens in Assam & its effects, College level disputes, Nepal Gorkhaland conflict, The Naga Accord, Discord in the nearby villages, ULFA & the distress in Kokrajhar and many more.

The idea behind this activity was to generate a sensitivity among students as to what damage simple words can make if not used properly and gauge their understanding of peaceful means in conflict resolution. The discussion went on for about an hour and a half with many case studies sprouting up that can be of potential danger to the society. With these discussions and a hope to vouch for peace under all positive and negative situations of a journalistic life, the two day seminar cum workshop ended.



Students creating memories during the workshop with Dr. Hussain

Media Coverage

The event was well covered by a number of media houses with detailed reports published in their leading dailies, including The Assam Tribune, The Sentinel, Niyomiya Barta, Dainik Purvoday, Dainik Asam, Purvanchal Prahri, Amar Asom, Northeast Now to name a few. These was published across English, Hindi, and Assamese newspapers. The reports showcased that how Gandhian philosophy is still relevant in practicing peaceful journalism even till this date and how ethics and responsible reporting can make a huge difference in conflict resolution of any nature. Electronic media channels like Assam Talks, Newslive and All India Radio Tezpur also aired the news.

A few snippets of the media clips are added as below for an immediate reference.

THE SENTINEL, DATE-23.02.2018, PG NO-05

Seminar on Gandhian philosophy and peace journalism

FROM A CORRESPONDENT

TEZPUR, Feb 22: A two-day seminar and workshop on Gandhian Philosophy and Peace Journalism began on Wednesday at the Department of Mass Communication and Journalism of Tezpur University. The programme, organized by the department in collaboration with Gandhi Smriti and Darshan Samiti, was inaugurated by Prof. PK Das, Dean, School of Humanities and Social Sciences of the university. In his introductory remarks, journalist PJ Baruah said that Gandhi introduced a novel way of seeking justice. Speaking on the topic 'Peace journalism in theory and practice: observations on India's Northeast', freelance journalist and author Rajeev Bhattacharyya shared his experiences of reporting in the conflict areas. He urged the journalists to stick to objectivity and accuracy.

Prof. Abhijit Borah, Head of the Department of MCJ, gave welcome address at the beginning of the seminar. In his speech, he highlighted the recent achievements of the department as the best mass communication department among Indian universities in the Times Higher Education Asia University Rankings 2018. Prof Borah also added that it was now a big responsibility on the part of the MCJ department to live up to this achievement.

THE ASSAM TRIBUNE, DATE-23.02.2018, PG NO-08

'Gandhian philosophy still relevant while practising journalism'

CORRESPONDENT

TEZPUR, Feb 22: "Gandhian philosophy is still relevant while practising journalism as Gandhiji was not only the father of the nation, but he was also the father of true journalism. The application of his philosophy needs greater courage and moral strength. He believed that Ahimsa has a universal application and it can be used in one's own family, society and the world at large. Through the technique of non-violence, an opponent can be changed morally, and by using this tool, Gandhiji defeated his opponent and gained India's freedom." This was observed by PJ Baruah, noted journalist and Executive Editor of *The Assam Tribune*, while giving the introductory remark in the inaugural ceremony of the two-day-long symposium cum workshop on the issue of 'Gandhian Philosophy and Peace Journalism' organised at the Mass Communication and Journalism Department of Tezpur University here.

In his speech, Baruah put stress on the positive role of the media so that it can bring peace and development in the society. "As we live in a conflict zone like the North-east, the media should always try to be very sincere in its reporting so that peace, communal fabric and the human integrity are not harmed."

He spoke on the subjects of 'Looking for Stories of Rebuilding Lives and Untold Stories of Rehabilitation Faultiness in Conflict-hit Areas' and 'Issues and Challenges of Covering Conflict: An Editor's Perspective'.

The event organised by the department in collaboration with the Gandhi Smriti and Darshan Samiti was formally inaugurated by Prof PK Das, Dean, School of Humanities and Social Sciences of the University

Das, in his speech, appreciated the positive and balanced reporting done by *The Assam Tribune*.

Speaking on the topic 'Peace Journalism in Theory and Practice: Observations on India's Northeast', freelance journalist and author Rajeev Bhattacharyya shared his experiences of reporting in the conflict zones. He urged the journalists to stick to objectivity and accuracy.

Prominent journalist Sushanta Talukdar, Editor of *The Nezine*, spoke on the subjects of 'Looking for Stories of Rebuilding Lives and Untold Stories of Rehabilitation Faultiness in Conflict-hit Areas' and 'Issues and Challenges of Covering Conflict: An Editor's Perspective', while giving concluding remarks on the event.

Dr Syed Murtaza Alfarid Hussain, Assistant Professor, Department of Mass Communication and Journalism, Assam University, stated that there is an urgent need of maintaining accuracy and positivity in reporting.

Biren Das, Registrar of Tezpur University, and a few students of the department, like Amlan Jyoti Das, Nidarshana Mahanta, Tirtharaj Gohain and RK Yaibiren Sana, also spoke on the occasion.

Prof Abhijit Borah, Head of the Mass Communication and Journalism Department, gave the welcome speech. Dr Anjuman Borah and Dr Junali Deka gave the introduction and the vote of thanks respectively.

NIYOMIYA BARTA
DATE - 24.02.2018, PG NO - 01

তেজপুৰ বিশ্ববিদ্যালয়ত আলোচনা-চক্ৰ, কৰ্মশালাৰ শুভাৰম্ভ

বৃহৎ উদ্যোগ গঢ়ি তুলিলে জৈৱ বৈচিত্র্য সমৃদ্ধ অসমলৈ আহিব চৰম ভাবুকি



বার্তা ব্যুৰ', তেজপুৰ, ২১ ফেব্রুৱাৰী ঃ 'টি আৰ পিৰ বাবে নহয়, সমাজ আৰু দেশৰ স্বার্থ আগত ৰাখিহে সাংবাদিকতা কৰা উচিত। সমস্যাৰ সন্মুখীন হোৱা তথা সংঘর্ষপীড়িত জনসাধাৰণৰ ১০ পৃষ্ঠাত

NIYOMIYA BARTA
DATE - 24.02.2018, PG NO - 10

বৃহৎ উদ্যোগ গঢ়ি তুলিলে জৈৱ বৈচিত্র্য সমৃদ্ধ অসমলৈ

সমস্যাৱলীৰ সমাধানৰ বাবে ঋণাত্মক নহয়, গঠনমূলক তথা ইতিবাচক চিন্তাচর্চাৰেহে সাংবাদিকতা কৰা প্রয়োজন।' আজি তেজপুৰ বিশ্ববিদ্যালয়ত এই মন্তব্য কৰে বিশিষ্ট সাংবাদিক তথা 'দ্য আছাম ট্রিবিউন' কাকতৰ কার্যবাহী সম্পাদক প্রশান্তজ্যোতি বৰুৱাই। উল্লেখ্য যে তেজপুৰ বিশ্ববিদ্যালয়ৰ গণসংযোগ আৰু সাংবাদিকতা বিভাগৰ উদ্যোগত আৰু নতুন দিল্লীৰ গান্ধী স্মৃতি আৰু দর্শন সমিতিৰ সহযোগত আজি বিশ্ববিদ্যালয়খনৰ সংশ্লিষ্ট বিভাগৰ সন্মিলনকক্ষত 'গান্ধী দর্শন আৰু শান্তি সাংবাদিকতা' শীর্ষক বিষয়ত দুদিনীয়া আলোচনা-চক্ৰ আৰু কর্মশালাৰ শুভাৰম্ভ কৰা হয়। উক্ত আলোচনা-চক্ৰ আৰু কর্মশালা মুকলি কৰি বিশিষ্ট সাংবাদিক বৰুৱাই আগবঢ়োৱা উদ্বোধনী ভাষণত আৰু কয় যে সংঘর্ষজর্জৰিত বুলি চিহ্নিত উত্তৰ-পূর্বাঞ্চলৰ বহুবিষয়ক লৈ চলি থকা বিতর্ক সমস্যাৱলীৰ কিয়দংশ মীমাংসা হৈছে যদিও আনবোৰ বিষয়ক লৈ কথা-বতৰা চলি আছে। এনে পৰিপ্রেক্ষিতত শান্তি বাতাৱৰণ সৃষ্টিৰে সমস্যাৱলীসমূহৰ

ASOMIYA PRADITIN
DATE - 24.02.2018, PG NO - 07

তেজপুৰ বিশ্ববিদ্যালয়ত 'গান্ধী দৰ্শন আৰু শান্তি সাংবাদিকতা' শীৰ্ষক আলোচনা চক্ৰ

'সমাজ-দেশৰ স্বাৰ্থৰ হকে সাংবাদিকতা কৰা উচিত'

প্ৰতিদিন বিশেষ সেৱা, তেজপুৰ, ২১ ফেব্ৰুৱাৰী ঃ টি আৰ পিৰ বাবে নহয়, সমাজ আৰু দেশৰ স্বাৰ্থ অক্ষত ৰাখিবে সাংবাদিকতা কৰা উচিত। সমস্যাৰ সম্মুখীন হোৱা তথা সংঘৰ্ষপীড়িত জনসাধাৰণৰ সমস্যাবলী সমাধানৰ বাবে গঠনমূলক নহয়, গঠনমূলক তথা ইতিবাচক চিন্তা-চৰ্চাযোগেহে সাংবাদিকতা কৰা প্ৰয়োজন। আজি তেজপুৰ বিশ্ববিদ্যালয়ত এই মন্তব্য কৰে বিশিষ্ট সাংবাদিক তথা দি অসম ট্ৰিবিউন কাকতৰ কাৰ্যবাহী সম্পাদক প্ৰশান্তজ্যোতি বৰুৱাই। উল্লেখ্য যে তেজপুৰ বিশ্ববিদ্যালয়ৰ গণসংযোগ আৰু সাংবাদিকতা বিভাগৰ উদ্যোগত আৰু নতুন দিল্লীৰ গান্ধী স্মৃতি আৰু দৰ্শন সমিতিৰ সহযোগত আজি বিশ্ববিদ্যালয়খনৰ সংশ্লিষ্ট বিভাগৰ সন্মিলন কক্ষত 'গান্ধী দৰ্শন আৰু শান্তি সাংবাদিকতা' শীৰ্ষক বিষয়ৰ দুদিনীয়া আলোচনা-চক্ৰ আৰু কৰ্মশালাৰ শুভাৰম্ভ কৰা হয়। উক্ত আলোচনা-চক্ৰ আৰু কৰ্মশালা মুকলি কৰি বিশিষ্ট সাংবাদিক বৰুৱাই উদ্বোধনী ভাষণত আৰু কয় যে সংঘৰ্ষজৰ্জৰিত বুলি চিহ্নিত উত্তৰ-পূৰ্বাঞ্চলৰ বহু বিষয়ক লৈ চলি থকা বিতৰ্ক, সমস্যাবলীৰ কিয়দংশ মীমাংসা হৈছে যদিও আনবোৰ বিষয়ক লৈ কথা-বতৰা চলি আছে। এনে পৰিপ্ৰেক্ষিতত শান্তিৰ বাতাবৰণ সৃষ্টিৰে সমস্যাসমূহৰ সমাধানৰ পথ প্ৰশস্ত কৰাত সাংবাদিকৰ বিশেষ ভূমিকা আছে। সেয়েহে কোনো ধৰণৰ উৰা-বাতৰিৰ পৰিবৰ্তে সত্যাসত্য পৰীক্ষা কৰি সচ্ছতা আৰু দায়বদ্ধতাৰে বাতৰি পৰিবেশনেৰে প্ৰকৃত প্ৰতিচ্ছবি প্ৰকাশ কৰাৰ ক্ষেত্ৰত সংবাদ মাধ্যমৰ গুৰুত্বপূৰ্ণ ভূমিকা আছে। সাম্প্ৰতিক কালৰ আধুনিক তথা-প্ৰযুক্তিয়ে সংবাদ ক্ষেত্ৰখন আৰু ওপৰলৈ নি বিশ্বজুৰি প্ৰভাৱ বিস্তাৰ কৰিলৈ সক্ষম হৈছে। সমান্তৰালভাৱে সমগ্ৰ বিশ্বত গান্ধীজীৰ দৰ্শন আৰু শান্তি সাংবাদিকতাৰ প্ৰাসংগিকতা বৃদ্ধি পাইছে। এনে প্ৰেক্ষাপটত সংবাদ মাধ্যমে সকলো দিশ চলি-জাৰি চাই বাতৰি প্ৰকাশ কৰাটো অতি গুৰুত্বপূৰ্ণ হৈ পৰিছে বুলি মন্তব্য কৰে বিশিষ্ট সাংবাদিক বৰুৱাই। উদ্বোধনী অনুষ্ঠানত বিশ্ববিদ্যালয়খনৰ প্ৰক্ষাপ কৰাটো অতি গুৰুত্বপূৰ্ণ হৈ পৰিছে বুলি মন্তব্য কৰে বিশিষ্ট সাংবাদিক বৰুৱাই। উদ্বোধনী ভাষণত আৰু কয় যে সংঘৰ্ষজৰ্জৰিত বুলি চিহ্নিত উত্তৰ-পূৰ্বাঞ্চলৰ বহু বিষয়ক লৈ চলি থকা বিতৰ্ক, সমস্যাবলীৰ কিয়দংশ মীমাংসা হৈছে যদিও আনবোৰ বিষয়ক লৈ কথা-বতৰা চলি আছে। এনে পৰিপ্ৰেক্ষিতত শান্তিৰ বাতাবৰণ সৃষ্টিৰে সমস্যাবলী সমাধানৰ পথ প্ৰশস্ত কৰাত সাংবাদিকৰ বিশেষ ভূমিকা আছে। সেয়েহে কোনো ধৰণৰ উৰা-বাতৰিৰ পৰিবৰ্তে সত্যাসত্য পৰীক্ষা কৰি সচ্ছতা আৰু দায়বদ্ধতাৰে বাতৰি পৰিবেশনেৰে প্ৰকৃত প্ৰতিচ্ছবি প্ৰকাশ কৰাৰ ক্ষেত্ৰত সংবাদ মাধ্যমৰ গুৰুত্বপূৰ্ণ ভূমিকা আছে। সাম্প্ৰতিক কালৰ আধুনিক তথা-প্ৰযুক্তিয়ে সংবাদ ক্ষেত্ৰখন আৰু ওপৰলৈ নি বিশ্বজুৰি প্ৰভাৱ বিস্তাৰ কৰিলৈ সক্ষম হৈছে। সমান্তৰালভাৱে সমগ্ৰ বিশ্বত গান্ধীজীৰ দৰ্শন আৰু শান্তি সাংবাদিকতাৰ প্ৰাসংগিকতা বৃদ্ধি পাইছে। এনে প্ৰেক্ষাপটত সংবাদ মাধ্যমে সকলো দিশ চলি-জাৰি চাই বাতৰি প্ৰকাশ কৰাটো অতি গুৰুত্বপূৰ্ণ হৈ পৰিছে বুলি মন্তব্য কৰে বিশিষ্ট সাংবাদিক প্ৰশান্তজ্যোতি বৰুৱাই। উদ্বোধনী অনুষ্ঠানত বিশ্ববিদ্যালয়খনৰ গণসংযোগ আৰু সাংবাদিকতা বিভাগৰ মুৰব্বী অধ্যাপক ড° প্ৰশান্ত কুমাৰ দাসে নিমন্ত্ৰিত অতিথিসকলক সম্বৰ্ধনা জ্ঞাপন কৰি আগবঢ়োৱা বক্তব্যত উত্তৰ-পূৰ্বাঞ্চলত গান্ধীজীৰ দৰ্শন আৰু শান্তি সাংবাদিকতাৰ প্ৰাসংগিকতা আছে বুলি মন্তব্য কৰে। তাঁহাতকৈ বিশ্ববিদ্যালয়খনৰ প্ৰাক্তন ডীন অধ্যাপক অভিজিৎ বৰাই আদৰণী ভাষণত বিশ্ববিদ্যালয়খনৰ অগ্ৰগতিৰ বিষয়ে দাপ্তি ধৰি আলোচনাচক্ৰ আৰু কৰ্মশালাৰ গুৰুত্বৰ বিষয়ে বক্তব্য আগবঢ়ায়। গণসংযোগ আৰু সাংবাদিকতা বিভাগৰ সহকাৰী অধ্যাপিকা অম্বুমান বৰাই অনুষ্ঠানটোৰ উদ্দেশ্য ব্যাখ্যা কৰি হিংসা আৰু সংঘৰ্ষপূৰ্ণ পৰিস্থিতিৰ সময়ৰ সংবাদ মাধ্যমে শান্তি বাৰ্তা কটিওৱাত কেনেদৰে ভূমিকা গ্ৰহণ কৰিব পাৰে, এই সম্পৰ্কত উল্লেখ কৰে। ইয়াৰ পিছত 'শান্তি সাংবাদিকতাৰ তত্ব আৰু ব্যৱহাৰিক দিশ' অধিবেশন' শীৰ্ষক বিষয়ত সমল ব্যক্তি হিচাপে ভাষণ প্ৰদান কৰে বিশিষ্ট সাংবাদিক আৰু লেখক বাৰ্তা ভট্টাচাৰ্যই। অসমস্তৰ্ভক্ত উত্তৰ-পূৰ্বাঞ্চলৰ বিশেষ বিশেষ অঞ্চলত চৰজমিন তদন্তৰে বাতৰি পৰিবেশন কৰা অভিজ্ঞতাৰ বিষয়ে ব্যক্ত কৰি ভট্টাচাৰ্যই প্ৰশ্নোত্তৰ পৰ্বত অংশগ্ৰহণ কৰা সাংবাদিকসকলৰ বিভিন্ন প্ৰশ্নৰ উত্তৰ দিয়ে। অপৰাহ্ণ ৩ বজাত আৰম্ভ হোৱা এই উদ্বোধনী অনুষ্ঠানত বিশ্ববিদ্যালয়খনৰ মাননীয়া উপাচাৰ্য ড° প্ৰশান্ত কুমাৰ দাসে নিমন্ত্ৰিত অতিথিসকলক সম্বৰ্ধনা জ্ঞাপন কৰি আগবঢ়োৱা বক্তব্যত উত্তৰ-পূৰ্বাঞ্চলত গান্ধীজীৰ দৰ্শন আৰু শান্তি সাংবাদিকতাৰ প্ৰাসংগিকতা আছে বুলি মন্তব্য কৰে। তাঁহাতে বিশ্ববিদ্যালয়খনৰ প্ৰাক্তন ডীন অধ্যাপক অভিজিৎ বৰাই আদৰণী ভাষণত বিশ্ববিদ্যালয়খনৰ অগ্ৰগতিৰ বিষয়ে দাপ্তি ধৰি আলোচনাচক্ৰ আৰু কৰ্মশালাৰ গুৰুত্বৰ বিষয়ে বক্তব্য আগবঢ়ায়। গণসংযোগ আৰু সাংবাদিকতা বিভাগৰ সহকাৰী অধ্যাপিকা অম্বুমান বৰাই অনুষ্ঠানটোৰ উদ্দেশ্য ব্যাখ্যা কৰি হিংসা আৰু সংঘৰ্ষপূৰ্ণ পৰিস্থিতিৰ সময়ৰ সংবাদ মাধ্যমে শান্তি বাৰ্তা কটিওৱাত কেনেদৰে ভূমিকা গ্ৰহণ কৰিব পাৰে, এই সম্পৰ্কত উল্লেখ কৰে। ইয়াৰ পিছত 'শান্তি সাংবাদিকতাৰ তত্ব আৰু ব্যৱহাৰিক দিশ— অধিবেশন' শীৰ্ষক বিষয়ত সমল ব্যক্তি হিচাপে ভাষণ প্ৰদান কৰে বিশিষ্ট সাংবাদিক আৰু লেখক বাৰ্তা ভট্টাচাৰ্যই। অসমস্তৰ্ভক্ত উত্তৰ-পূৰ্বাঞ্চলৰ বিশেষ বিশেষ অঞ্চলত চৰজমিন তদন্তৰে বাতৰি পৰিবেশন কৰা অভিজ্ঞতাৰ বিষয়ে ব্যক্ত কৰি ভট্টাচাৰ্যই প্ৰশ্নোত্তৰ পৰ্বত অংশগ্ৰহণ কৰা সাংবাদিকসকলৰ বিভিন্ন প্ৰশ্নৰ উত্তৰ দিয়ে। অপৰাহ্ণ ৩ বজাত আৰম্ভ হোৱা এই উদ্বোধনী অনুষ্ঠানত বিশ্ববিদ্যালয়খনৰ মাননীয়া উপাচাৰ্য ড° প্ৰশান্ত কুমাৰ দাসে নিমন্ত্ৰিত অতিথিসকলক সম্বৰ্ধনা জ্ঞাপন কৰি



সাংবাদিক ভট্টাচাৰ্যই ভাৰতৰ পিতৃ গান্ধীজীৰ সন্দৰ্ভত কয় যে গান্ধীজী এগৰাকী মহান সাংবাদিকেই নহয়, তেওঁ এগৰাকী মহান লেখকো আছিল। শান্তি সাংবাদিকতাৰ অন্যতম প্ৰবৰ্তক গান্ধীজীৰ দৰ্শন আৰু শান্তি সাংবাদিকতাই ক্ৰমান্বয়ে প্ৰাসংগিকতা লাভ কৰিছে বুলি উল্লেখ কৰি ভট্টাচাৰ্যই নীতিগত সাংবাদিকতা বৰ্তাই ৰাখিবলৈ আহ্বান জনাই জনাগণ সবাইকে উদ্বাত্ত বুলি মন্তব্য কৰে। গণসংযোগ আৰু সাংবাদিকতা বিভাগৰ ছাত্ৰ অম্লানজ্যোতি দাসে 'গান্ধী আৰু শান্তি সাংবাদিকতা' আৰু ছাত্ৰী নন্দনা মহন্তই 'বিতৰ্কিত বিষয়ত সংবাদ মাধ্যমৰ ভূমিকা' শীৰ্ষক বিষয়ত প্ৰতিবেদন আগবঢ়ায়। আনহাতে, বিশিষ্ট সাংবাদিক সুশান্ত তালুকদাৰে বিদ্রোহী সমস্যা জৰ্জৰিত অঞ্চলত পুনৰ সংস্থাপন বিভিন্ন দিশ সম্পৰ্কে আলোকপাত কৰে। ইপিনে, অসম বিশ্ববিদ্যালয়ৰ গণসংযোগ আৰু সাংবাদিকতা বিভাগৰ অধ্যাপক ড° টুমৰ মুৰ্তজা আলফাবিদ ছাহেবে সামৰণী ভাষণ দিয়ে।

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সমাজ-দেশৰ স্বাৰ্থ সন্মুখত ৰাখিহে সাংবাদিকতা কৰা উচিত ঃ পি জে বৰুৱা

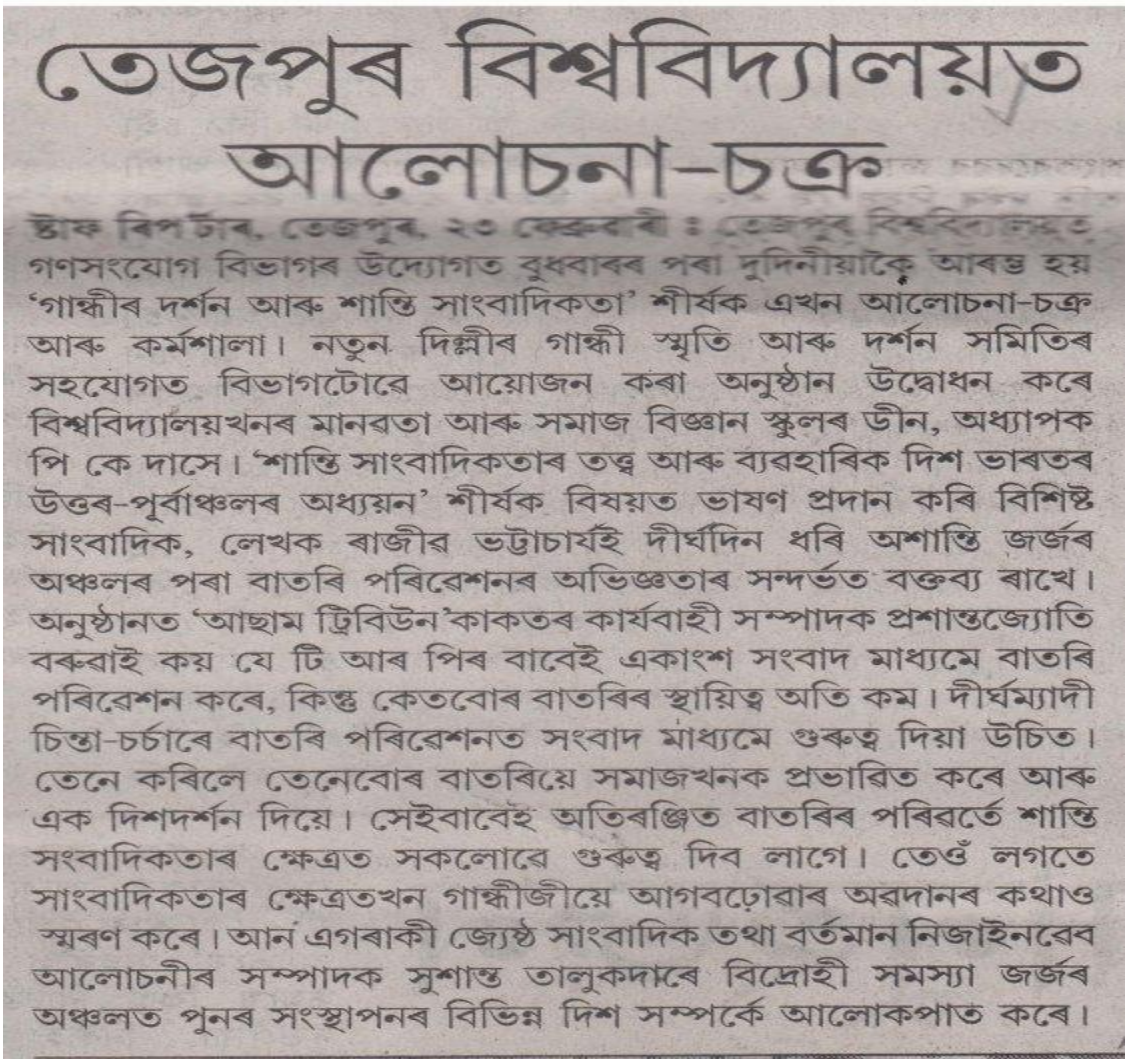
জিলাৰ বাতৰি দিওঁতা

তেজপুৰ, ২১ ফেব্ৰুৱাৰী ঃ 'টি আৰ পিৰ বাবে নহয়, সমাজ আৰু দেশৰ স্বাৰ্থ সন্মুখত ৰাখিহে সাংবাদিকতা কৰা উচিত। সমস্যাৰ সন্মুখীন হোৱা তথা সংঘৰ্ষপীড়িত জনসাধাৰণৰ সমস্যাবলীৰ সমাধানৰ বাবে গঠনমূলক তথা ইতিবাচক চিন্তা- চৰ্চাযোগে সাংবাদিকতা কৰাৰ প্ৰয়োজন।' আজি তেজপুৰ বিশ্ববিদ্যালয়ত এই মন্তব্য কৰে বিশিষ্ট সাংবাদিক তথা 'দ্য অছাম ট্ৰিবিউন' কাকতৰ কাৰ্যবাহী সম্পাদক প্ৰশান্তজ্যোতি বৰুৱাই। উল্লেখ্য যে তেজপুৰ বিশ্ববিদ্যালয়ৰ গণসংযোগ আৰু সাংবাদিকতা বিভাগৰ উদ্যোগত আৰু নতুন দিল্লীৰ গান্ধী স্মৃতি আৰু দৰ্শন সমিতিৰ সহযোগত আজি বিশ্ববিদ্যালয়খনৰ সংশ্লিষ্ট বিভাগৰ সন্মিলন কক্ষত 'গান্ধী দৰ্শন আৰু শান্তি সাংবাদিকতা' শীৰ্ষক বিষয়ৰ দুদিনীয়া আলোচনা-চক্ৰ আৰু কৰ্মশালাৰ শুভাৰম্ভ কৰা হয়। উক্ত আলোচনা-চক্ৰ আৰু কৰ্মশালা মুকলি কৰি বিশিষ্ট সাংবাদিক বৰুৱাই উদ্বোধনী ভাষণত আৰু কয় যে সংঘৰ্ষজৰ্জৰিত বুলি চিহ্নিত উত্তৰ-পূৰ্বাঞ্চলৰ বহু বিষয়ক লৈ চলি থকা বিতৰ্ক, সমস্যাবলীৰ কিয়দংশ মীমাংসা হৈছে যদিও আনবোৰ বিষয়ক লৈ কথা-বতৰা চলি আছে। এনে পৰিপ্ৰেক্ষিতত শান্তিৰ বাতাবৰণ সৃষ্টিৰে সমস্যাবলী সমাধানৰ পথ প্ৰশস্ত কৰাত সাংবাদিকৰ বিশেষ ভূমিকা আছে। সেয়েহে কোনো ধৰণৰ উৰা-বাতৰিৰ পৰিবৰ্তে সত্যাসত্য পৰীক্ষা কৰি সচ্ছতা আৰু দায়বদ্ধতাৰে বাতৰি পৰিবেশনেৰে প্ৰকৃত প্ৰতিচ্ছবি প্ৰকাশ কৰাৰ ক্ষেত্ৰত সংবাদ মাধ্যমৰ গুৰুত্বপূৰ্ণ ভূমিকা আছে। সাম্প্ৰতিক কালৰ আধুনিক তথা-প্ৰযুক্তিয়ে সংবাদ ক্ষেত্ৰখন আৰু ওপৰলৈ নি বিশ্বজুৰি প্ৰভাৱ বিস্তাৰ কৰিলৈ সক্ষম হৈছে। সমান্তৰালভাৱে সমগ্ৰ বিশ্বত গান্ধীজীৰ দৰ্শন আৰু শান্তি সাংবাদিকতাৰ প্ৰাসংগিকতা বৃদ্ধি পাইছে। এনে প্ৰেক্ষাপটত সংবাদ মাধ্যমে সকলো দিশ চলি-জাৰি চাই বাতৰি প্ৰকাশ কৰাটো অতি গুৰুত্বপূৰ্ণ হৈ পৰিছে বুলি মন্তব্য কৰে বিশিষ্ট সাংবাদিক প্ৰশান্তজ্যোতি বৰুৱাই। উদ্বোধনী অনুষ্ঠানত বিশ্ববিদ্যালয়খনৰ গণসংযোগ আৰু সাংবাদিকতা বিভাগৰ মুৰব্বী অধ্যাপক ড° প্ৰশান্ত কুমাৰ দাসে নিমন্ত্ৰিত অতিথিসকলক সম্বৰ্ধনা জ্ঞাপন কৰি আগবঢ়োৱা বক্তব্যত উত্তৰ-পূৰ্বাঞ্চলত গান্ধীজীৰ দৰ্শন আৰু শান্তি সাংবাদিকতাৰ প্ৰাসংগিকতা আছে বুলি মন্তব্য কৰে। আনহাতে, বিশ্ববিদ্যালয়খনৰ প্ৰাক্তন ডীন অধ্যাপক অভিজিৎ বৰাই আদৰণী ভাষণত বিশ্ববিদ্যালয়খনৰ অগ্ৰগতিৰ বিষয়ে দাপ্তি ধৰি আলোচনাচক্ৰ আৰু কৰ্মশালাৰ গুৰুত্বৰ বিষয়ে বক্তব্য আগবঢ়ায়। গণসংযোগ আৰু সাংবাদিকতা বিভাগৰ সহকাৰী অধ্যাপিকা অম্বুমান বৰাই অনুষ্ঠানটোৰ উদ্দেশ্য ব্যাখ্যা কৰি হিংসা আৰু সংঘৰ্ষপূৰ্ণ পৰিস্থিতিৰ সময়ত সংবাদ মাধ্যমে শান্তি বাৰ্তা কটিওৱাত কেনেদৰে ভূমিকা গ্ৰহণ কৰিব পাৰে, সেই সম্পৰ্কত কয়। ইয়াৰ পিছত 'শান্তি সাংবাদিকতাৰ তত্ব আৰু ব্যৱহাৰিক দিশ— অধিবেশন' শীৰ্ষক বিষয়ত সমল ব্যক্তি হিচাপে ভাষণ দিয়ে বিশিষ্ট সাংবাদিক আৰু লেখক বাৰ্তা ভট্টাচাৰ্যই। অসমস্তৰ্ভক্ত উত্তৰ-পূৰ্বাঞ্চলৰ বিশেষ বিশেষ অঞ্চলত চৰজমিন তদন্তৰে বাতৰি পৰিবেশন কৰা অভিজ্ঞতাৰ বিষয়ে ব্যক্ত কৰি ভট্টাচাৰ্যই প্ৰশ্নোত্তৰ পৰ্বত অংশগ্ৰহণ কৰা সাংবাদিকসকলৰ বিভিন্ন প্ৰশ্নৰ উত্তৰ দিয়ে। অপৰাহ্ণ ৩ বজাত আৰম্ভ হোৱা এই উদ্বোধনী অনুষ্ঠানত বিশ্ববিদ্যালয়খনৰ মাননীয়া উপাচাৰ্য ড° প্ৰশান্ত কুমাৰ দাসে নিমন্ত্ৰিত অতিথিসকলক সম্বৰ্ধনা জ্ঞাপন কৰি



আগবঢ়োৱা বক্তব্যত উত্তৰ-পূৰ্বাঞ্চলত গান্ধীজীৰ দৰ্শন আৰু শান্তি সাংবাদিকতাৰ প্ৰাসংগিকতা আছে বুলি মন্তব্য কৰে। তাঁহাতকৈ বিশ্ববিদ্যালয়খনৰ প্ৰাক্তন ডীন অধ্যাপক অভিজিৎ বৰাই আদৰণী ভাষণত বিশ্ববিদ্যালয়খনৰ অগ্ৰগতিৰ বিষয়ে দাপ্তি ধৰি আলোচনাচক্ৰ আৰু কৰ্মশালাৰ গুৰুত্বৰ বিষয়ে বক্তব্য আগবঢ়ায়। গণসংযোগ আৰু সাংবাদিকতা বিভাগৰ সহকাৰী অধ্যাপিকা অম্বুমান বৰাই অনুষ্ঠানটোৰ উদ্দেশ্য ব্যাখ্যা কৰি হিংসা আৰু সংঘৰ্ষপূৰ্ণ পৰিস্থিতিৰ সময়ত সংবাদ মাধ্যমে শান্তি বাৰ্তা কটিওৱাত কেনেদৰে ভূমিকা গ্ৰহণ কৰিব পাৰে, সেই সম্পৰ্কত কয়। ইয়াৰ পিছত 'শান্তি সাংবাদিকতাৰ তত্ব আৰু ব্যৱহাৰিক দিশ— অধিবেশন' শীৰ্ষক বিষয়ত সমল ব্যক্তি হিচাপে ভাষণ দিয়ে বিশিষ্ট সাংবাদিক আৰু লেখক বাৰ্তা ভট্টাচাৰ্যই। অসমস্তৰ্ভক্ত উত্তৰ-পূৰ্বাঞ্চলৰ বিশেষ বিশেষ অঞ্চলত চৰজমিন তদন্তৰে বাতৰি পৰিবেশন কৰা অভিজ্ঞতাৰ বিষয়ে ব্যক্ত কৰি ভট্টাচাৰ্যই প্ৰশ্নোত্তৰ পৰ্বত অংশগ্ৰহণ কৰা সাংবাদিকসকলৰ বিভিন্ন প্ৰশ্নৰ উত্তৰ দিয়ে। অপৰাহ্ণ ৩ বজাত আৰম্ভ হোৱা এই উদ্বোধনী অনুষ্ঠানত বিশ্ববিদ্যালয়খনৰ মাননীয়া উপাচাৰ্য ড° প্ৰশান্ত কুমাৰ দাসে নিমন্ত্ৰিত অতিথিসকলক সম্বৰ্ধনা জ্ঞাপন কৰি

AMAR ASOM , DATE-24.02.2018, PG NO-06



তেজপুৰ বিশ্ববিদ্যালয়ত আলোচনা-চক্ৰ

ষ্টাফ ৰিপৰ্টাৰ, তেজপুৰ, ২৩ ফেব্রুৱাৰী : তেজপুৰ বিশ্ববিদ্যালয়ত গণসংযোগ বিভাগৰ উদ্যোগত বুধবাৰৰ পৰা দুদিনীয়াকৈ আৰম্ভ হয় 'গান্ধীৰ দৰ্শন আৰু শান্তি সাংবাদিকতা' শীৰ্ষক এখন আলোচনা-চক্ৰ আৰু কৰ্মশালা। নতুন দিল্লীৰ গান্ধী স্মৃতি আৰু দৰ্শন সমিতিৰ সহযোগত বিভাগটোৱে আয়োজন কৰা অনুষ্ঠান উদ্বোধন কৰে বিশ্ববিদ্যালয়খনৰ মানৱতা আৰু সমাজ বিজ্ঞান স্কুলৰ ডীন, অধ্যাপক পি কে দাসে। 'শান্তি সাংবাদিকতাৰ তত্ত্ব আৰু ব্যৱহাৰিক দিশ ভাৰতৰ উত্তৰ-পূৰ্বাঞ্চলৰ অধ্যয়ন' শীৰ্ষক বিষয়ত ভাষণ প্ৰদান কৰি বিশিষ্ট সাংবাদিক, লেখক ৰাজীৱ ভট্টাচাৰ্যই দীৰ্ঘদিন ধৰি অশান্তি জৰ্জৰ অঞ্চলৰ পৰা বাতৰি পৰিৱেশনৰ অভিজ্ঞতাৰ সন্দৰ্ভত বক্তব্য ৰাখে। অনুষ্ঠানত 'আছাম ট্ৰিবিউন'কাকতৰ কাৰ্যবাহী সম্পাদক প্ৰশান্তজ্যোতি বৰুৱাই কয় যে টি আৰ পিৰ বাবেই একাংশ সংবাদ মাধ্যমে বাতৰি পৰিৱেশন কৰে, কিন্তু কেতবোৰ বাতৰিৰ স্থায়িত্ব অতি কম। দীৰ্ঘম্যাদী চিন্তা-চৰ্চাৰে বাতৰি পৰিৱেশনত সংবাদ মাধ্যমে গুৰুত্ব দিয়া উচিত। তেনে কৰিলে তেনেবোৰ বাতৰিয়ে সমাজখনক প্ৰভাৱিত কৰে আৰু এক দিশদৰ্শন দিয়ে। সেইবাবেই অতিৰঞ্জিত বাতৰিৰ পৰিৱৰ্তে শান্তি সংবাদিকতাৰ ক্ষেত্ৰত সকলোৱে গুৰুত্ব দিব লাগে। তেওঁ লগতে সাংবাদিকতাৰ ক্ষেত্ৰতখন গান্ধীজীয়ে আগবঢ়োৱাৰ অৱদানৰ কথাও স্মৰণ কৰে। আন এগৰাকী জ্যেষ্ঠ সাংবাদিক তথা বৰ্তমান নিজাইনৰেব আলোচনীৰ সম্পাদক সুশান্ত তালুকদাৰে বিদ্রোহী সমস্যা জৰ্জৰ অঞ্চলত পুনৰ সংস্থাপনৰ বিভিন্ন দিশ সম্পৰ্কে আলোকপাত কৰে।

DAINIK PURVODAY, DATE-24.02.2018, PG NO-05

तेजपुर : पत्रकारिता पर सेमिनार आयोजित

तेजपुर, 23 फरवरी (पू.सं.)। तेजपुर विश्वविद्यालय के मास कम्युनिकशन एंड जर्नलिज्म विभाग में 21 व 22 फरवरी को गांधीवादी विचारधारा और शांति पत्रकारिता विषय पर दो दिवसीय संगोष्ठी तथा कार्यशाला का आयोजन किया गया। मास कम्युनिकेशन और जर्नलिज्म विभाग द्वारा गांधी स्मृति और दर्शन समिति के संयुक्त तत्वावधान में आयोजित इस संगोष्ठी का उद्घाटन स्कूल ऑफ ह्युमिनिटीज एंड सोशल साइसेंज के डीन प्रो. पीके दास ने गत 21 फरवरी को किया।

इस अवसर पर असम ट्रिब्यून के कार्यकारी संपादक पीजे बरुवा ने कहा कि गांधी ने न्याय पाने के लिए एक विशेष तरीके से परिचय कराया। गांधीजी ने कहा था, 'मजहब इंसान को बैर रखना नहीं सिखता, वह तो लोगों को प्रेम के धागों में बांधने के लिए है। गांधीजी ने सिखाया था कि सबसे बड़ा नैतिक कानून यही है कि हम मानव जाति की भलाई के लिए निरंतर कार्य करते रहें। गांधी जी का मानना था कि हिंसा की बनियाद पर कोई भी चीज टिक नहीं सकती।

'प्रीस जर्नलिज्म इन थ्योरी एंड प्रैक्टिसः ऑवजरवेशंस ऑन इंडियाज नॉर्थ ईस्ट' विषय पर अपने अनुभवों के बारे में बताते हुए फ्रीलांस जर्नलिस्ट तथा लेखक राजीव भट्टाचार्य ने पत्रकारों को उद्देश्यपूर्ण और सटीक रिपोर्टिंग करने की सलाह दी। इस अवसर पर दो अन्य प्रतिष्ठित पत्रकार सुशांत तालुकदार तथा पीजी बरुवा ने 'लुकिंग फॉर स्टोरीज ऑफ रिनिल्डिंग लाइब्ज एंड अनटोल्ड स्टोरीज ऑफ रिहेविलटेशन पाल्टीनेस इन कन्फलीक्ट-हिट-एरीयाज' विषय और 'इस्यूजेज एंड चैलेंजेज ऑफ कवरिंग कनफिल्कट : एडीटर्स पर्सपेक्टिव' विषय पर अपने ज्ञानवर्धक विचार व्यक्त किए।

मास कम्युनिकेशन एंड जर्नलिज्म विभाग, असम विश्वविद्यालय के असिसटेंट प्रो. डॉ. मुर्तजा अलफ्रेड हुसैन ने प्रथम दिन समापन भाषण दिया और द्वितीय दिन पीस जर्नलिज्म कार्यशाला का संचालन किया। तेजपुर विश्वविद्यालय के रजिस्ट्रार डॉ. बीरेन दास ने इस अवसर पर अपने विचार व्यक्त किए। सेमिनार की शुरुआत में एमजीजे विभागाध्यक्ष प्रो. अभिजीत बोरा ने सभी उपिस्थित लोगों का स्वागत करते हुए टाइम्स हायर एजुकेशन एशिया यूनिवर्सिटीज रैंकिंग 2018 में भारतीय विश्वविद्यालयों में तेजपुर विश्वद्यिालय के मास कम्युनिकेशन विभाग को सर्वश्रेष्ठ मास कम्युनिकेशन विभाग घोषित किए जाने पर प्रसन्नता जाहिर करते हुए कहा कि इस महत्वपूर्ण उपलब्धि से हमारी जिम्मेदारियां और भी बढ़ गई हैं, जिन पर हमें खरा उतरना है। एमए के छात्र एम्लान ज्योतिदास, निदर्शना महंत, तीर्थराज गोहाईं और आरके थाविरेन साना ने भी कार्यशाला में अपने विचार व्यक्त किए। डॉ. अंजुमन बोरा ने कार्यक्रमों का परिचय कराया जबकि डॉ. जुलानी डेका ने धन्यवाद ज्ञापन प्रस्तुत किया।

PURVANCHAL PRAHARI,
DATE-24.02.2018, PG NO-05

तेजपुर विवि में पत्रकारिता पर संगोष्ठी संपन्न

तेजपुर। तेजपुर विश्वविद्यालय के मास कम्यूनिकेशन एंड जर्नलिज्म विभाग में गत 21 तथा 22 फरवरी को 'गांधीवादी विचारधारा और शांति पत्रकारिता' विषय पर दो दिवसीय संगोष्ठी तथा कार्यशाला का आयोजन किया गया। मास कम्यूनिकेशन और जर्नलिज्म विभाग द्वारा गांधी स्मृति और दर्शन समिति के संयुक्त तत्वावधान में आयोजित उक्त संगोष्ठी का उद्घाटन स्कूल ऑफ ह्यूमेनिटीज एंड सोशल साइंसेज के डीन प्रो. पीके दास ने गत 21 फरवरी को किया। इस अवसर पर असम ट्रिब्यून के कार्यकारी संपादक पीजे बरुवा ने गांधी के न्याय पाने के लिए एक विशेष तरीके से परिचय कराया। गांधी जी ने कहा था कि मजहब इंसान को बैर रखना नहीं सिखाता, वह तो लोगों को प्रेम के धागे में बांधने के लिए है। गांधी जी ने सिखाया था कि सबसे बड़ा नैतिक कानून यही है कि हम मानव जाति की भलाई के लिए निरंतर कार्य करते रहें। गांधी जी का मानना था कि हिंसा की बुनियाद पर कोई भी चीज टिक नहीं सकती। 'पीस जर्नलिज्म इन थ्योरी एंड प्रैक्टिस : ऑब्जरवेशंस ऑन इंडियाज नार्थ ईस्ट' विषय पर अपने अनुभवों के बारे में बताते हुए फ्रीलांस जर्नलिस्ट तथा लेखक राजीव भट्टाचार्य ने पत्रकारों को उद्देश्यपूर्ण और सटीक रिपोर्टिंग करने की सलाह दी। इस अवसर पर दो अन्य प्रतिष्ठित पत्रकार सुशांत तालुकदार तथा पीजी बरुवा ने 'लुकिंग फॉर स्टोरीज ऑफ रिबिल्डिंग लाइन्ज एंड अनटोल्ड स्टोरिज ऑफ रिहेबिलिटेशन फाल्टीनेस इन कन्फ्लीक्ट-हिट-एरियाज' विषय पर और 'इश्यूजेज एंड चैलेंजेज ऑफ कवरिंग कनफिटक्ट : एन एडीटर्स पर्सपेक्टिव' विषय पर अपने ज्ञानवर्द्धक विचार व्यक्त किए। मास कम्यूनिकेशन एंड जर्नलिज्म विभाग, असम विश्वविद्यालय के एसिसटेंट प्रोफेसर डॉ. मुर्तजा अलफ्रेड हुसैन ने प्रथम दिन समापन भाषण दिया और द्वितीय दिन 'पीस जर्नलिज्म' कार्यशाला का संचालन किया। तेजपुर विश्वविद्यालय के रजिस्ट्रार डॉ. बीरेन दास ने इस अवसर पर अपने विचार व्यक्त किए। सेमिनार की शुरुआत में एमसीजे विभागाध्यक्ष प्रो. अभिजीत बोरा ने सभी उपस्थित लोगों का स्वागत करते हुए टाइम्स हायर एजुकेशन एशिया यूनिवर्सिटीज रैंकिंग-2018 में भारतीय विश्वविद्यालयों में तेजपुर विश्वविद्यालय के मास कम्यूनिकेशन विभाग को सर्वश्रेष्ठ मास कम्युनिकेशन विभाग घोषित किए जाने पर प्रसन्नता जाहिर करते हुए कहा कि इस महत्वपूर्ण उपलब्धि से हमारी जिम्मेदारियां और भी बढ़ गई हैं, जिन पर हमें खरा उतरना है। एमए के छात्र अम्लान ज्योति दास, निदर्शना महंत, तीर्थराज गोहाईं और आरके याविरेन साना में भी कार्यशाला में अपने विचार व्यक्त किए। डॉ. अंजुमन बोरा ने कार्यक्रमों का परिचय कराया जबकि डॉ. जोनाली डेका ने धन्यवाद ज्ञापन प्रस्तुत किया। (निसं)

Conclusion & Recommendations

The two day event was accomplished as per expectations as it saw a lot of knowledge sharing, opinion analysis, debates, discussions and discourses that inspired the young media aspirants to lead an ethical and responsible life and work for peace. It acquainted the participants with a variety of conflict situations and how they must and must not react under specific circumstances. It is one of the most important duties of a journalist to work towards a peaceful society by deploying the means of mass communication at his or her disposal. People usually trust mass media and the news and information circulated by them; this trust can also revive their faith in peaceful means of conflict resolution.

After the workshop, the participants understood that violence can be of many forms and not only its physical manifestation. Cultural and structural violence too act as major triggers resulting in brutal conflicts across the globe. Peace has be inculcated as an attitude amongst the common people, and especially in the media fraternity who should avoid glorifying discord and violence of any nature. It's a herculean responsibility on the forth pillar of democracy to build the path of progress and prosperity without compromising on social justice and tranquility. It the innocent people of any country who suffer the most in case of any conflict, just as we see in case of Syria. War and bloodshed is not the only solution to reconcile any disagreement; it is peaceful compatibility and mutual compromises that brings about common good for all.

The expert speakers of the session shared their views and anecdotes as to how they had confronted conflict situations and maintained peace and objectivity at the same time. Propagating peace does not mean manipulating facts or telling lies; verification of facts and their accuracy is of utmost importance at any given point of time.

After the successful completion of the seminar & workshop, it was unanimously recommended by all the participants that "peace journalism" should be included in the course content of all the institutions imparting degree and diplomas in mass communication. Strong academic foundation of the aspiring journalists can yield favorable results as they shall be exposed to the knowhow of how

to react during conflict situations with objectivity and glorify the path of cordial resolutions, as against to incitement and provocation of the parties in disagreement.
Another important recommendation is that workshops on similar themes should be periodically conducted for the working journalists on field. Many a times soaring competition, thirst for power, political affiliations, etc. makes media & journalism a mere business for its executors. A deliberation on peace journalism and real time problem solving discussions can go a long way in improving the quality of mass communication that the world is witnessing in the present times.

Also, reading about the history & work of various peacemakers who have brought about a difference in the way we see conflict and war is a must read for the media fraternity. These include eminent peace personalities such as Mahatma Gandhi, Mother Teresa, Nelson Mandela, Martin Luther King, Daisaku Ikeda and many more. The world is in dire need of familiarization with peace and handling conflict situations peacefully rather than furiously.

The seminar and workshop was indeed an eye opening session which boosted the sensitivity of the participants about the much unexplored avenue of peace journalism.

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AFO/C4D/2014/ 453
17 July 2014

Prof. S. K. Dutta,
Head of Department Mass Communication and Journalism,
Tezpur University
Napaam, Tezpur
Sonitpur – 784172

Dear Mr. Dutta,

Tezpur University: Post Graduate MA degree course in Communication for Development (C4D)

With reference to your letter dated 11 July 2014 regarding **'Submission of proposal for converting postgraduate diploma program to C4D postgraduate program'**, we are happy to confirm our support for the following activities:

I. Pre – Consultation for state level review of C4D curriculum (modules)- 28-29 July 2014
II. State Consultation on MA in C4D program roll out in Tezpur University & C4D Curriculum Development (August 2014)

In this regard, we have reviewed the budget submitted by you and confirm out support to a maximum of Rs 3,28,200/-. The detailed budget break-up is annexed for your reference.

Sl	Activity	Place	Total Cost
1	Pre- Consultation for state level review of C4D curriculum	Tezpur	18,500
2	Consultation for C4D Curriculum development	Tezpur	300,700
3	Project Management support	Tezpur	9,000
		TOTAL	**328,200**

We would request you to kindly let us have the duly completed FACE form to enable us to release 80% of the approved budget for the same. On completion of the activities we would request you to kindly let us have the duly completed FACE form, Statement of Expenditure (SoE) with list of participants, as well as report which would enable us to liquidate this amount.

As part of our standard operating procedure, we need to undertake the financial assessment of Implementing Partners. In this regard, UNICEF operation and finance team may visit your office on periodic basis, upon mutually convenient dates, and review the financial processes and systems with your accounts and finance staff.

Looking forward to a fruitful partnership to take ahead the agenda of Social and Behaviour Change Communication (SBCC) in the state.

Yours sincerely,

Rushabh Hemani
OIC, Field Office

Cc: Ms Joya Chravarty, Assistant Prof, MCJ, Department

EVENT REPORT

NORTH EAST YOUTH CONCLAVE

6-8 September 2019

The Department of Mass Communication & Journalism, Tezpur University in collaboration with Gandhi Smriti and Darshan Samiti, New Delhi organized a three day North East Youth Conclave on the theme 'Role of Youth in Contributing to a Culture of Peace and Non-violence' during 6-8 September 2019. A total of 150 participants from 18 institutions across North East India participated in the Conclave. The conclave witnessed a most wonderful confluence of youthful energy, vibrancy and enthusiasm that demonstrated the power and potential of youth people to create change in the world.

DAY 1: INAUGURAL FUNCTION

The North East Youth Conclave began on the morning of 6th September 2019. The inaugural ceremony was graced by Senior Journalist Samudra Gupta Kashyap as the Chief Guest, prominent social activist Dibyajyoti Saikia as special invitee and Gulshan Gupta, the North East Coordinator of Gandhi Smriti and Darshan Samiti as part of the guest panel. The dais was also graced by Dr. Joya Chakraborty, Head, Department of MCJ-TU and Prof. Abhijit Borah, former Head and Professor in the Department.

The session began with the national anthem followed by a Borgeet presentation. The Head of the Department of Mass Communication and Journalism, Dr. Joya Chakraborty delivered the welcome address where she greeted the 150 youth representatives from 18 different universities across northeast India that included Tripura University, Sikkim University, Manipur University of Culture, North East Hill University and many others.



Glimpses of the inaugural session: Dibyajyoti Saikia lighting the lamp (left) and borgeet presentation (right)

Thereafter, Gulshan Gupta shared his insights on Gandhi Smriti and Darshan Samiti. Speaking on the occasion, he quoted, "By birth, we are all non-violent." Following this, the inaugural address was delivered by Professor Abhijit Bora, Department of Mass Communication and Journalism, where he highlighted an important aspect of the theme by saying, "You are the only community that can give peace a chance." In line with the spirit of the occasion, Samudra Gupta Kashyap enlightened the gathering with his inspiring words. He talked about the freedom struggle in North East India, about it being the biodiversity hotspot of the country and the role of Mahatma Gandhi in peace building exercises among the youth. He said, "Once you put Gandhi in your heart, he will never go away. Anyone and everyone can become a Gandhi."

The event concluded with the vote of thanks by Dr. Anjuman Borah, Assistant Professor of the Department of Mass Communication and Journalism and the conclave coordinator. Recalling her mentor, Dr. Daisaku Ikeda, she ended the session with the words, "Peace is a decision."

DAY 1, SESSION I: UNDERSTANDING A CULTURE OF PEACE AND NON-VIOLENCE IN THE CONTEXT OF NORTH-EAST

Mr. Dibyajyoti Saikia conducted the first session of the day. A core member of the team of Anna Hazare, Mr. Saikia is the founder secretary general of the national level social organization, Brothers.

Keeping in sync with the theme of the conclave, he started off the first session by speaking about Mahatma Gandhi and how *satyagraha* has been a tool of non-violence and has always been effective. Talking mainly on non-violence, he said that only refraining from hitting or destroying is not non-violence but having a sense of positivity in mind and soul is the true essence of it. Interacting with the youth representatives, he asked them about the major conflict issues prevailing in the Northeastern region. He mentioned that humanity should be above any religion or caste. As he neared the end of his session, he emphasized the importance of youth towards the progress of the country. He said that there is no point of progress if violence is still being practiced. Thereafter, he concluded the session by highlighting the role of youth to take forward the virtue of non-violence.

DAY 1, SESSION II: TOWARDS A SCIENCE OF HUMAN STRENGTHS AND VIRTUES

Dr. Pranjal Buragohain, Associate Professor, Department of Education, Dibrugarh University conducted the second session of the North East Youth Conclave. The session was centered around the topic, towards a science of Human Strengths and Virtues, wherein he spoke to the representatives about positive psychology and the role it plays to promote and establish the conception of peace and non-violence, especially in the lives of youth.

Applauding the participants' enthusiasm to have a talk on peace in today's times, Dr. Pranjal Buragohain talked on how in modern times, religion and politics have given rise to conflicts, and

has become a major contributor to violence in the society. He said, “Every human’s intention is to want something better in life,” emphasizing upon the five dimensions responsible for a positive and peaceful outlook on life, namely- positive emotions, positive engagement, positive purpose, positive relationships and positive accomplishment.



Glimpses from the first day session with Dr. Pranjal Buragohain

Towards the end of the session, he cited a few scientific real-life examples of how inculcating such positive practices have led to not only a higher standard of living, but have also significantly contributed to less aggression or violence, in people’s personal lives. He concluded by mentioning his idea, that is, “Peace within one’s own self contributes largely to peace in society.”

DAY 1, SESSION III: EDUCATION FOR CULTIVATING PEACE, HARMONY AND SELF-IMPROVEMENT

The third technical session of the day was also conducted by Dr. Pranjal Buragohain, who made it an activity based session, and began by introducing a couple of group activities where unknown participants were asked to sit in groups and talk to each other in order to get to know their partners and find out a point of strength and weakness of each other. Similar activities went on further, seeking to boost interaction between the participants and being productive and fun in the process.

At the end of the day, there was a cultural evening where the students of TU and participating universities showcased their skills and enthralled the audience, which was followed by dinner, marking the end of the day.





Glimpses from the final session of the first day of the conclave

DAY 2, SESSION IV: UNDERSTANDING THE CHEMISTRY OF A SOCIAL EMOTION

On the second day of the event, the first session was conducted by Dr. Pranjal Buragohain. He started off by making everybody understand the chemistry of a social emotion and how emotions are needed in youth for prevailing peace and harmony. Mr. Pranjal shared some of his personal life experiences and the emotional aspects in relation with the theme of the youth conclave. The speaker conducted various activities to increase the level of one's well being and the sense of

gratitude towards someone. Sharing stories from his childhood and hostel life, he expressed how emotions have helped him grow, acknowledging the presence of largely two emotions- 'gratitude' and 'generosity'. These were certain values that he had imbibed from his teacher, to whom he shall remain ever grateful for having taught him the value of those. Talking about his teacher further, he mentioned about the practice of writing a letter of gratitude to him, thanking him for each and every help that he had received from his teacher, and how humbled he felt when he received a reply from his teacher, promising him that he shall continue helping others in the future. Proceeding further, he mentioned the three mantras of a good sustainable life- firstly, acknowledge the gift, secondly, recognise the value of the gift and lastly, appreciate the intention of the donor. In his words, gratitude is a social reinforcer, gratitude is a social motivator, gratitude is a social barometer and gratitude is a social repenter.

DAY 2, SESSION V: BUILDING AN EMOTION THROUGH SKILLS FOR PEACE AND HARMONY

The second technical session of the day was continued by Dr. Pranjal Burhagohain, which focused on developing skills that builds up the feeling of gratitude within an individual. To do so, he conducted various activities and asked the participants to continue the same for not only their well-being but also for the greater good of the society. "Well-being of my life is my choice," he said, to emphasize on which, he shared his own experiences with his students in the past wherein he found that students continuously writing journals had a higher level of gratitude and well-being.

Focusing on the emotion of gratitude, Dr. Burhagohain conducted a total of four exercises to foster the growth of the emotion- one, maintaining a gratitude journal that kept a record of all the good experiences of their life, for according to him, the mind is sensitive to bad experiences; two, writing gratitude letters; three, the WWW, i.e the 'what, when, where' exercise, to keep a record of all the good experiences which inevitably empower the positive emotions, and fourth, the "Hero-of-the-Day" exercise wherein he asked the participants to name a person who had

changed their life somehow or added value to it, with the objective of realising the value of a person in one's life.

He concluded the session with a practice called 'gratitude prayer', a cathartic experience where the participants were asked to imagine their mothers and offer heartfelt gratitude for each of her deeds, followed by a question answer session between the speaker and the participants.

DAY 2, SESSION VI: CONFLICT RESOLUTION- TOOLS AND STRATEGIES-I

The speakers at the event were Ms. Suravi Sarkar and Mr. Santu Guchait, Communication for Development specialists and Ms. Anokhi Parikh, an independent trainer, from Banglanatakdotcom. The session began with an activity of self-introspection followed by an analysis of the concept of conflict and peace-making. Thereafter, the entire mass of participants was divided into three groups, headed by each of the three speakers.

As the teams dispersed to their respective locations, the sessions began. The one headed by Ms. Suravi Sarkar approximately consisted of around 20 participants. The participants were shown how conflict ensues in social media and is miscommunicated in the virtual world, through a group activity where all the participants had to make a perfect square without verbally communicating. The exercise entailed a lot of related issues like gender-based violence, leadership qualities, push factors of violence, the essence of global peace, the core of violence etc. The discussions were punctuated by real-life situations where the participants had to apply their reason on issues such as mob mentality etc. and try to understand the whats and whys of conflict. Moving along the lines of understanding the concept of conflict, certain conflict resolution strategies were devised, namely- competing, avoiding, accommodating, compromising and collaborating.




Glimpses from one of the afternoon sessions with Mr Santu Guchait

With another group, the session had Mr. Santu Guchait who got the participants involved in an activity that asked the questions, "What are the main causes of conflict and non violence?" and "What are the three things that describes you?". He also made the participants perform an exercise where he asked them to hold a rope and make a perfect square of it without using verbal communication. He then made a human loop and tangled it. Then he asked the participants to untangle without letting go off their hands.

Through the exercises he showed the complex nature of conflict and how it can only be resolved through united effort.

DAY 2, SESSION VII: CONFLICT RESOLUTION - TOOLS AND STRATEGIES-II

The post tea session with Ms. Suravi Sarkar began with the discussion of conflict resolution strategies where the efficiency of each of them based on the degree of assertiveness or cooperativeness they allowed for, was evaluated by the speaker and the audience. In the course of the discussion, the speaker mentioned, "every problem has a solution; no matter however critical the problem may be, it is born with a solution." As a part of conflict controlling measures, she discussed the importance of preventing future conflicts and not letting prevailing conflicts spread. Majorly an interactive activity based session, participants were explained how conflicts ensued through miscommunication through a game of Chinese whispers, followed by a brief discussion on the skills for conflict resolution namely- leadership, interpersonal

relationship, listening, empathy, mediation, kindness, negotiation, assertiveness and cooperation. In the process of explaining the importance of these skills, the speaker divided the audience into groups of two and gave them a chance to interact, in an attempt to get to know each other. Further, they were asked to write thank you letters to their partners. Thereafter, the participants were asked to recollect a conflict situation of their respective lives and write about it in brief, anonymously for future sessions. Finally, the session ended with a feedback session where the participants put forth their comments on the session they just attended.



Students being divided into groups in the first afternoon session of the second day

In Mr. Santu Guchait's group, the last session of the second day saw the participants come together in groups of seven where each student discussed the various conflicts they faced and how they resolved them. Taking cues from the exercise, he said that there are 5 ways to resolve a conflict, namely- Competing, Avoiding, Collaborating, Accommodating and Compromising.

Ms. Anokhi Parikh began with an exercise of making a square with a rope. This was a non-verbal exercise where the group members could not communicate with each other verbally and rather use hand signals and body language. Chart papers and ribbons were assigned to note the skills involved while making the square. Skills such as coordination, leadership, teamwork, etc. were mentioned by several participants. The motive behind this exercise was to explain how

peace and non-violence cannot be attained single handedly. Push factors detailing why violence occurs in the society was the second phase of the activity. Participants were handed small chits to jot down the factors they perceived as leading the society towards violence.

As the technical sessions of the day ended, all the participants assembled in the screening hall of the host department, to enjoy a rich cultural evening packed with an array of spellbinding performances by the students of Tezpur University and other participating universities.



A glimpse of the cultural evening on Day 2

DAY 3, SESSION VIII: TOOLS AND SKILL PRACTICE PREPARATION OF PRESENTATION FOR 'APPRECIATING DIVERSITY'

The final day of the event started with a technical session conducted by Ms. Suravi Sarkar and Mr. Santu Guchait, Communication for Development specialists from banglanatakdotcom. The session was in continuation of the previous day's activity of understanding self identity. In this session, the students were asked to describe their social identities, that is, how they portray themselves in the social media. The activity threw some light on the contrast between our real self and virtual self, and how the virtual self is a projection of what we would like to see our real selves as being. The participants were then asked to find similarities between the qualities of

their virtual self within their real self. They also mentioned that it is our duty to react or comment to a post responsibly since social media is a volatile space.

Thereafter, all the youth representatives were categorized into group as story writers, instruments players, vocalists, actors, and so on. This activity was done to form a team which would later perform a drama.

Following this, a drama was performed by the team as a part of the valedictory session with the assistance of Ms. Suravi Sarkar and Mr. Santu Guchait. Racism, gender based violence (witch hunting), and cyb



Picture from the final day session: the drama being rehearsed

VALEDICTORY SESSION

The valedictory session of the North East Youth Conclave began on the afternoon of the third day of the conclave, that is, August 8th, 2019. The session was hosted by Abhilash Bapanasha and Sangita Kalita, students of the Department of Mass Communication and Journalism, Tezpur University.

At the very beginning of the session, Dr. Anjuman Borah, the conclave coordinator, shared her remarks and experiences of the three-day conclave. She mentioned how in a very small period of time everything was arranged. The conclave got an overwhelming response at the very beginning

itself when 18 institutions responded by sending their youth representatives. She also acknowledged the contribution of Samudra Gupta Kashyap as he successfully set the tone for the entire conclave. She thanked the research panel who did their best to keep the sessions informative and engaging and ensured that each participant took home an experience. Appreciating the bond between all the participants, she said the conclave was a "beautiful confluence of youthful minds". She also expressed her gratitude to all the teams who worked backstage to make the event a success. The teams were categorised as Reporting, Video documentation, Registration and Reception, Poster, Photography, Social media and Cultural.

This was followed by a drama presentation that was prepared under the guidance of the resource persons from banglanatakdotcom, Ms. Suravi Sarkar and Mr. Santu Guchait. It involved a video telecast of the previous day's (Session VI and VII) exercises. The theme of the performance was surrounded around Racism (racial conflict faced by people belonging to the Mongoloid race in India), Social evils and superstitions, and Cyber bullying. It dealt with different kinds of conflicts, how one deals with them in real life and how one can resolve differences by means of peace and justice. It showed that addressing small issues and working towards resolving the underlying factors of conflict, one can restore peace and harmony in the world. The message that the act sent across was that "*we are one, we are human, we come with nothing and we will go back with nothing, so don't discriminate*".



Stills from the cyberbullying skit of the final day.

After the drama, Ms. Chinmoyee Deka and Mr. Abdul Wadood from the reporting team presented a brief session-wise report on the three-day conclave where all the various activities that were carried out throughout the Conclave were re-lived and appreciated.

The resource persons then shared their experiences. Ms. Suravi Sarkar called it just the beginning and promised further and continued association with the Department of Mass Communication and Journalism, Tezpur University. She said that in order to live a peaceful life, one must stay positive and spread positivity. Mr. Santu Guchait asked the participants to pat themselves on their backs and say "you did well". One of the visiting participants ensured the creation of an instagram account which will connect all the participants in the conclave so that they can stay connected among themselves and also with banglanatakdotcom.

This was followed by a participatory puppetry video presentation of about six minutes. Students of the nearby Panchmile HS School were the puppeteers. It was titled "The Lesson" and addressed two important themes for creating peace – 'Respect' and 'Love'.

Following this, representative from Gandhi Smriti and Darshan Samiti, Mr. Gulshan Gupta expressed his gratitude towards everyone and said, "We are about to begin the journey towards peace". He also shared a story about Guru Nanak that went like this: *Guru Nanak was once travelling with his disciple "Mardana" when he came across a village that was filled with hypocrites. He was disappointed but he still blessed the village on his farewell, saying, "May you always stay and flourish in this same place." Mardana was shocked but could not ask anything. Then he reached the next village where people were waiting to receive them. He spent three days there and received a lot of love and warmth. On the last day he blessed them too, by saying, "May you get displaced from here." Mardana could not stop himself and asked, "Why such contrasting blessings?" Guru ji replied, "People of the first village have their eyes closed with ego so wherever they go they will spread their ego but people from the second village will spread peace and love wherever they go"*. Mr. Gupta equated Nanak's lesson to the message of the Conclave and said that all the participants must now spread the message of peace and the importance of the role that youth can play in peace building, wherever they go.

The session then welcomed the Chief Guest for the evening, Professor Dilip Kumar Saikia, Pro Vice Chancellor of Tezpur University and felicitated him by presenting him a *gamusa* and a sapling. He congratulated the department for hosting a successful conclave and said, “Peace is the need of the hour and you (youth) are the main source of peace”.

The feedback session saw many students coming up to share their experiences. A student from Sikkim University said, “Thanks everyone. I learnt a lot and hope for a continued brotherhood”. Another student praised the accommodation and well planned conveyance facilities provided by the department and said that the Conclave gave him an opportunity to see and present their own as well as others’ skills. A participant from the Manipur University of Culture said, “Hope to come again. We are the youth so we should plan and come together to work for peace”. A Tripura University participant said that the Conclave was a great platform to make friends and that whatever they learnt from it they can apply in real life as well. A student of North East Hill University expressed his satisfaction with the department and called it one of the most advanced Mass Communication departments.

Later, the certificates were distributed to all the visiting participants by the Pro Vice Chancellor and the Head of the Department, Mass Communication and Journalism.



The pro-vice chancellor handing out certificates to participants

Then there was a special gesture by the students of Manipur University of Culture where they gifted a traditional Manipuri memento, *Lengyan Phee* to the Pro Vice Chancellor and the Head, Department of MCJ.

This was followed by the Vote of Thanks proposed by Mr. Joy Chakraborty, a student of the third semester, M.A. in Mass Communication and Journalism.

The session came to a close with the singing of the National Anthem, followed by a photo session.



Group photo of the event

*****

Collaborating for Peaceniks

Ananya Bhattacharya <ananya@banglanatak.com> 30 October 2018 at 20:30
To: Anjuman Borah <anjuman.borah@gmail.com>
Cc: Shalini Majumdar <shalini.banglanatak@gmail.com>, Devdan Mookerjee <devdan@banglanatak.com>

Dear Anjuman

Greetings from Kolkata ! Hope this email finds you well.

Writing to you to explore possibility of having your institute as a partner in our Peaceniks initiative. We launched this program in February 2018 in collaboration with the US Consulate General Kolkata. Peaceniks aims at creating a regional network for regular exchange, dialogue, collaboration and concerted action by youth to build peace. 44 young people from India, Bangladesh and Nepal have participated so far. In India the youth are from West Bengal, Assam, Bihar and Jharkhand. Information on activities in 2018 are available in Peaceniks YouTube channel and Facebook channels.

In 2019 we will continue the work with the Peaceniks champions. They will be using digital tools, art, music and theatre in building peace and countering the growing violent extremism. Our plan is to partner with media institutes/colleges and encourage the students to use tools like internet TV/radio, social media and art for building safe communities. We will partner with Radio Bongonet as internet radio channel. We plan partnerships with Community Radios in college campus. We will be using blog and youtube channel.

We request participation of students of C4D Department in this initiative to encourage students to act as changemakers leading action for building safe and peaceful communities. This entails

- Encouraging students led action in promoting peace by organizing awareness programs using cultural and digital media, conducting research on identifying good practices and sharing via digital media.
- Facilitating participation of 2 students in capacity building workshops in March and May/June. The workshop in March will be held on 28-29 March 2019. Mr. Raymond Caldwell, an artistic director in the field of theatre from U.S.A., will train the students in performative storytelling skills. In May/June the workshop will focus on enhancing knowledge on effective use of digital media and art and theatre in digital space.
- The participating students will work with their peers to stop violence and build safe communities using digital and cultural media. These may be awareness programs, round tables, debates, sharing of content via the Peaceniks YouTube channel, social media, Community Radio and Internet Radio/TV for wider use.
- Mentoring and guidance will be provided to media students and young journalists through virtual conferences/webinars.
- Students participating in the workshops will also have the opportunity to participate in cross border collaboration with youth in Nepal and Bangladesh.
- The students will showcase their innovations and work in Youth for Peace Summit at the American Center in Kolkata in August 2019.

Looking forward to hearing from you

Warmly
Ananya

--
Ananya Bhattacharya

Director
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A REPORT ON THE PARTICIPATORY AUDIO AND VIDEO PRODUCTION WORKSHOP

Resource Person- Debarun Dutta

Dated – 22nd January – 26th January 2020

OVERVIEW: The Department of Mass Communication and Journalism, Tezpur Universitry has initiated a workshop from 22nd to 26th of January. It was conducted by Mr. Debarun Dutta, the executive director of Drishti. The workshop gave broad understanding about participatory audio/video, community media.

The workshop was basically done to understand about the concerns of community and development through of participatory audio/video.

The workshop included planning a proper strategy to conduct facilitation to the students, which also involved basic technical training for the students of Fakhruddin Ali Ahmed High School, Tezpur.

The workshop was very beneficial for the students of Borghat and Panchmile, where they were able to learn about the basics of videography and interviewing techniques. On the other hand, we also learned so many useful things from the workshop. It is a very good step to educate the students about the power of a camera and how they can use it as a tool to solve their problems.

The intended output of the workshop was to create three short films on three different issues in the community/school. Of which, topics were decided by the students themselves, we worked as a facilitator throughout the process of making the movies.

A participatory video is a form of participatory media in which a group or community creates its film. The idea behind this is that making a video is easy and accessible, and is a great way of bringing people together to explore issues, voice concerns or simply to be creative and tell stories. It is therefore primarily about the process, though high quality and accessible films (products) can be created using these methods if that is the desired outcome. This process can be very empowering, enabling a group or community to take their action to solve their problems, and also to communicate their needs and ideas to decision-makers and/or other groups and communities. As such, PV can be a highly effective tool to engage and mobilize marginalized people and to help them to implement their forms of sustainable development based on local needs.

1st session begins on 22nd January morning at 10 AM where we discussed poverty and tried to understand it from different angles and then we discussed underprivileged people and also shared our thoughts on that. The 2nd session started at 2:30 PM where we made a pitch document on the issues of the community and school. We made three different documents by three groups and our group made a pitch document on the development of Napam M.V. School and we also made a presentation on that document. We made some blunders in the initial attempt, which Debarun Dutta corrected and from which we acquired some important tips.

The next day 23rd January we interacted with the students of Napam M.V school and F.A.A High school. In the morning session, we interacted with the students of Napam. There was a total of 10 students all were girls and these students gave us some very useful information about their school. The issues that

they highlighted were no electric fan facility in their classroom, no drinking water facility in their school and the students also mentioned the poor washroom facility in the school.

The next session (4th session) was started at 3.30 p.m. where we interacted with students of Borghat. These students gave us some serious issues to work on, they told us about the practice of child marriage in their community, noise pollution by their neighbourhood and alcoholic, etc. Some of the students themselves had experienced these issues in their community and they shared their own experiences in the interaction. After selecting three issues we made three groups of students for shooting and scripting three short films. We made our planning for training the students later that evening.

On 24th January, we started to train the students. It was our 5th session of the workshop and in the first phase, we taught the students about the basics of videography through presentations which include different shot sizes, framing, angles, etc. Later we also give them some tips for interviewing people and then we gave them our phones to shoot different shots and to interview each other they were very interested to perform the given tasks and they completed it very easily.

The next day (25th January) the students shot the entire video. There were two boys Biplab and Muksidul both of them handled the camera beautifully and a girl of class X named Himadri interviewed the students and the teachers. We were very happy to see them working very nicely because we just taught them the earlier they and they were doing it with the least flaws.

Later that day we visited a youth club of LGB institute of mental health. Where Debarun Dutta had a session with the club members and he showed us some participatory videos.

We started our editing of the video on 26th January. It was a very long and lengthy process but fortunately, we completed the editing before evening and we screened all three films in the conference hall.

The key challenge of the workshop has to be time management because we had a very short deadline and making a participatory short film takes a lot of time. We have to train the students properly to shoot the film and then we have to complete the shooting and editing in 24 hours. We somehow managed everything by working overtime and we learned a life lesson from this workshop.

From this workshop especially in my case, I learned to never underestimate anyone. These school kids of class IX and X are very talented according to their age. Especially the two boys who handled the camera and the girl who interviewed everyone was doing their work so calmly and perfectly that I never felt to interrupt them at any time. They were doing the same thing that we taught them earlier in the training session and in some cases better than we had expected.

"3.7.1 Number of collaborative activities with other institutions/ research establishments/industry for research and academic development of faculty and students per year (10)

3.7.1.1: Total number of Collaborative activities with other institutions/ research establishment/industry for research and academic development of faculty and students year wise during the last five years

लेसर एवं फोटोनिक्स केन्द्र
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CENTRE FOR LASERS & PHOTONI
INDIAN INSTITUTE OF TECHNOLO
KANPUR 208 016, IN

Dated 4th October, 2019

To Whomsoever It May Concern

Dr. Pabitra Nath is associated with the on-going BIRAC (Biotechnology Ignition Grant) project titled "Smartphone/camera based up-gradation of existing fluorescence spectroscopic devices for early cancer diagnosis" in the Bio-Photonics Lab. of Physics & CELP Department, IIT Kanpur, the Project Coordinator being Ms. Shikha Ahirwar and the Advisor being Prof. Asima Pradhan. Dr. Pabitra Nath is associated with the project as the Scientific Advisor with his expertise in the area of smartphone based bio-sensing instrumentation.

Asima Pradhan
Bio-Photonics Lab
Department of Physics & CELP

तेजपर विश्वविद्यालय / TEZPUR UNIVERSITY

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Date: 14/11/12

From:
Dr P.C. Sarmah
Scientist & Head
Electronics & Instrumentation Division

TO WHOM IT MAY CONCERN

This is to certify that Dr. Pal itra Nath, Assistant Professor, Department of Physics, Tezpur University, Napaam, has been involved as a Co-principal Investigator of the project "Development of optical fiber based multiplex sensor network system for landslide monitoring and implementation" funded by CSIR under Network project (Project amount Rs. 1.3 crore) for the period of 2012-17 as approved by the Director CSIR-NEIST. Dr Nath has been associated in the formulation of the total project plan to be implemented by the group in a phase manner. This research project is under the programme "Engineering of disaster management" in collaboration with other CSIR laboratories.

1.CSIR-Central Building Research Institute, Roorkee
2.CSIR-Central Road Research Institute, New Delhi
3.CSIR-Central Scientific Instrumentation Organisation, Punjab
4.CSIR-Central Mathematical Modelling And Computer Simulation, Bangalore
5.CSIR-National Environmental Engineering Research Institute, Chennai
6.CSIR-North East Institute of Science & Technology, Jorhat

14/11/12

(P.C. Sarmah)

Gmail

Rupjyoti Gogoi <rupjyotigogoi@gmail.com>

...Extension of Visiting Associateship at IUCAA....

2 messages

Visitor Services <vs@iucaa.in> Wed, Jul 27, 2016 at 9:16 PM
Cc: kandu <kandu@iucaa.in>, Visitor Services <vs@iucaa.in>

Dear Colleagues,

I am indeed happy to inform you that your term as a Visiting Associate at IUCAA has been extended for a further period of three years with effect from August 1, 2016 till July 31, 2019. This is just an advance intimation.

A detailed communication in this regard will be sent to you in due course.

Kindly acknowledge receipt.

With kind regards,

Sincerely,

Prof. Kandaswamy Subramanian
Dean, Visitor Academic Programmes,
IUCAA
Pune - 411 007.

Rupjyoti Gogoi <rupjyotigogoi@gmail.com> Thu, Jul 28, 2016 at 10:14 AM
To: Visitor Services <vs@iucaa.in>
Cc: kandu <kandu@iucaa.in>

Dear Sir,
Thank you so much.

Regards,
Rupjyoti
[Quoted text hidden]
--
Dr Rupjyoti Gogoi,
Assistant Professor,
Department of Physics,
Tezpur University, Napaam-784028, Assam, India
Phone: (Office) +91-3712-275577 (Direct line),
(Mobile) +91-94355-59488

Imaging Subsurface Structure of an Urban Area Based on Diffuse-Field Theory Concept Using Seismic Ambient Noise

Nilutpal Bora,[1] Rajib Biswas,[2] and Peter Malischewsky[3]

Abstract—Single station ambient noise measurement in campaign mode has of late gained a huge popularity among geoscientists. Herein, we present results of ambient vibration analysis, executed in a highly populated urban area covering 47 survey points. The resonance frequency estimates range from 0.5 to 3 Hz, as found from H/V. Taking H/V curve as input for retrieving subsurface information, we deploy diffuse field assumption (DFA) theory. The obtained shear wave velocity from the inversion of H/V curve through DFA approach provides evidence of the complex nature of the subsurface geological structures. Identifying six characteristic 2D cross-sections of the entire area, we attain prevalence of a low-velocity intermediate layer with velocity ranging from 128–192 m/s. On the contrary, a relatively high-velocity layer is also obtained (279–471 m/s) which can be treated as sedimentary deposits (may be for some sites as basin basement). The attained results, when extended to a 3D shear wave profile, tally excellently with estimated frequency distribution and its corresponding links with depth wise strata, accompanied by a topographical profile of the surveyed locations. All findings are comprehensively analyzed and interpreted as a proof of concept of implementation of DFA approach towards retrieving subsurface information.

Keywords: Site effects, seismic noise, spectral ratio techniques, shear-wave velocity profile, earthquake data analysis, soil characterization.

1. Introduction

Inferring seismic response from ambient noise has become a widely established technique which renders quick acquisition as well as cost-effectiveness. Numerous literatures are available which directly deals with effectiveness of quantification of ambient vibrations for delineating local seismic response (e.g. Nakamura 1989; SESAME 2004; D'Amico et al. 2008; Albarello and Lunedei 2011; Gallipoli et al. 2011 ; Paolucci et al. 2015; Farrugia et al. 2016). Local site settings play a major role in enhancing seismic damage; although, there are contributions from proximity of source location as well as path effect. In general terms, local site effects may be well-defined as amendment of features of receiving seismic wavefield, arising due to specific characteristics of site geology. There are two ways to quantify site effects; firstly, by using seismic array arrangement (multi-station) (Maranó et al. 2017) and secondly, by simply using a single-station deployment (e.g., Mucciarelli 1998; Parolai et al. 2005; Picozzi et al. 2005; Foti et al. 2011). Related to the latter category, ambient noises are accrued through three component sensor. From accrued noise, a division of amplitude of spectral ratios (HVSR) of horizontal (H) component by vertical (V) component is executed. As per reports of Field and Jacob (1993), Bindi et al. (2000), and Lunedei and Albarello (2010), shape of HVSR possesses no direct link with amplification. It is mention worthy that HVSR shape is dependent on frequency. Even, one cannot infer direct information of subsoil formation, as affirmed by Lachet and Bard (1994), Bard (1999) and Haghshenas et al. (2008). All these facts relate a complex nature of HVSR curve (e.g., Fäh et al. 2001; Albarello and Lunedei 2011 ; Sanchez-Sesma et al. 2011, Lunedei and Malischewsky 2015). Nonetheless, it is argued that the HVSR curve produces a sharp peak owing to existence of a sharp impedance contrast (e.g., Malischewsky and Scherbaum, 2004; Bonnefoy-Claudet et al. 2006). Again, the HVSR peak amplitude is linked to impedance contrast, giving rise to resonance (e.g., Albarello and Lunedei 2011). As pointed out by Lunedei and Malischewsky (2015),

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[3] Institute of Geosciences, Friedrich-Schiller University Jena, Jena 07749, Germany.

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Disaster Prevention Research Institute
Kyoto University

Gokasho, Uji,
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Tel +81 774 38 4052
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Hiroshi KAWASE
E-mail: kawase@sere.dpri.kyoto-u.ac.jp

August 19, 2019

TO WHOM IT MAY CONCERN:

This letter is to confirm that Prof. Dr. R. Biswas, Department of Physics, Tezpur University, India and myself agreed to be co-investigators in the Joint Collaborative research project on the innovative use of observed microtremor and earthquake data for strong motion prediction.

The Joint Collaborative project is proposed to the Japan Society for the Promotion of Science (JSPS) on the Japanese side. We are committed to the fruitful outcome of our endeavor.



Prof. Dr. Hiroshi Kawase
Disaster Prevention Research Institute, Kyoto University,
Gokasho, Uji, 611-0011, Japan

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Tectonophysics

journal homepage: www.elsevier.com/locate/tecto

Regional variation of coda *Q* in Kopili fault zone of northeast India and its implications

Nilutpal Bora[a], Rajib Biswas[a,*], Anna A. Dobrynina[b]

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[b] Institute of the Earth's crust SB RAS, 128 Lermontov Street, 664033 Irkutsk, Russia

ARTICLE INFO

Keywords:
Coda-wave
Wave propagation
Frequency dependency
Seismic attenuation
Crustal properties

ABSTRACT

Kopili fault has been experiencing higher seismic and tectonic activity during the recent years. These kind of active tectonics can be inspected by examining coda-wave attenuation and its dependence with frequency. Exploiting single back-scattering model, we have endeavored to measure coda *Q* and its associated parameters such as frequency dependent factor (*n*) and attenuation coefficient (γ) covering seven lapse-time windows spanning from *30* to *90* s and central frequencies *1.5*, *3.5*, *6*, *9* and *12* Hz. The average estimated values of Q_C increases with frequency and lapse time window from *114* at frequency *1.5* Hz to *1563* at frequency *12* Hz for *30* s window length, and from *305* at frequency *1.5* Hz to *2135* at frequency *12* Hz for *90* s window length. The values of Q_o and n are also estimated for the entire Kopili fault zone. For this study region, the Q_o values vary from *62* to *348* and *n* varies from *0.57* to *1.51* within the frequency range *1.5* to *12* Hz. Furthermore, depth variation of attenuation of this region reveals that there is velocity anomaly at depth *210–220 km* as there arises sharp changes in γ and *n* which are supported by available data, reported by other researcher for this region. Finally, we have tried to separate the intrinsic and scattering attenuation for this area. It is observed that the entire region is dominated by mainly scattering attenuation, but we can see an increase in intrinsic attenuation with depths in two stations namely TZR and BKD. Furthermore, the obtained results are comparable with the available global data.

1. Introduction

The energy of seismic wave at various distances from earthquake source is severely affected by geological medium. Attenuation is one of the vital parameters that characterizes the medium through which seismic wave propagates. It is defined as the dissipation of seismic energy while traversing from the earthquake source to the receiver. This dissipation continues until the seismic wave disappears due to loss of energy (or loss of amplitude). The energy of seismic waves decays due to the geometrical spreading, intrinsic and scattering attenuation (Sedaghati and Pezeshk, 2016; Padhy et al., 2011). The decrease in amplitude of the seismic waves, caused by the geometrical spreading in a homogeneous and isotropic medium, is inversely proportional to the distance travelled. While intrinsic attenuation converts the seismic energy into heat due to inelastic absorption, scattering or elastic attenuation redistributes the energy at random heterogeneities present in the earth's crust. It should be pointed out that the total energy in the wavefield remains constant in case of scattering attenuation, whereas the intrinsic attenuation causes disappearance of the wave due to loss of energy. Therefore, attenuation of seismic waves in the earth's crust is a key phenomenon for understanding the diversity in the earth's physical states and seismic potential of the region (Singh and Hermann, 1983; Parvez et al., 2008). The attenuation property of the medium is usually measured by a dimensionless quantity, called the quality factor *Q*, which is defined as the ratio of wave energy to the energy lost per harmonic oscillations (Knopoff and Hudson, 1964; Aki and Chouet, 1975) while propagating through the intermediate medium. The attenuation of medium is inversely proportional to the quality factor (*Q*) which means that, seismic waves are highly attenuated for regions having lower *Q* values.

Attenuation parameter (*Q*) can be characterized by *P*-wave quality factor (Q_p), *S*-wave quality factor (Q_S), and coda-wave quality factor (Q_C). Several researchers have carried out extensive research by using various methods in order to investigate the mechanism of coda wave attenuation in the earth's lithosphere. Aki (1969) and later, Aki and Chouet (1975) first estimated Q_C from the tail portion of the seismogram. They (Aki, 1969; Aki and Chouet, 1975; Rautian and Khalturin, 1978) estimated Q_C based on the fact that coda wave was caused by

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Available online 09 November 2017
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Appendix 1.2 – Collaborative Research Student Agreement





Collaborative Research Student Agreement

This agreement should be read in conjunction with the Memorandum of Agreement (Partnership Agreement) between The Queen's University Belfast (QUEEN'S) and Tezpur University (TEZPUR), signed on 3 September 2019.

1. The Partners

(i) Name of School: *QUEEN'S UNIVERSITY BELFAST*

(ii) Name of School: *TEZPUR UNIVERSITY*

2. The Student

(iii) Name of student: *Jyotirup Sarma*

(iv) Student's nationality: *Indian*

(v) Outline of student's qualifications: BSc Physics (1st class, Cotton University, India), *MSc Physics (1st class, Tezpur University)*

3. The Research

Research Title: **Ion acceleration and related phenomena in laser-plasma interaction**

The interaction of high power laser with matter in the plasma state has opened up a new and promising field of research in physics. The study of this field not only enhances the understanding of how radiation couples energy into matter, but can also provide novel information of relevance to astrophysical events , e.g. in areas such as particle acceleration, creation of particle-antiparticle, magnetic field generation. Through laser-irradiation of suitable targets, it is possible to achieve high accelerating gradients potentially leading to reduced accelerator size and cost as compared to conventional accelerators. In the last few decades, laser technology has made tremendous progress.

A current aim of research on ion acceleration is to attain significantly high (>100 MeV) mono-energetic ions with good conversion efficiency by varying various plasma and laser parameters. In this project we intend to investigate ion acceleration in overdense and near critical plasma using both linearly and circularly polarized light and explore the effects of plasma parameters, target size, structure and shape, density etc. and optimise these parameters to obtain monoenergetic ion beam by using Particle In Cell (PIC) simulations. Targetetd three dimensional PIC simulations will be highly beneficial to explore the complex physics underlying these intense laser-plasma interaction. Multi-species gaseous/cryogenic targets may also be used to increase the conversion efficiency of laser to ions.

All related Intellectual Property Rights and/or Copyright arising from the Research shall be assigned as per standard protocol at QUEEN'S for postgraduate research.

4. The Supervisors

The following Supervisors shall be assigned to the Student:

Marco Borghesi, of SCHOOL of Mathematics and Physics– QUEEN'S (Principal Supervisor)
Nilakshi Das, of Department of Physics – TEZPUR (Co Supervisor)
Satyabrata Kar, of SCHOOL of Mathematics and Physics – QUEEN'S (Third Supervisor)

Both Supervisors commit themselves to duly exercise their duties as PhD Co-supervisors with the Student. Any required changes to these personnel shall be managed in accordance with established protocols at QUEEN'S. Supervisor's Rights and Responsibilities as outlined in Appendix 1.4 and QUEEN'S Study Regulations for Research Degree Programme: http://www.qub.ac.uk/directorates/AcademicStudentAffairs/AcademicAffairs/GeneralRegulations/StudyRegulations/StudyRegulationsforResearchDegreeProgrammes/.

5. Registration

The Student shall be registered at QUEEN'S for the duration of the research period, commencing *16 September 2019,* in accordance with normal requirements and, as such, shall be entitled to the privileges and subject to the duties of students of the University. The Student will be registered to the School within which the Principal Supervisor is based.

The Student must register at the start of their research programme and at the beginning of every subsequent academic year at QUEEN'S. Registration in the second and subsequent years is subject to satisfactory progress reports.

The Student shall comply with the University's training requirements, the training requirements of their funding body and with any compulsory or recommended training requirements put in place by the School in which they are registered at QUEEN'S.

In registering to the University, the Student will be subject to, and must abide by, all the rules and regulations of QUEEN'S and the immigration rules for the United Kingdom.

6. Residency Requirements

It is expected that the Student shall spend the following periods of time at each institution:

September 2019 - September 2022: Queen's University Belfast

Any subsequent change(s) to the above dates shall be mutually agreed and approved in accordance with normal protocols in place at each institution, as required. At a minimum, students are expected to be in attendance at QUEEN'S annually to take part in formal annual reviews of progress and to undertake the oral examination after the thesis has been submitted.

The Student will be responsible for housing, travel, medical insurance (if applicable), and any related subsistence costs incurred during these periods and in relation to the duration of the PhD programme more generally.

12. Approval and Signature

Signed for and on behalf of:

THE QUEEN'S UNIVERSITY OF BELFAST

Professor Mark Price
Faculty Pro-Vice-Chancellor
Date: 03/09/2019

Professor Stan Scott
University Co-ordinator
Date: 03/09/2019

Professor Marco Borghesi
Principal Supervisor
Date: 3/09/2019

TEZPUR UNIVERSITY

Name
Pro-Vice-Chancellor
Date: 9-9-2019

Name Dhruba Kr Bhattacharyya
University Co-ordinator
Date: 9/9/19

Name Prof. Nilakshi Das.
Co-Supervisor
Date: 09-09-2019

Read and acknowledged by the Student:

Jyotirup Sarma

Name JYOTIRUP SARMA
Date: 09-09-2019

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Physics Letters A ••• (••••) •••–•••

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Free energy and spatial periodicity of generalized cnoidal ion holes

Nilakshi Das [a,*], Prathana Borah [a], Hans Schamel [b]

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[b] Physikalisches Institut, Universität Bayreuth, D-95440 Bayreuth, Germany

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Nonlinear Vlasov equation
Moderate and stronger slope singularity
Pseudopotential
Trapping nonlinearity

ABSTRACT

The free energy density of an ion hole excited coherent state plasma is calculated. The underlying ion trapping nonlinearity is shown to support expressions which appear one order earlier in the expansion scheme compared with conventional, linearly based derivations. This energy argument hence provides a further hint for the necessity of a full nonlinear Vlasov equation in describing coherent wave phenomena found ubiquitously in space, lab and fusion plasmas. The most prominent holes, the privileged cnoidal ion holes, being characterized by regularly trapped ions, can achieve states of lower energy than the ones of the unperturbed plasma and are hence attractors in the nonlinear dynamical evolution being approached in long term runs. Such negative energy holes can also be found by an extended description of holes that admits singularities of the trapped ion distribution f_{it}, namely a moderate slope singularity at the separatrix of f_{it}. A stronger singularity in form of a jump of f_{it} at the boundary of the trapped ion region, on the other hand, results in a positive energy expression and is already for this reason less probable. As a biproduct, a novel expression is obtained for the wave length of a cnoidal ion hole in terms of the other relevant parameters.

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1. Introduction

The formation of phase space holes in subcritically current-driven plasmas [1–8] has raised the question about their identity. Even if one focuses in the theoretical description on the kinetically upgraded pseudopotential method, developed by Schamel [10] and disregards a BGK analysis [9] because of its shortcomings, there remains an ambiguity in concern with the chosen trapped particle distribution $f_{st}, s = e, i$. Holes which on the one hand rest on regularly trapped particles are smoothest in phase space and seem to be most extinguished and most probable [11]. To emphasize this property the terminology "privileged" [12] has been introduced in referring to them as "privileged cnoidal hole modes". Another favorizing argument for their relevance is the fact that this continuum spectrum of hole modes leaves the plasma in a lower energy state [13,14,3] giving room for an enriched variety of potential wave excitations including positive energy, fluid-like modes.

On the other hand the well-known fact that the Vlasov–Poisson system admits singular solutions opens the possibility for modes which possess a more singular character with respect to $f_{st}; s = e, i$. A first step in this direction has been made by [15] with the consequence that a slope singularity of $f_{st}; s = e, i$ prevents a well-defined phase velocity of the structure, a property being inherent in all BGK solutions [12].

The purpose of the present analysis is a three-fold extension. Concentrating on slow ion holes with negligible trapped electron effects we (i) re-derive the free energy expression of a plasma excited by such a hole, using this time a negative plasma potential analysis [16], and show equivalence with earlier expressions, (ii) extend the analysis by incorporating singularities in f_{it}, and (iii) present for the first time a fundamental expression for the wavelength of a periodic structure in terms of the other plasma characterizing parameters.

2. Basic equations

In our previous paper [12] we have derived the governing equations for the privileged spectrum of ion hole modes in current-driven plasmas and its extension by incomplete ion trapping. In this paper we focus on the energy and further details associated with these coherent equilibria. With reference to this paper, the distributions for electrons and ions, respectively, are given for negative potential $-\psi < \phi < 0$ with $\psi << 1$, which are the solution of Vlasov equation as in [16]. The distribution functions for electrons with negligible trapped electron singularity effects ($\gamma_e = 0$, $\kappa_e = 1$) and for ions are, respectively, given as

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ii) Sanction Letter: If the progress is found to be satisfactory the renewal sanction for the year will be issued in the beginning of that financial year in April/May.

iii) Claim: On receipt of the renewal sanction, the PI shall claim the funds sanctioned by submitting the following documents to **Shri G. N. Pandey, Programme Officer, BRNS Secretariat, First Floor, Central Complex, BARC, Trombay, Mumbai-400 085**:

a) Claim in **Form-II** (enclosed) quoting reference of the renewal sanction.
b) Utilisation Certificate (UC) as on **31st March** of the preceding financial year in **Form-III** (enclosed) duly audited by the Internal Auditor of the University/ Institution or a Chartered Accountant.
c) Statement of Accounts (SA) as on **31st March** of the preceding financial year in **Form-IV** (enclosed) duly audited by the Internal Auditor of the University/ Institution or a Chartered Accountant. **Interest earned in previous year should be reflected in the Statement of Accounts.**
d) Copy of appointment order and joining report of the staff appointed for the project along with minutes of the Selection Committee.
e) An inventory of equipment **in Form-V**.

These forms are enclosed with the sanction letter (first year) also.

D. At the end of Terminal Year the final Settlement Grant will be released on fulfillment of the following requirements:

a) Claim Form-II.
b) The final Consolidated Statement of Accounts (SA) and Consolidated Utilization Certificate (UC) **duly audited by a Chartered Accountant or a Statutory (Govt.) Auditor.**
c) Final Consolidated Progress Report **in Form-VII** (enclosed).

5. AAO (Bills II), DAE, Anushakti Bhavan, CSM Marg, Mumbai - 400 001 – With a request that the amount granted for the first year of the project may be released immediately.

6. Member Secretary (ATC): Dr. P.V.A. Padmanabhan, L&PTD, BARC, Mumbai-400 085.

7. Co-Investigator (CI): Dr. G. A. Ahmed, Associate Professor, Department of Physics, Tezpur University, Napaam, Assam – 784 028.

8. Principal Collaborator (PC): Dr. Kartik Patel, L&PTD, BARC, Trombay, Mumbai 400 085.

You or your nominee may please be the DAE representative for selection of Research Fellow/ Research Associate for the project.

Gandey

(G.N. Pandey)

**** Note:**

1. All the documents as applicable be sent in time to avoid delays and unnecessary correspondence.
2. Please quote the Sanction No.**2012/34/61/BRNS** in all your correspondence with BRNS.
3. **In case you have not utilized/could not utilize the 1st year grant during the financial year 2012-2013 and would like to start your project from 1st April 2013, you may send us a request for revalidation of the 1st year grant from 2012-2013 to 2013-2014.**

better, advisable

STATE CHILD PROTECTION SOCIETY, ASSAM
(A Government of Assam Organisation)
46, Survey Bus Stop, Beltola, Guwahati- 28
Ph:- 0361 (2229275/ 2265385), Fax:- 0361-2229275,
Email: scpstraining.iecassam1@gmail.com website: www.scpsassam.org

No.SCPS (G) 439/2018-19/28 Dtd Guwahati the 12 May, 2019

From: **Hemen Das, ACS**
Secretary, Social Welfare cum
Member Secretary
State Child Protection Society, Assam.

Office of the Pro Vice-Chancellor
Receipt No 111
Date 24/5/19

To: **Prof. Vinod Kumar Jain**
Vice-Chancellor
Tezpur University

Sub: **Conducting Study on Child Marriage**

Sir,

Kindly recall our informal discussion on 23rd March 2019 held during the occasion of Assam Science Festival at your University, regarding collaborating with your university on the realms of child rights issues.

State Child Protection Society (SCPS), Assam is a registered Society under the aegis of the Department of Social Welfare, Government of Assam. The primary responsibility of SCPS is to roll out the National Flagship Programme of Government of India, namely the - Child Protection Services (CPS) in the State. CPS aims at providing a holistic Child Protection Mechanism both at the Preventive and Rehabilitative aspect for Children in Need of Care and Protection and Children in Conflict with Law.

State Child Protection Society, Assam would like to request your support to conduct a study on Child Marriage through your esteemed University's Social Work Department. In this regard, I would like to request you to kindly accept our proposal.

Prof C. Sarma
HOD, Social Works
for needful pl.
24-5-2019

Yours faithfully

Member Secretary
State Child Protection Society,
Assam

Memo No. No. SCPS (G) 439/2018-19/28 Dated Guwahati the May, 2019

Copy to:
1. The Addl. Chief Secretary to the Govt. of Assam, Social Welfare Department, cum Chairperson, SCPS, Assam, Dispur – 06 for kind information.

Member Secretary
State Child Protection Society,
Assam

24/5/2019

Dr. Anupam Sarmah
Team Leader, BHL & WAL, WWF-India
Parvati Nagar, P.O. Tezpur 784 001
asarmah@wwfindia.net

06 March, 2021

To,
Ms. Namami Sharma, Ph.D.
Assistant Professor, Department of Social Work
Tezpur University, Assam

Subject: Visit to western Arunachal Pradesh in connection with a collaborative study

Dear Dr. Sharma,

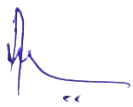
It is our pleasure to invite you for a field visit to western Arunachal Pradesh, in connection with the collaborative study *Forest, Farms and Markets: An exploratory study*.

We are proposing this visit between 10 to 15 March 2021 to Mandala Phudung, Thembang and Nyukmadung villages of West Kameng district where we are proposing to hold few group discussions in these villages to understand current agriculture practices, dependency over natural resources and future intervention strategies.

As you know, there will be no financial obligation on either side of us, but we would be happy to take care of your travel and accommodations during this entire trip.

We really appreciate your guidance and valuable input in this study so far and looking forward to your participation in this field exercise.

Sincerely,



(Anupam Sarmah)

President: Mr. Jamshyd N. Godrej
Vice President: Mr. Arvind Wable
Mr. Arun Maira
Mr. N. Kumar
Ms. Chanda Singh