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ON THE NUMBER OF SOLUTIONS OF A GENERALIZED COMMUTATOR EQUATION IN FINITE GROUPS

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Abstract. We consider some generalized commutator equations in a finite group and show that the number of solutions of such equations are characters of that group. We also obtain explicit formula for this character, considering the equation $[\dots[[[x_1,x_2],x_3],\ x_4],\dots,x_n]=g$, for some well-known classes of finite groups in terms of orders of the group, its center and its commutator subgroup.

1. Introduction

Let $F(x_1,x_2,\dots,x_n)$ denote the free group on n generators $x_1,x_2,\dots,x_n$. Let $w=w(x_1,\cdots,x_n)$ be a non-trivial reduced word in $F(x_1,x_2,\dots,x_n)$. For any finite group G the word $w(x_1,x_2,\dots,x_n)$ defines a map $w\colon G^n\to G$ given by $(g_1,g_2,\dots,g_n)\mapsto w(g_1,g_2,\dots,g_n)$ called a word map induced by the word $w(x_1,x_2,\dots,x_n)$. For $g\in G$ we have

$$|w^{-1}(g)|=\Big|\Big\{(a_1,a_2,\dots,a_n)\in G^n : w(a_1,a_2,\dots,a_n)=g\Big\}\Big|.$$

Consider the map $\zeta_G^w\colon G\to\mathbb{N}\cup\{0\}$ given by $g\mapsto|w^{-1}(g)|$. Note that $\zeta_G^w(g)$ is the number of solutions of the equation $w(x_1,x_2,\dots,x_n)=g$. For any word w, the map ζ_G^w is a class function on G but not necessarily a character. Equations in finite groups have been studied by many authors since the days of Frobenius (see for example [2,5,22,27,28,30,31] etc.). If

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$w(x_1, x_2) = [x_1, x_2] := x_1^{-1}x_2^{-1}x_1x_2$ then Frobenius [10] showed that

$$\zeta_G^w = \sum_{\chi \in \mathrm{Irr}(G)} \frac{|G|}{\chi(1)} \chi, \tag{1.1}$$

which is a character of G. This result has been used as one of the main ingredients to settle down the longstanding Ore's Conjecture (see [16]). It is also used in order to study generalized commutativity degrees of finite groups (see [6,8,24]). In [32], Tambour derived explicit formulas for ζ_G^w considering $w(x_1, x_2, \dots, x_n) = [x_1, x_2] \cdots [x_{n-1}, x_n]$ and $w(x_1, x_2, \dots, x_n) = x_1x_2 \cdots x_nx_1^{-1}x_2^{-1} \cdots x_n^{-1}$. In particular, he deduced that ζ_G^w is a character of G for these words. Extending the results of Tambour, Das and Nath [5] showed that ζ_G^w is a character of G if w is an admissible word. Further, Parzanchevski and Schul [22] have considered some classes of words and extended the results of Das and Nath [5]. In Section 2 of this paper, we consider the word $w(x_1, x_2, \dots, x_m) = [w_1(x_1, \dots, x_n), w_2(x_{n+1}, \dots, x_m)]$ and show that ζ_G^w is a character of G under some condition. In particular, we prove the following result.

THEOREM 1.1. *Let G be a finite group and*

$$w(x_1, \dots, x_m) := \big[w_1(x_1, \dots, x_n), w_2(x_{n+1}, \dots, x_m)\big].$$

If $\zeta_G^{w_1}$ is a character of G and $\zeta_G^{w_2}$ is a constant map then ζ_G^w is a character of G and

$$\zeta_G^w = \sum_{\chi \in \mathrm{Irr}(G)} \frac{|G|^{m-n}}{\chi(1)} \langle \zeta_G^{w_1}\chi, \chi \rangle \chi.$$

This extends a classical result of Frobenius mentioned above. Note that the word $w(x_1, x_2, \dots, x_m) = [w_1(x_1, \dots, x_n), w_2(x_{n+1}, \dots, x_m)]$ also includes words that are not considered in [22]. In Section 3, we consider the word $w_n(x_1, \dots, x_n) = [\dots[[[x_1, x_2], x_3], x_4], \dots, x_n] = [w_{n-1}(x_1, \dots, x_{n-1}), x_n]$ for any n and using Theorem 1.1 we obtain exact formulas for $\zeta_G^{w_n}$ in case of nilpotent Camina groups and its generalizations. In the next paragraph we recall the definition and the main properties of these groups.

Two groups G and H are said to be n-isoclinic, if there exist isomorphisms ϕ and ψ such that

(1) ϕ is an isomorphism from $G/Z_n(G)$ to $H/Z_n(H)$;
(2) ψ is an isomorphism from $\gamma_{n+1}(G)$ to $\gamma_{n+1}(H)$;
(3) ψ is compatible with ϕ, i.e.

$$\psi([\dots[[g_1, g_2], g_3], \dots, g_{n+1}]) = \big[\dots[[h_1, h_2], h_3], \dots, h_{n+1}\big],$$

for any $h_i \in \phi(g_i Z_n(G))$, $g_i \in G$, $1 \leq i \leq n+1$. In other words, we have the commutative diagram

$$\begin{array}{ccc} \frac{G}{Z_n(G)} \times \cdots \times \frac{G}{Z_n(G)} & \xrightarrow{\phi \times \cdots \times \phi} & \frac{H}{Z_n(H)} \times \cdots \times \frac{H}{Z_n(H)} \\ \Big\downarrow a_{(n+1,G)} & & \Big\downarrow a_{(n+1,H)} \\ \gamma_{n+1}(G) & \xrightarrow{\quad\quad\psi\quad\quad} & \gamma_{n+1}(H) \end{array}$$

where

$$a_{(n+1,G)}\big(\big(g_1 Z_n(G), \ldots, g_{n+1} Z_n(G)\big)\big) = \big[\ldots[g_1, g_2], \ldots, g_{n+1}\big]$$

and

$$a_{(n+1,H)}\big(\big(h_1 Z_n(H), \ldots, h_{n+1} Z_n(H)\big)\big) = \big[\ldots[h_1, h_2], \ldots, h_{n+1}\big].$$

Such a pair (ϕ, ψ) is called an n-isoclinism between G and H. If $n = 1$, then (ϕ, ψ) is called an isoclinism between G and H and the groups are called isoclinic. The notion of isoclinism of groups was introduced by P. Hall [9]. In Section 3.1, we prove the following generalization of [24, Lemma 3.5] and derive an explicit formula for $\zeta_G^{w_n}(g)$ if G is a finite nilpotent generalized Camina group.

THEOREM 1.2. *Let G and H be two finite groups and (ϕ, ψ) be an n-isoclinism from G to H. If $g \in \gamma_{n+1}(G)$, then*

$$\zeta_G^{w_{n+1}}(g) = (|G|/|H|)^{n+1} \zeta_H^{w_{n+1}}(\psi(g)).$$

A group G is called a *Camina group* if $gG' = \mathrm{Cl}(g)$ for all $g \in G \setminus G'$. Note that the fact $gG' = \mathrm{Cl}(g)$ for all $g \in G \setminus G'$ is equivalent with the fact $\chi(g) = 0$ for all $\chi \in \mathrm{nl}(G)$ and $g \notin G'$, where $\mathrm{nl}(G)$ is the set of all nonlinear irreducible characters of G. A pair (G, N) is said to be a *Camina pair* if N is a proper non-trivial normal subgroup of G such that $gN \subseteq \mathrm{Cl}(g)$ for all $g \in G \setminus N$. We remark that $Z(G) < N < G'$ if (G, N) is a Camina pair and G is a Camina group if and only if (G, G') is a Camina pair. A group G is said to be a *generalized Camina group* if $gG' = \mathrm{Cl}(g)$ for all $g \in G \setminus G'Z(G)$. A pair (G, N) is called a *generalized Camina pair* (abbreviated as GCP) if N is a normal subgroup of the group G and $gG' = \mathrm{Cl}(g)$ for all $g \notin N$ equivalently $\chi(g) = 0$ for all $\chi \in \mathrm{nl}(G)$ and $g \notin N$ (see [12]). Note that G is a generalized Camina group if and only if $(G, G'Z(G))$ is a generalized Camina pair. The groups with $(G, Z(G))$, a GCP were studied under the name of VZ-groups by Lewis in [13]. In Section 3.2, we derive an explicit formula for $\zeta_G^{w_n}(g)$ considering groups for which $(G, Z(G))$ is a Camina pair and $(G/Z(G), Z(G/Z(G)))$ is a generalized Camina pair. In the last section, we derive an explicit formula for $\zeta_G^{w_n}(g)$ for groups having unique nonlinear irreducible character.

Let $P_w^{(n)}$ be the probability distribution associated to the word map w induced by $w(x_1, x_2, \ldots, x_n)$ on G. Then $P_w^{(n)}(g) = \frac{\zeta_G^w(g)}{|G|^n}$. In recent years, many authors have studied the probability distribution associated to the word maps induced by different kinds of words (see [1,7,18,19,21] etc.). As an application of our results, one may obtain the probability distribution associated to the word map induced by the word $w_n(x_1, x_2, \ldots, x_n) = [[\ldots[[x_1, x_2], x_3]\ldots], x_n]$ for the different families of finite groups considered in Section 3. It may be mentioned here that $P_{w_n}^{(n)}(g)$ is also considered in [8] (see Theorem 7.4). The case when $g = 1$, the identity element of G, is studied by Moghaddam et al. [17] in the name of nth nilpotency degree of G. If $n = 2$ then we have $P_{w_n}^{(2)}(g) = P_g(G)$, a notion introduced and studied by Pournaki and Sobhani [24].

Throughout this paper G denotes a finite non-abelian group, $\mathrm{Irr}(G)$ and $\mathrm{lin}(G)$ denote the set of all irreducible characters and linear irreducible characters of G, respectively. We write $\mathrm{cd}(G)$ to denote the set of all irreducible character degrees of G. For any normal subgroup N of G, $\mathrm{Irr}(G|N)$ denotes $\mathrm{Irr}(G) \setminus \mathrm{Irr}(G/N)$. Also, for any character χ of G we write $\chi\downarrow_N$ to denote the restriction of χ on N. If χ is a character of any subgroup H then we write $\chi\uparrow_H^G$ to denote the induced character of G induced by χ. The inner product of two characters ϕ and ψ of G is given by $\langle \phi, \psi \rangle = \frac{1}{|G|}\sum_{g \in G} \phi(g)\overline{\psi(g)}$. For any integer $i \geq 1$ define inductively $Z_0(G) = \{1\}$, $Z_1(G) = Z(G)$ the center of G, $Z_i(G)/Z_{i-1}(G) =$ the center of $G/Z_{i-1}(G)$. Also $\gamma_1(G) = G$, $\gamma_2(G) = [G, G] = G'$ the commutator subgroup of G and $\gamma_{i+1}(G) = [\gamma_i(G), G]$.

2. Proof of Theorem 1.1

Suppose $\chi \in \mathrm{Irr}(G)$ is afforded by the irreducible representation ρ_χ of G. Consider the element

$$z = \sum_{\substack{x_1,\ldots,x_n,\\ x_{n+1},\ldots,x_m \in G}} \left[w_1(x_1, \ldots, x_n), w_2(x_{n+1}, \ldots, x_m)\right].$$

Then, we have

$$\rho_\chi(z) = \sum_{x_1,\ldots,x_n \in G} \rho_\chi\big(w_1(x_1, \cdots, x_n)^{-1}\big)\, A_\chi(x_1, \ldots, x_n) \tag{2.1}$$

where

$$A_\chi(x_1, \cdots, x_n) \tag{2.2}$$

$$= \sum_{x_{n+1},\dots,x_m \in G} \rho_\chi\big(w_2(x_{n+1},\dots,x_m)^{-1} w_1(x_1,\dots,x_n) w_2(x_{n+1},\dots,x_m)\big).$$

Since $\zeta_G^{w_2}$ is a constant function, $A_\chi(x_1,\cdots,x_n)$ commutes with $\rho_\chi(g)$ for all $g \in G$. Hence, by Schur's lemma (see [11, Lemma 2.25]), we have

$$A_\chi(x_1,\dots,x_n) = \alpha I \tag{2.3}$$

for some $\alpha \in \mathbb{C}$. Now taking trace on the both sides of (2.3) and using (2.2) we get

$$\alpha = \frac{|G|^{m-n}}{\chi(1)} \chi(w_1(x_1,\cdots,x_n)). \tag{2.4}$$

Again taking trace on both side of (2.1) and using (2.3), (2.4) we get

$$\chi(z) = \frac{|G|^{m-n}}{\chi(1)} \sum_{h\in G} \zeta_G^{w_1}(h)\chi(h)\chi(h^{-1}) = \frac{|G|^{m-n+1}}{\chi(1)} \langle \zeta_G^{w_1}\overline{\chi}, \overline{\chi}\rangle. \tag{2.5}$$

By definition of ζ_G^w, we have

$$\chi(z) = |G|\langle \zeta_G^w, \overline{\chi}\rangle. \tag{2.6}$$

Therefore, by (2.5) and (2.6), we have

$$\zeta_G^w = \sum_{\chi\in \mathrm{Irr}(G)} \frac{|G|^{m-n}}{\chi(1)} \langle \zeta_G^{w_1}\chi, \chi\rangle\ \chi. \tag{2.7}$$

Since $\zeta_G^{w_1}$ is a character of G, the coefficient $\frac{|G|^{m-n}}{\chi(1)}\langle \zeta_G^{w_1}\chi, \chi\rangle$ is a non-negative integer. Hence ζ_G^w is a character of G. □

3. Explicit formula for $\zeta_G^{w_n}$

In this section, we consider the word

$$w_n(x_1,\dots,x_n) := [\dots[[[x_1,x_2],x_3],x_4],\dots,x_n]$$

for $n \geq 2$. Then $\zeta_G^{w_n}(g)$ is the number of solutions of the equation

$$[\dots[[[x_1,x_2],x_3],x_4],\dots,x_n] = g.$$

Note that the word w_n can be defined recursively as

$$w_2(x_1,x_2) = [x_1,x_2] = x_1^{-1}x_2^{-1}x_1x_2,$$

$$w_3(x_1, x_2, x_3) = [w_2(x_1, x_2), x_3],$$
$$\dots$$
$$w_n(x_1, \dots, x_n) = [w_{n-1}(x_1, \dots, x_{n-1}), x_n].$$

Thus w_n is a particular type of the words considered in Section 2. By Theorem 1.1, it follows that $\zeta_G^{w_n}$ is a character of G and

$$\zeta_G^{w_n} = |G| \sum_{\chi \in \mathrm{Irr}(G)} \frac{C^{w_n}(\chi)\chi}{\chi(1)} \tag{3.1}$$

where $C^{w_n}(\chi) = \langle \zeta_G^{w_{n-1}} \chi, \chi \rangle$. Note that $C^{w_{m+1}}(\chi) = \chi(1)^2 |G|^{m-1}$ whenever m is greater than the nilpotency class of G. For $n = 2$, $C^{w_2}(\chi) = 1$ and so equation (3.1) reduces to (1.1); which is the classical result of Frobenius. For $n \geq 3$ no explicit formula is known for $C^{w_n}(\chi)$. However, the following proposition shows that $C^{w_n}(\chi) = |G|^{n-2}$ if $\chi \in \mathrm{lin}(G)$.

PROPOSITION 3.1. *Let G be a finite group and $w_n(x_1, \dots, x_n) := [\dots[[x_1, x_2], x_3], \dots, x_n]$ with $n \geq 3$. Then we have the following.*

(1) *If $\chi \in \mathrm{lin}(G)$, then $C^{w_n}(\chi) = \langle \zeta_G^{w_{n-1}}, 1_G \rangle$.*

(2) $\langle \zeta_G^{w_{n-1}}, 1_G \rangle = |G| \langle \zeta_G^{w_{n-2}}, 1_G \rangle$. *In particular,* $\langle \zeta_G^{w_{n-1}}, 1_G \rangle = |G|^{n-2}$.

PROOF. (1) Let $\chi \in \mathrm{lin}(G)$. Then

$$C^{w_n}(\chi) = \langle \zeta_G^{w_{n-1}} \chi, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \zeta_G^{w_{n-1}}(g) |\chi(g)|^2 = \langle \zeta_G^{w_{n-1}}, 1_G \rangle.$$

(2) Using (3.1), we have

$$\langle \zeta_G^{w_{n-1}}, 1_G \rangle = \frac{1}{|G|} \sum_{g \in G} \zeta_G^{w_{n-1}}(g) = \frac{1}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \left(\sum_{g \in G} \frac{|G|}{\chi(1)} \langle \zeta_G^{w_{n-2}} \chi, \chi \rangle \chi(g) \right)$$
$$= \frac{1}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \left(\frac{|G|}{\chi(1)} \langle \zeta_G^{w_{n-2}} \chi, \chi \rangle |G| \langle \chi, 1_G \rangle \right) = |G| \langle \zeta_G^{w_{n-2}}, 1_G \rangle.$$

Therefore, $\langle \zeta_G^{w_{n-1}}, 1_G \rangle = |G|^{n-2}$. $\square$

In view of the above proposition, (3.1) becomes

$$\zeta_G^{w_n} = |G|^{n-1} \sum_{\chi \in \mathrm{lin}(G)} \chi + |G| \sum_{\chi \in \mathrm{nl}(G)} \frac{C^{w_n}(\chi)\, \chi}{\chi(1)}. \tag{3.2}$$

In the following subsections we compute $C^{w_n}(\chi)$ where $\chi \in \mathrm{nl}(G)$ for $n \geq 3$ and hence derive an explicit formula for $\zeta_G^{w_n}(g)$ for some classes of finite groups.

3.1. n-isoclinism and generalized Camina group. We begin with the proof of Theorem 1.2. Let the pair (ϕ, ψ) be an n-isoclinism between G and H. Consider the group $\overline{G} = G/Z_n(G)$ and $\overline{H} = H/Z_n(H)$. Then

$$\frac{1}{|Z_n(G)|^{n+1}}\zeta_G^{w_{n+1}}(g)$$

$$= \frac{1}{|Z_n(G)|^{n+1}}\big|\big\{(g_1, g_2, \ldots, g_{n+1}) \in G^{n+1} \mid [\ldots[[g_1, g_2], g_3], \ldots, g_{n+1}] = g\big\}\big|$$

$$= \frac{1}{|Z_n(G)|^{n+1}}\big|\big\{(g_1, g_2, \ldots, g_{n+1}) \in G^{n+1} \mid$$

$$a_{(n+1,G)}\big([\ldots[[\bar{g}_1, \bar{g}_2], \bar{g}_3], \ldots, \bar{g}_{n+1}]\big) = g\big\}\big|$$

$$= \big|\big\{(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_{n+1}) \in \overline{G}^{n+1} \mid a_{(n+1,G)}\big([\ldots[[\bar{g}_1, \bar{g}_2], \bar{g}_3], \ldots, \bar{g}_{n+1}]\big) = g\big\}\big|$$

$$= \big|\big\{(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_{n+1}) \in \overline{G}^{n+1} \mid$$

$$\psi\big(a_{(n+1,G)}\big([\ldots[[\bar{g}_1, \bar{g}_2], \bar{g}_3], \ldots, \bar{g}_{n+1}]\big)\big) = \psi(g)\big\}\big|$$

(since ψ is an isomorphism)

$$= \big|\big\{(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_{n+1}) \in \overline{G}^{n+1} \mid$$

$$a_{(n+1,H)}\big([\ldots[[\phi(\bar{g}_1), \phi(\bar{g}_2)], \phi(\bar{g}_3)], \ldots, \phi(\bar{g}_{n+1})]\big) = \psi(g)\big\}\big|$$

(using property (3) of n-isoclinism)

$$= \big|\big\{(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_{n+1}) \in \overline{H}^{n+1} \mid$$

$$a_{(n+1,H)}\big([\ldots[[\bar{h}_1, \bar{h}_2], \bar{h}_3], \ldots, \bar{h}_{n+1}]\big) = \psi(g)\big\}\big|$$

(since ϕ is an isomorphism)

$$= \frac{1}{|Z_n(H)|^{n+1}}\big|\big\{(h_1, h_2, \ldots, h_{n+1}) \in H^{n+1} \mid$$

$$a_{(n+1,H)}\big([\ldots[[\bar{h}_1, \bar{h}_2], \bar{h}_3], \ldots, \bar{h}_{n+1}]\big) = \psi(g)\big\}\big|$$

$$= \frac{1}{|Z_n(H)|^{n+1}}\big|\big\{(h_1, h_2, \ldots, h_{n+1}) \in H^{n+1} \mid$$

$$[\ldots[[h_1, h_2], h_3], \ldots, h_{n+1}] = \psi(g)\big\}\big| = \frac{1}{|Z_n(H)|^{n+1}}\,\zeta_H^{w_{n+1}}(\psi(g)).$$

Hence the result follows. $\square$

Now we derive an explicit formula for $\zeta_G^{w_n}(g)$ if G is a finite nilpotent generalized Camina group. Dark and Scoppola [4] completed the classification of Camina groups. Indeed, they proved the following theorem.

THEOREM 3.2 [4]. *Let G be a finite group. Then G is Camina group if and only if one of the following holds*:

(1) *G is a Camina p-group of nilpotence class 2 or 3.*

(2) *G is a Frobenius group with a cyclic Frobenius complement.*

(3) *G is a Frobenius group whose Frobenius complement is isomorphic to the quaternions.*

In [14], Lewis showed that a finite group G is a generalized Camina group if and only if G is isoclinic to a Camina group. Hence by the above theorem, if G is a finite nilpotent generalized Camina group, then G is isoclinic to a Camina p-group for some prime p. So, in view of Theorem 1.2, it is enough to compute $\zeta_G^{w_n}(g)$ for a Camina p-group. If G is a finite Camina p-group of nilpotency class 2, then $(G, Z(G))$ is a GCP. The following result gives some properties of groups such that $(G, Z(G))$ is a GCP.

THEOREM 3.3 [26, Section 3]. *Let G be a finite group such that $(G, Z(G))$ is a GCP. Then we have the following.*

(1) $\mathrm{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$.

(2) *The number of nonlinear irreducible characters of G is $|Z(G)| - |Z(G)/G'|$.*

(3) *There is a bijection $\widehat{\Phi}\colon \mathrm{Irr}(Z(G)|G') \to \mathrm{nl}(G)$ defined by*

$$\widehat{\Phi}(\lambda)(g) := \begin{cases} |G : Z(G)|^{1/2}\lambda(g) & \text{if } g \in Z(G) \\ 0 & \text{otherwise.} \end{cases}$$

If G is a finite Camina p-group of nilpotency class 2 then $\zeta_G^{w_n}(g)$ can be obtained by the following theorem.

THEOREM 3.4. *Let G be a finite group such that $(G, Z(G))$ is a GCP. If $g \in G'$ then*

$$\zeta_G^{w_2}(g) = \begin{cases} \frac{|G|^2}{|G'|}\left(1 - \frac{1}{|G:Z(G)|}\right) & \text{if } g \neq 1 \\ \frac{|G|^2}{|G'|}\left(1 + \frac{|G'|-1}{|G:Z(G)|}\right) & \text{if } g = 1 \end{cases}$$

and

$$\zeta_G^{w_n}(g) = \begin{cases} 0 & \text{if } g \neq 1 \\ |G|^n & \text{if } g = 1 \end{cases} \quad \text{if } n \geq 3.$$

PROOF. If $(G, Z(G))$ is a GCP then by [12, Lemma 2.4] we have that G' is a subgroup of $Z(G)$. That is, the nilpotency class of G is 2. Hence the case when $n \geq 3$ follows.

The case when $n = 2$ follows from [24, Corollary 2.3] and the fact that $\mathrm{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$ (which follows from Theorem 3.3(1)). $\square$

Now we compute $\zeta_G^{w_n}(g)$ if G is a finite Camina p-group of nilpotency class 3. We need the following result which follows from [25, Section 3].

THEOREM 3.5. *Let G be a finite Camina p-group of nilpotency class* 3. *Then*

(1) *$Z(G) \leq G'$; G/G', $G'/Z(G)$ and $\gamma_3(G) = [G, G'] = Z(G)$ are elementary abelian p-groups with $|G : G'| = p^{2m}$, $|G' : Z(G)| = p^m$ and $|G : Z(G)| = p^{3m}$, where m is even.*

(2) *$\mathrm{nl}(G) = \mathrm{Irr}(G|Z(G)) \sqcup \mathrm{nl}(G/Z(G))$, $\mathrm{cd}(G) = \{1, p^m, p^{3m/2}\}$,*

$$|\,\mathrm{Irr}(G|Z(G))| = |Z(G)| - 1$$

and $|\mathrm{nl}(G/Z(G))| = \frac{|G'|}{|Z(G)|} - 1$.

(3) *If $\chi \in \mathrm{Irr}(G|Z(G))$ then $\chi(G \setminus Z(G)) = 0$ and $\chi\downarrow_{Z(G)} = p^{3m/2}\lambda$, where $\lambda \in \mathrm{Irr}(Z(G))$, not the principal character.*

(4) *If $\chi \in \mathrm{nl}(G/Z(G))$ then $\chi(G \setminus G') = 0$ and $\chi(g) = p^m(\lambda \circ \eta)(g)$, where $g \in G'$, $\lambda \in \mathrm{Irr}(G'/Z(G))$, not the principal character, and $\eta \colon G \to G/\gamma_3(G)$ is the natural homomorphism.*

Using Theorem 3.5 part (3) and (4), we get the following result.

PROPOSITION 3.6. *Let G be a finite Camina p-group of nilpotency class* 3. *Then*

$$C^{w_3}(\chi) = \begin{cases} \langle \zeta_G^{w_2} \downarrow_{Z(G)}, 1_{Z(G)} \rangle & \text{if } \chi \in \mathrm{Irr}(G|Z(G)), \\ \langle \zeta_G^{w_2} \downarrow_{G'}, 1_{G'} \rangle & \text{if } \chi \in \mathrm{nl}(G/Z(G)). \end{cases}$$

The next theorem gives expression for $\zeta_G^{w_3}$ if G is a finite Camina p-group of nilpotency class 3.

THEOREM 3.7. *Let G be a finite Camina p-group of nilpotency class* 3. *Then*

$$\zeta_G^{w_3} = |G|^2 \sum_{\chi \in \mathrm{lin}(G)} \chi + \left(\frac{|G|^3}{|G'|} + \frac{|G|^2(|G'| - |Z(G)|)}{|Z(G)|} \right) \sum_{\chi \in \mathrm{Irr}(G|Z(G))} \frac{\chi}{\chi(1)}$$
$$+ \frac{|G|^3}{|G'|} \sum_{\chi \in \mathrm{nl}(G/Z(G))} \frac{\chi}{\chi(1)}.$$

PROOF. By Theorem 3.5(2) we have $\mathrm{nl}(G) = \mathrm{Irr}(G|Z(G)) \sqcup \mathrm{nl}(G/Z(G))$. Therefore, using (3.2), we have

$$\zeta_G^{w_3} = |G|^2 \sum_{\chi\in \mathrm{lin}(G)} \chi + |G| \sum_{\chi\in \mathrm{Irr}(G|Z(G))} \frac{C^{w_3}(\chi)\,\chi}{\chi(1)} + |G| \sum_{\chi\in \mathrm{nl}(G/Z(G))} \frac{C^{w_3}(\chi)\,\chi}{\chi(1)}.$$

By Proposition 3.6 the above expression becomes

$$\zeta_G^{w_3} = |G|^2 \sum_{\chi\in \mathrm{lin}(G)} \chi + \langle \zeta_G^{w_2}\downarrow_{Z(G)}, 1_{Z(G)}\rangle |G| \sum_{\chi\in \mathrm{Irr}(G|Z(G))} \frac{\chi}{\chi(1)} + \langle \zeta_G^{w_2}\downarrow_{G'}, 1_{G'}\rangle |G| \sum_{\chi\in \mathrm{nl}(G/Z(G))} \frac{\chi}{\chi(1)}. \tag{3.3}$$

Now, using Theorem 3.5 part (3) and (4), and the the expression of $\zeta_G^{w_2}$, we get

$$\langle \zeta_G^{w_2}\downarrow_{Z(G)}, 1_{Z(G)}\rangle = |G|\langle \varphi\downarrow_{Z(G)}, 1_{Z(G)}\rangle + |G| \sum_{\lambda\in \mathrm{Irr}(Z(G))\setminus\{1_{Z(G)}\}} \langle \lambda, 1_{Z(G)}\rangle + |G||\mathrm{nl}(G/Z(G))|.$$

where $\varphi = \sum_{\chi\in\mathrm{lin}(G)}\chi$. Since $Z(G) \subseteq G'$ and φ is the regular character of G/G', $\langle \varphi\downarrow_{Z(G)}, 1_{Z(G)}\rangle = |G|/|G'|$. Also, $\langle \lambda, 1_{Z(G)}\rangle = 0$ for any $1_{Z(G)} \neq \lambda \in \mathrm{Irr}(Z(G))$. Hence,

$$\langle \zeta_G^{w_2}\downarrow_{Z(G)}, 1_{Z(G)}\rangle = \frac{|G|^2}{|G'|} + |G||\mathrm{nl}(G/Z(G))| = \frac{|G|^2}{|G'|} + \frac{|G|(|G'| - |Z(G)|)}{|Z(G)|}.$$

We have

$$\zeta_G^{w_2}\downarrow_{G'} = |G| \sum_{\chi\in \mathrm{lin}(G)} \chi\downarrow_{G'} + |G| \sum_{\chi\in \mathrm{Irr}(G|Z(G))} \frac{\chi\downarrow_{G'}}{\chi(1)} + |G| \sum_{\chi\in \mathrm{nl}(G/Z(G))} \frac{\chi\downarrow_{G'}}{\chi(1)}.$$

It is easy to see that $\langle \chi\downarrow_{G'}, 1_{G'}\rangle = \langle \phi\downarrow_{G'}, 1_{G'}\rangle = 0$ for $\chi \in \mathrm{Irr}(G|Z(G))$ and $\phi \in \mathrm{nl}(G/Z(G))$. Therefore, $\langle \zeta_G^{w_2}\downarrow_{G'}, 1_{G'}\rangle = |G|^2/|G'|$. Now putting the values of $\langle \zeta_G^{w_2}\downarrow_{Z(G)}, 1_{Z(G)}\rangle$ and $\langle \zeta_G^{w_2}\downarrow_{G'}, 1_{G'}\rangle$ in (3.3) we get the required expression for $\zeta_G^{w_3}$. □

The following lemma is useful in proving the final result of this subsection.

LEMMA 3.8. *Let G be a finite Camina p-group of nilpotency class 3. Then*

$$\sum_{\chi\in\mathrm{Irr}(G|Z(G))}\frac{\chi(g)}{\chi(1)}=\begin{cases}-1 & \text{if } 1\neq g\in Z(G)\\ 0 & \text{if } g\in G'\setminus Z(G)\end{cases}$$

and

$$\sum_{\chi\in\mathrm{nl}(G/Z(G))}\frac{\chi(g)}{\chi(1)}=\begin{cases}\frac{|G'|-|Z(G)|}{|Z(G)|} & \text{if } 1\neq g\in Z(G)\\ -1 & \text{if } g\in G'\setminus Z(G).\end{cases}$$

PROOF. Follows from Theorem 3.5 part (3) and (4). $\square$

We conclude this subsection with the following result.

THEOREM 3.9. *Let G be a finite Camina p-group of nilpotency class 3. If $g\in G'$ then*

$$\zeta_G^{w_2}(g)=\begin{cases}\frac{|G|(|G|-|G'|)}{|G'|}+\frac{|G|(|G'|-|Z(G)|)}{|Z(G)|} & \text{if } 1\neq g\in Z(G)\\ \frac{|G|(|G|-|G'|)}{|G'|} & \text{if } g\in G'\setminus Z(G)\\ \frac{|G|^2}{|G'|}+\frac{|G||G'|}{|Z(G)|}+|G|(|Z(G)|-2) & \text{if } g=1\end{cases}$$

and

$$\zeta_G^{w_3}(g)=\begin{cases}\frac{|G|^2(|G'|-|Z(G)|)(|G|-|G'|)}{|G'||Z(G)|} & \text{if } 1\neq g\in Z(G)\\ 0 & \text{if } g\in G'\setminus Z(G)\\ \frac{|G|^3(|Z(G)|^2+|G'|-|Z(G)|)}{|Z(G)||G'|}+\frac{|G|^2(|G'|-|Z(G)|)(|Z(G)|-1)}{|Z(G)|} & \text{if } g=1.\end{cases}$$

PROOF. If $g=1$ then (1.1) gives $\zeta_G^{w_2}(1)=|G||\operatorname{Irr}(G)|$. Hence, the result follows from Theorem 3.5(2). The case when $g\neq 1$ follows from [20, Proposition 5.4].

Again, if $g=1$ then by Theorem 3.7 we have

$$\zeta_G^{w_3}(1)=|G|^2\frac{|G|}{|G'|}$$
$$+\left(\frac{|G|^3}{|G'|}+\frac{|G|^2(|G'|-|Z(G)|)}{|Z(G)|}\right)|\operatorname{Irr}(G|Z(G))|+\frac{|G|^3}{|G'|}|\mathrm{nl}(G/Z(G))|$$
$$=\frac{|G|^3(|Z(G)|^2+|G'|-|Z(G)|)}{|Z(G)||G'|}+\frac{|G|^2(|G'|-|Z(G)|)(|Z(G)|-1)}{|Z(G)|},$$

using Theorem 3.5(2). If $1 \neq g \in G'$ then $\sum_{\chi \in \mathrm{lin}(G)} \chi(g) = |G|/|G'|$. Therefore, by Theorem 3.7, we have

$$\zeta_G^{w_3}(g) = \frac{|G|^3}{|G'|} + \left(\frac{|G|^3}{|G'|} + \frac{|G|^2(|G'| - |Z(G)|)}{|Z(G)|}\right) \sum_{\chi \in \mathrm{Irr}(G|Z(G))} \frac{\chi(g)}{\chi(1)} + \frac{|G|^3}{|G'|} \sum_{\chi \in \mathrm{nl}(G/Z(G))} \frac{\chi(g)}{\chi(1)}. \tag{3.4}$$

If $1 \neq g \in Z(G)$ then by (3.4) and Lemma 3.8, we have

$$\begin{aligned}\zeta_G^{w_3}(g) &= \frac{|G|^3}{|G'|} + \left(\frac{|G|^3}{|G'|} + \frac{|G|^2(|G'| - |Z(G)|)}{|Z(G)|}\right)(-1) + \frac{|G|^3(|G'| - |Z(G)|)}{|G'||Z(G)|} \\ &= \frac{|G|^2(|G'| - |Z(G)|)(|G| - |G'|)}{|G'||Z(G)|}.\end{aligned}$$

If $g \in G' \setminus Z(G)$ then by (3.4) and Lemma 3.8, we have $\zeta_G^{w_3}(g) = 0$. $\square$

3.2. Groups for which $(G, Z(G))$ is a Camina pair and $(G/Z(G), Z(G/Z(G)))$ is a GCP. In [15], Lewis began the study of those groups G for which $(G, Z(G))$ is a Camina pair and, proved that such a group G must be a p-group for some prime p. Note that if G is a finite group such that $(G, Z(G))$ is a Camina pair and $(\frac{G}{Z(G)}, Z(\frac{G}{Z(G)}))$ is a GCP then the nilpotency class of G is 3. In the following theorem, we quote some facts about the groups for which $(G, Z(G))$ is a Camina pair.

THEOREM 3.10 [25, Section 4]. *Let G be a finite group such that $(G, Z(G))$ is a Camina pair. Then we have the following.*

(1) *If $\chi \in \mathrm{Irr}(G|Z(G))$, then $\chi(G \setminus Z(G)) = 0$ and $\chi(1) = |G/Z(G)|^{1/2}$.*

(2) *There is a bijection $\Phi : \mathrm{Irr}(Z(G)) \setminus \{1_{Z(G)}\} \to \mathrm{Irr}(G|Z(G))$ such that*

$$\Phi(\lambda)(g) := \begin{cases} |G/Z(G)|^{1/2}\lambda(g) & \text{if } g \in Z(G) \\ 0 & \text{otherwise.} \end{cases}$$

(3) $|\mathrm{Irr}(G|Z(G))| = |Z(G)| - 1$.

If $(G, Z(G))$ is a Camina pair and $(G/Z(G), Z(G/Z(G)))$ is a GCP, then $Z(G) \subseteq G'$ and $(G/Z(G))' = G'/Z(G) \subseteq Z(G/Z(G))$. Therefore, as a corollary of Theorem 3.3, we have the following result.

COROLLARY 3.11. *Let G be a finite group such that $(G, Z(G))$ is a Camina pair and $(\frac{G}{Z(G)}, Z(\frac{G}{Z(G)}))$ is a GCP. Then we have the following.*

(1) *There is a bijection* $\Psi : \mathrm{Irr}(Z(G/Z(G)) \mid G'/Z(G)) \to \mathrm{nl}(G/Z(G))$ *such that*

$$\Psi(\lambda)(g) := \begin{cases} |G : Z_2(G)|^{1/2}\lambda(g) & \text{if } g \in Z_2(G)/Z(G) \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the degree of every nonlinear irreducible characters of $G/Z(G)$ *is equal to* $|G : Z_2(G)|^{1/2}$.

(2) $|\mathrm{nl}(G/Z(G))| = \frac{|Z_2(G)|(|G'|-|Z(G)|)}{|G'||Z(G)|}$ *and*

$$\mathrm{nl}(G) = \mathrm{Irr}(G|Z(G)) \sqcup \mathrm{nl}(G/Z(G)).$$

Proposition 3.12. *Let* G *be a finite group such that* $(G, Z(G))$ *is a Camina pair and* $\big(\frac{G}{Z(G)}, Z(\frac{G}{Z(G)})\big)$ *a GCP. Then*

$$C^{w_3}(\chi) = \begin{cases} \langle \zeta_G^{w_2} \downarrow_{Z(G)}, 1_{Z(G)}\rangle & \text{if } \chi \in \mathrm{Irr}(G|Z(G)) \\ \langle \zeta_G^{w_2} \downarrow_{Z_2(G)}, 1_{Z_2(G)}\rangle & \text{if } \chi \in \mathrm{nl}(G/Z(G)). \end{cases}$$

Proof. By Theorem 3.10(1) one can easily compute that $\langle \zeta_G^{w_2}\chi, \chi\rangle = \langle \zeta_G^{w_2} \downarrow_{Z(G)}, 1_{Z(G)}\rangle$, where $\chi \in \mathrm{Irr}(G|Z(G))$. Again, by Corollary 3.11, we have $\langle \zeta_G^{w_2}\chi, \chi\rangle = \langle \zeta_G^{w_2} \downarrow_{Z_2(G)}, 1_{Z_2(G)}\rangle$, where $\chi \in \mathrm{nl}(G/Z(G))$. □

The following theorem gives an expression for $\zeta_G^{w_3}$ if G is a finite group such that $(G, Z(G))$ is a Camina pair and $(G/Z(G), Z(G/Z(G)))$ a GCP.

Theorem 3.13. *Let* G *be a finite group such that* $(G, Z(G))$ *is a Camina pair and* $\big(\frac{G}{Z(G)}, Z(\frac{G}{Z(G)})\big)$ *a GCP. Then*

$$\zeta_G^{w_3} = |G|^2 \sum_{\chi\in \mathrm{lin}(G)} \chi + \left(\frac{|G|^3}{|G'|} + \frac{|G|^2|Z_2(G)|(|G'|-|Z(G)|)}{|G'||Z(G)|}\right) \sum_{\chi\in \mathrm{Irr}(G|Z(G))} \frac{\chi}{\chi(1)} + \frac{|G|^3}{|Z_2(G)|} \sum_{\chi\in \mathrm{nl}(G/Z(G))} \frac{\chi}{\chi(1)}.$$

Proof. By Corollary 3.11(2) we have $\mathrm{nl}(G) = \mathrm{Irr}(G|Z(G)) \sqcup \mathrm{nl}(G/Z(G))$. Therefore, using (3.2), we get

$$\zeta_G^{w_3} = |G|^2 \sum_{\chi\in \mathrm{lin}(G)} \chi + |G| \sum_{\chi\in \mathrm{Irr}(G|Z(G))} \frac{C^{w_3}(\chi)\chi}{\chi(1)} + |G| \sum_{\chi\in \mathrm{nl}(G/Z(G))} \frac{C^{w_3}(\chi)\chi}{\chi(1)}.$$

By Proposition 3.12 the above expression becomes

$$\zeta_G^{w_3} = |G|^2 \sum_{\chi\in \mathrm{lin}(G)} \chi + \langle \zeta_G^{w_2} \downarrow_{Z(G)}, 1_{Z(G)}\rangle |G| \sum_{\chi\in \mathrm{Irr}(G|Z(G))} \frac{\chi}{\chi(1)} + \langle \zeta_G^{w_2} \downarrow_{Z_2(G)}, 1_{Z_2(G)}\rangle |G| \sum_{\chi\in \mathrm{nl}(G/Z(G))} \frac{\chi}{\chi(1)}. \tag{3.5}$$

Let $\phi = \sum_{\chi\in \mathrm{lin}(G)} \chi$. Then $\langle \phi \downarrow_{Z_2(G)}, 1_{Z_2(G)}\rangle = |G|/|Z_2(G)|$, since $G' \subseteq Z_2(G)$. Therefore

$$\left\langle \zeta_G^{w_2} \downarrow_{Z_2(G)}, 1_{Z_2(G)}\right\rangle = |G|\langle \phi \downarrow_{Z_2(G)}, 1_{Z_2(G)}\rangle = \frac{|G|^2}{|Z_2(G)|},$$

observing that $\langle \chi \downarrow_{Z_2(G)}, 1_{Z_2(G)}\rangle = 0$ for $\chi \in \mathrm{Irr}(G|Z(G)) \sqcup \mathrm{nl}(G/Z(G))$.

Also, $\langle \phi \downarrow_{Z(G)}, 1_{Z(G)}\rangle = |G|/|G'|$, since $Z(G) \subseteq G'$ and ϕ is the regular character of G/G'. Observe that $\langle \chi \downarrow_{Z(G)}, 1_{Z(G)}\rangle = 0$ for $\chi \in \mathrm{Irr}(G|Z(G))$ and $\langle \chi \downarrow_{Z(G)}, 1_{Z(G)}\rangle = \chi(1)$ for $\chi \in \mathrm{nl}(G/Z(G))$. Therefore,

$$\begin{aligned}\left\langle \zeta_G^{w_2} \downarrow_{Z(G)}, 1_{Z(G)}\right\rangle &= \frac{|G|^2}{|G'|} + |G||\mathrm{nl}(G/Z(G))| \\ &= \frac{|G|^2}{|G'|} + \frac{|G||Z_2(G)|(|G'| - |Z(G)|)}{|G'||Z(G)|},\end{aligned}$$

using Corollary 3.11(2). Now putting the values of $\langle \zeta_G^{w_2} \downarrow_{Z(G)}, 1_{Z(G)}\rangle$ and $\langle \zeta_G^{w_2} \downarrow_{Z(G)}, 1_{Z(G)}\rangle$ in (3.5) we get the required expression for $\zeta_G^{w_3}$. □

The following lemma is useful in proving the final result of this subsection.

LEMMA 3.14. *Let G be a finite group such that $(G, Z(G))$ is a Camina pair and $(\frac{G}{Z(G)}, Z(\frac{G}{Z(G)}))$ a GCP. Then*

$$\sum_{\chi\in \mathrm{Irr}(G|Z(G))} \frac{\chi(g)}{\chi(1)} = \begin{cases} -1 & \text{if } 1 \neq g \in Z(G) \subseteq G' \\ 0 & \text{if } g \in G' \setminus Z(G) \end{cases}$$

and

$$\sum_{\chi\in \mathrm{nl}(G/Z(G))} \frac{\chi(g)}{\chi(1)} = \begin{cases} \frac{|Z_2(G)|(|G'|-|Z(G)|)}{|G'||Z(G)|} & \text{if } 1 \neq g \in Z(G) \subseteq G' \\ -\frac{|Z_2(G)|}{|G')|} & \text{if } g \in G' \setminus Z(G). \end{cases}$$

PROOF. Follows from Theorem 3.10(1), (2) and Corollary 3.11. □

We conclude this subsection by the following result.

THEOREM 3.15. *Let G be a finite group such that $(G, Z(G))$ is a Camina pair and $(\frac{G}{Z(G)}, Z(\frac{G}{Z(G)}))$ a GCP. If $g \in G'$ then*

$$\zeta_G^{w_2}(g) = \begin{cases} \frac{|G|[|Z(G)|(|G|+|G'|(|Z(G)|-1))+|Z_2(G)|(|G'|-|Z(G)|)]}{|G'||Z(G)|} & \text{if } g = 1 \\ \frac{|G|[|Z(G)|(|G|-|G'|)+|Z_2(G)|(|G'|-|Z(G)|)]}{|G'||Z(G)|} & \text{if } 1 \neq g \in Z(G) \subseteq G' \\ \frac{|G|(|G|-|Z_2(G)|)}{|G'|} & \text{if } g \in G' \setminus Z(G) \end{cases}$$

and

$$\zeta_G^{w_3}(g) = \begin{cases} \frac{|G|^2[|G||Z(G)|^2+(|G'|-|Z(G)|)(|G|+|Z_2(G)|(|Z(G)|-1))]}{|G'||Z(G)|} & \text{if } g = 1 \\ \frac{|G|^2(|G'|-|Z(G)|)(|G|-|Z_2(G)|)}{|G'||Z(G)|} & \text{if } 1 \neq g \in Z(G) \subseteq G' \\ 0 & \text{if } g \in G' \setminus Z(G). \end{cases}$$

PROOF. If $g = 1$, then by (1.1) we have

$$\zeta_G^{w_2}(1) = |G||\operatorname{Irr}(G)| = \frac{|G|[|Z(G)|(|G| + |G'|(|Z(G)| - 1)) + |Z_2(G)|(|G'| - |Z(G)|)]}{|G'||Z(G)|},$$

noting that $\operatorname{Irr}(G) = \operatorname{lin}(G) \sqcup \operatorname{Irr}(G|Z(G)) \sqcup \operatorname{nl}(G/Z(G))$. If $1 \neq g \in G'$ then by (1.1) we have

$$\zeta_G^{w_2}(g) = \frac{|G|^2}{|G'|} + |G| \sum_{\chi \in \operatorname{Irr}(G|Z(G))} \frac{\chi(g)}{\chi(1)} + |G| \sum_{\chi \in \operatorname{nl}(G/Z(G))} \frac{\chi(g)}{\chi(1)}. \tag{3.6}$$

Now, if $1 \neq g \in Z(G) \subseteq G'$, then by (3.6) and Lemma 3.14 we have

$$\zeta_G^{w_2}(g) = \frac{|G|^2}{|G'|} + |G|(-1) + \frac{|G||Z_2(G)|(|G'| - |Z(G)|)}{|G'||Z(G)|} = \frac{|G|[|Z(G)|(|G| - |G'|) + |Z_2(G)|(|G'| - |Z(G)|)]}{|G'||Z(G)|}.$$

If $g \in G' \setminus Z(G)$, then by (3.6) and Lemma 3.14 we have

$$\zeta_G^{w_2}(g) = \frac{|G|^2}{|G'|} + \frac{-|G||Z_2(G)|}{|G'|} = \frac{|G|(|G| - |Z_2(G)|)}{|G'|}.$$

For the second part, if $g = 1$, then by Theorem 3.13, we have

$$\zeta_G^{w_3}(1) = \frac{|G|^3}{|G'|}$$

$$+\left(\frac{|G|^3}{|G'|}+\frac{|G|^2|Z_2(G)|(|G'|-|Z_2(G)|)}{|G'||Z(G)|}\right)|\operatorname{Irr}(G|Z(G))|+\frac{|G|^3|\mathrm{nl}(G/Z(G))|}{|Z_2(G)|}$$
$$=\frac{|G|^2[|G||Z(G)|^2+(|G'|-|Z(G)|)(|G|+|Z_2(G)|(|Z(G)|-1))]}{|G'||Z(G)|},$$

using Theorem 3.10(3) and Corollary 3.11(2).

If $1\neq g\in G'$, then by Theorem 3.13 we have

$$\zeta_G^{w_3}(g)=\frac{|G|^3}{|G'|}+\left(\frac{|G|^3}{|G'|}+\frac{|G|^2|Z_2(G)|(|G'|-|Z(G)|)}{|G'||Z(G)|}\right)\sum_{\chi\in\operatorname{Irr}(G|Z(G))}\frac{\chi(g)}{\chi(1)}$$
$$+\frac{|G|^3}{|Z_2(G)|}\sum_{\chi\in\mathrm{nl}(G/Z(G))}\frac{\chi(g)}{\chi(1)}. \tag{3.7}$$

If $1\neq g\in Z(G)\subseteq G'$, then by (3.7) and Lemma 3.14 we have

$$\zeta_G^{w_3}(g)=\frac{|G|^3}{|G'|}+\left(\frac{|G|^3}{|G'|}+\frac{|G|^2|Z_2(G)|(|G'|-|Z(G)|)}{|G'||Z(G)|}\right)(-1)$$
$$+\frac{|G|^3(|G'|-|Z(G)|)}{|G'||Z(G)|}=\frac{|G|^2(|G'|-|Z(G)|)(|G|-|Z_2(G)|)}{|G'||Z(G)|}.$$

If $g\in G'\setminus Z(G)$, then by (3.7) and Lemma 3.14, we have $\zeta_G^{w_3}(g)=0$. □

3.3. Groups with unique nonlinear irreducible character. The class of groups considered in this subsection is a subclass of the groups G with $|\operatorname{cd}(G)|=2$, where $\operatorname{cd}(G)=\{\chi(1):\chi\in\operatorname{Irr}(G)\}$. Groups having $|\operatorname{cd}(G)|=2$ are considered in [24] for the word $w_2(x_1,x_2)=[x_1,x_2]$. Note that an explicit formula for $\zeta_G^{w_2}(g)$ can be obtained from [24, Theorem 2.2] for these groups. In this subsection, we derive an explicit formula for $\zeta_G^{w_n}(g)$ where $n\geq 3$ for groups having a unique nonlinear irreducible character. We start with the following theorem which is a consequence of the main theorem in [3].

THEOREM 3.16. *Let G be a finite group having a unique nonlinear irreducible character. Then one of the following holds*:

(1) *G is an extra-special 2-group.*

(2) *G is a Frobenius group of order $p^m(p^m-1)$ for some prime power p^m with an abelian Frobenius kernel of order p^m, a cyclic Frobenius complement and $Z(G)=\{1\}$. Furthermore, the Frobenius kernel is equal to G' and G acts transitively, by conjugation, on $G'\setminus\{1\}$.*

Theorem 3.16 is also proved independently in [29] and [23]. Note that the groups in this theorem are Camina groups. Further, groups of type (1) have nilpotency class 2 whereas groups of type (2) are not nilpotent. For groups of type (1), an explicit formula of $\zeta_G^{w_n}$ can be obtained by Theorem 3.4. In this section, we derive an explicit formula of $\zeta_G^{w_n}$ for groups of type (2) considering $n \geq 3$.

We need the following lemma.

Lemma 3.17. *Let G be a finite group with unique nonlinear irreducible character ϕ with $Z(G) = \{1\}$. Then we have the following.*

(1) *ϕ vanishes outside G' and $\phi(1) = p^m - 1$.*

(2) *$\phi^2 = \sum_{\lambda \in \mathrm{lin}(G)} \lambda + (p^m - 2)\phi$.*

(3) *$\phi(g) = -1$ for all $g \in G' \setminus \{1\}$.*

Proof. By Theorem 3.16, G is a Frobenius group of order $p^m(p^m - 1)$ for some prime power p^m.

(1) Since G is a Camina group, ϕ vanishes outside G'. The second part follows from the relation $|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2$.

(2) Since ϕ^2 is a character of G, $\phi^2 = \sum_{\chi \in \mathrm{Irr}(G)} c_\chi \chi$, where $c_\chi = \langle \phi^2, \chi \rangle \in \mathbb{N} \cup \{0\}$. Let $\lambda \in \mathrm{lin}(G)$. Then $c_\lambda = \langle \phi^2, \lambda \rangle = \langle \phi, \overline{\phi}\lambda \rangle = \langle \phi, \phi\lambda \rangle = \langle \phi, \phi \rangle = 1$. Therefore,

$$\phi^2 = \sum_{\lambda \in \mathrm{lin}(G)} \lambda + c_\phi \phi.$$

Now, by degree computation of both sides of the above expression, we get $c_\phi = p^m - 2$. Hence, the result follows.

(3) Since G is a Frobenius group with Frobenius kernel G', any nonlinear irreducible character of G is induced from a non-trivial irreducible character of G' (see [11, Theorem 6.34 (a)]). But G has a unique nonlinear irreducible character ϕ. Therefore, $\phi\downarrow_{G'} = \sum_{\gamma \in (\mathrm{Irr}(G') \setminus \{1_{G'}\})} \gamma$, where $1_{G'}$ is the trivial character of G'. Now take $g \neq 1 \in G'$. Then $\big(\sum_{\gamma \in \mathrm{Irr}(G')} \gamma\big)(g) = 0$ and hence $\phi(g) = -1$. □

Theorem 3.18. *Let G be a finite group with unique nonlinear irreducible character ϕ and $Z(G) = \{1\}$. If $n \geq 3$ then*

$$\zeta_G^{w_n} = |G|^{n-1} \sum_{\lambda \in \mathrm{lin}(G)} \lambda + \frac{|G|}{p^m - 1} C^{w_n}(\phi)\phi$$

and $C^{w_n}(\phi) = \frac{|G|^{n-1}}{|G'|} + \frac{|G|(p^m-2)}{p^m-1} C^{w_{n-1}}(\phi)$.

PROOF. The expression for $\zeta_G^{w_n}$ follows from (3.2) applying Lemma 3.17(1). Using this we have

$$C^{w_n}(\phi) = \big\langle \zeta_G^{w_{n-1}}\phi, \phi\big\rangle = \left\langle \left(|G|^{n-2} \sum_{\lambda \in \mathrm{lin}(G)} \lambda + \frac{|G|}{p^m - 1} C^{w_{n-1}}(\phi)\phi \right)\phi, \phi \right\rangle$$

$$= |G|^{n-2} \sum_{\lambda \in \mathrm{lin}(G)} \langle \phi, \phi\rangle + \frac{|G|}{p^m - 1} C^{w_{n-1}}(\phi)\langle \phi^2, \phi\rangle.$$

Hence, using Lemma 3.17(2) and the fact that $|\mathrm{lin}(G)| = |G : G'|$, the result follows. $\square$

THEOREM 3.19. *Let G be a finite group with a unique nonlinear irreducible character ϕ with $Z(G) = \{1\}$. If $g \in G'$ and $n \geq 3$ then*

$$\zeta_G^{w_n}(g) = \begin{cases} \frac{|G|^n}{|G'|} + \frac{|G|^n((p^m-1)^{n-2} - (p^m-2)^{n-2})}{|G'|(p^m-1)^{n-3}} + \frac{|G|^{n-1}(p^m-2)^{n-2}}{(p^m-1)^{n-2}} & \text{if } g = 1 \\ \frac{|G|^n}{|G'|} - \frac{|G|^n((p^m-1)^{n-2} - (p^m-2)^{n-2})}{|G'|(p^m-1)^{n-2}} - \frac{|G|^{n-1}(p^m-2)^{n-2}}{(p^m-1)^{n-1}} & \text{if } g \neq 1. \end{cases}$$

PROOF. Let $g \in G'$ and $n \geq 3$. Then, using Theorem 3.18 and Lemma 3.17(3), we have

$$\zeta_G^{w_n}(g) = \begin{cases} \frac{|G|^n}{|G'|} + |G| C^{w_n}(\phi) & \text{if } g = 1 \\ \frac{|G|^n}{|G'|} - \frac{|G|}{(p^m-1)} C^{w_n}(\phi) & \text{if } g \neq 1. \end{cases}$$

By the recursive formula for $C^{w_n}(\phi)$, given in Theorem 3.18, we have

$$C^{w_n}(\phi) = \frac{|G|^{n-1}}{|G'|}\left(1 + \frac{p^m - 2}{p^m - 1} + \frac{(p^m-2)^2}{(p^m-1)^2} + \cdots + \frac{(p^m-2)^{n-3}}{(p^m-1)^{n-3}}\right) + \frac{|G|^{n-2}(p^m-2)^{n-2}}{(p^m-1)^{n-2}}$$

$$= \frac{|G|^{n-1}((p^m-1)^{n-2} - (p^m-2)^{n-2})}{|G'|(p^m-1)^{n-3}} + \frac{|G|^{n-2}(p^m-2)^{n-2}}{(p^m-1)^{n-2}}.$$

Putting $C^{w_n}(\phi)$ in this formula for $\zeta_G^{w_n}(g)$ we get the required result. $\square$

COROLLARY 3.20. *Let G be a finite group with unique nonlinear irreducible character ϕ with $Z(G) = \{1\}$. If $g \in G'$ then*

$$\zeta_G^{w_3}(g) = \begin{cases} \frac{2|G|^3}{|G'|} + \frac{|G|^2(p^m-2)}{p^m-1} & \text{if } g = 1 \\ \frac{|G|^3}{|G'|} - \frac{|G|^3}{|G'|(p^m-1)} - \frac{|G|^2(p^m-2)}{(p^m-1)^2} & \text{if } g \neq 1. \end{cases}$$

It is easy to see that the nonlinear irreducible character comes by induction of a linear character of the abelian commutator subgroup for the group of type (2) in Theorem 3.16. So it is natural to ask for some bounds of $C^{w_n}(\chi)$ if all the nonlinear irreducible characters come by induction of some linear character of a fixed abelian normal subgroup N of G. We conclude this paper by the following upper bound of $C^{w_n}(\chi)$.

PROPOSITION 3.21. *Let G be a finite group with* $\mathrm{cd}(G) = \{1, m\}$ *and let G have an abelian normal subgroup N of index m such that every nonlinear irreducible character of G is induced from some irreducible character of N. Then for $n \geq 3$, $C^{w_n}(\phi) \leq \frac{|G|^n}{|N|^2}$ whenever $\phi \in \mathrm{nl}(G)$.*

PROOF. Let N be a normal subgroup of index m. Then G/N has no nonlinear irreducible character and hence G/N is abelian. Therefore $G' \leq N$.

Let $\phi \in \mathrm{nl}(G)$ then $\psi \uparrow_N^G = \phi$ for some $\psi \in \mathrm{Irr}(N)$. Therefore $\phi(g) = 0$ for all $g \in G \setminus N$. Thus

$$C^{w_n}(\phi) = \langle \zeta_G^{w_{n-1}} \phi, \phi \rangle = \frac{1}{|G|} \sum_{g \in N} \zeta_G^{w_{n-1}}(g) |\phi(g)|^2 \leq \frac{\phi(1)^2}{|G|} \sum_{g \in N} \zeta_G^{w_{n-1}}(g).$$

Since $n \geq 3$, the image of w_{n-1} is contained in $G' \leq N$. Therefore,

$$\sum_{g \in N} \zeta_G^{w_{n-1}}(g) = \sum_{g \in G} \zeta_G^{w_{n-1}}(g) = |G|^{n-1}.$$

Hence, the result follows noting that $\phi(1) = m = |G : N|$. $\square$

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