

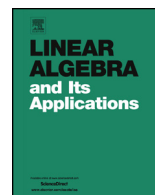


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Convex and quasiconvex functions on trees and their applications



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ABSTRACT

We introduce convex and quasiconvex functions on trees and prove that for a tree the eccentricity, transmission and weight functions are strictly quasiconvex. It is shown that the Perron vector of the distance matrix is strictly convex whereas the Perron vector of the distance signless Laplacian is quasiconvex for a tree. In the class of all trees with a given number of pendent vertices, we prove that the distance Laplacian and distance signless Laplacian spectral radius are both maximized at a dumbbell. Among all trees with fixed maximum degree, we prove that the broom is the unique tree that maximizes the distance Laplacian and distance signless Laplacian spectral radius. We find the unique graph that maximizes the distance spectral radius in the class of all unicyclic graphs of girth g on n vertices. Also we find the unique graph that maximizes the distance signless Laplacian and the distance Laplacian spectral radius in the class of all unicyclic graphs on n vertices.

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1. Introduction

All our graphs are finite, undirected, connected and simple. Let G be a graph on vertices $\{1, 2, \dots, n\}$. At times, we use $V(G)$ and $E(G)$ to denote the set of vertices and the set of edges of G , respectively. For $i, j \in V(G)$, the *distance* between i and j , denoted by $d_G(i, j)$ or simply d_{ij} , is the length of a shortest path from i to j in G . The *distance matrix* of G , denoted by $\mathcal{D}(G)$ is the $n \times n$ matrix with (i, j) -th entry d_{ij} .

Like the Laplacian and signless Laplacian matrix of a graph, Aouchiche and Hansen[1] first introduced the distance Laplacian and distance signless Laplacian of a graph. The transmission, denoted by $Tr(v)$ of a vertex v is the sum of the distances from v to all other vertices in G . Let $Tr(G)$ be the $n \times n$ diagonal matrix with i -th diagonal entry $Tr(i)$. The distance Laplacian and the distance signless Laplacian of G are defined as the matrix $\mathcal{D}^L(G) = Tr(G) - \mathcal{D}(G)$ and $\mathcal{D}^Q(G) = Tr(G) + \mathcal{D}(G)$, respectively, see [1].

For a column vector $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ we have

$$x^T \mathcal{D}^L(G)x = \sum_{1 \leq i < j \leq n} d_{ij}(x_i - x_j)^2 \text{ and } x^T \mathcal{D}^Q(G)x = \sum_{1 \leq i < j \leq n} d_{ij}(x_i + x_j)^2. \quad (1.1)$$

It follows that $\mathcal{D}^L(G)$ and $\mathcal{D}^Q(G)$ are positive semidefinite. Thus the eigenvalues of distance Laplacian and distance signless Laplacian matrix of a graph are all real. The largest eigenvalue of the distance, distance Laplacian and the distance signless Laplacian matrix are known as the distance spectral radius, distance Laplacian spectral radius and distance signless Laplacian spectral radius, respectively. We use $\rho^{\mathcal{D}}$, ρ^L and ρ^Q to denote the distance, distance Laplacian and distance signless Laplacian spectral radius, respectively.

Note that for a connected graph G , both $\mathcal{D}(G)$ and $\mathcal{D}^Q(G)$ are irreducible nonnegative matrices. Thus by the Perron–Frobenius Theorem, both $\rho^{\mathcal{D}}$ and ρ^Q are simple, and there is a positive eigenvector of $\mathcal{D}(G)$ and $\mathcal{D}^Q(G)$ corresponding to $\rho^{\mathcal{D}}$ and ρ^Q , respectively. Such eigenvectors corresponding to $\rho^{\mathcal{D}}$ and ρ^Q are called *Perron vector* of $\mathcal{D}(G)$ and $\mathcal{D}^Q(G)$, respectively. By an eigenvector we mean a unit eigenvector.

Recently, the study of distance, distance Laplacian and distance signless Laplacian spectral radius has attracted many researchers. The problem of finding all graphs with maximal or minimal distance spectral radius among certain class of graphs has been studied extensively. For related results, one may refer to [2,4–6,9] and the references therein.

Subhi and Powers in [8] proved that for $n \geq 3$ the path P_n has the maximum distance spectral radius among trees on n vertices. Stevanović and Ilić in [7] generalized this result, and proved that among trees with fixed maximum degree Δ , the broom graph has maximum distance spectral radius and proved that the star S_n is the unique graph with minimal distance spectral radius among all trees on n vertices.

If vertices i and j are adjacent, we write $i \sim j$. Vertices i, j, k in graph G are said to be consecutive if j is adjacent to both i and k . An edge independent set of a graph G is a set of edges such that any two distinct edges of the set are not incident on a common vertex.

The matching number of G , denoted by $m(G)$, is the maximum of the cardinalities of all edge independent sets. For a connected graph G of order n , its matching number $m(G)$ satisfies $1 \leq m(G) \leq \lfloor \frac{n}{2} \rfloor$. By \mathcal{T}_n^m we denote the class of all trees on n vertices with matching number m .

If u, v are vertices in a graph G , by $G - u$ and $G - \{u, v\}$ we mean the graphs obtained from G by deleting the vertex u and the edge $\{u, v\}$ respectively. Also by $G + \{u, v\}$ we mean the graph obtained from G by adding the edge $\{u, v\}$. A *quasipendent vertex* is a vertex which is adjacent to a pendent vertex. If v is a vertex of a tree T , then the components of $T - v$ are called the *branches* of T at v .

The *dumbbell* $D(n, a, b)$ consists of the path P_{n-a-b} together with a independent vertices adjacent to one pendent vertex of P and b independent vertices adjacent to the other pendent vertex, where $a, b \geq 1$. By $\mathcal{T}_{n,k}$ we denote the class of all trees on n vertices and k pendent vertices.

We now outline the contents of this paper. In section 2 of this article, we introduce some convex and quasiconvex functions on trees and prove that for a tree the eccentricity, transmission and weight functions are strictly quasiconvex. In section 3, we prove that the Perron vector of distance matrix of a tree is strictly convex and the Perron vector of distance signless Laplacian matrix of a tree is strictly quasiconvex. In section 4, we have obtained the unicyclic graph that maximizes the distance spectral radius among all unicyclic graphs of fixed girth and order. In section 5, we prove that the tree that maximizes the distance Laplacian spectral radius among all trees with a given number of pendent vertices is a dumbbell. Moreover, we obtain the tree that maximizes the distance Laplacian spectral radius among all trees with fixed maximum degree. We obtain the unique unicyclic graph that maximizes the distance Laplacian spectral radius among all unicyclic graph. In section 6, we prove that the tree that maximizes the distance signless Laplacian spectral radius among all trees with a given number of pendent vertices is a dumbbell. Moreover, we obtain the tree that maximizes the distance signless Laplacian spectral radius among all trees with fixed maximum degree. We obtain the unique unicyclic graph that maximizes the distance signless Laplacian spectral radius among all unicyclic graphs.

2. Convex and quasiconvex functions on trees

Definition. Let T be a tree with $V(T) = \{1, \dots, n\}$, $n \geq 3$, and let $f : V(T) \rightarrow [0, +\infty)$. We say that f is *convex* if for any distinct $i, j, k \in V(T)$ with $i \sim j$, $j \sim k$ we have $2f(j) \leq f(i) + f(k)$ and *strictly convex* if $2f(j) < f(i) + f(k)$. We say that f is *quasiconvex* if for any distinct $i, j, k \in V(T)$ with $i \sim j$, $j \sim k$ we have $f(j) \leq \max\{f(i), f(k)\}$ and *strictly quasiconvex* if $f(j) < \max\{f(i), f(k)\}$.

Lemma 2.1. *Let T be a tree with $V(T) = \{1, \dots, n\}$ and let $f : V(T) \rightarrow [0, +\infty)$ be either strictly convex or strictly quasiconvex. Then f attains its minimum, either at a unique vertex, or at two adjacent vertices. Furthermore:*

- (i) If f attains its minimum at the unique vertex i , then for any path $i = i_1 - i_2 - \dots - i_k$, starting at i , $f(i_1) < f(i_2) < \dots < f(i_k)$. (We say that f is strictly increasing along any path starting at i .)
- (ii) If f attains its minimum at the two adjacent vertices i and j , then f is strictly increasing along any path starting at i , and not containing j , or starting at j , and not containing i .

Proof. First suppose that f is strictly convex. Observe that if $i, j, k \in V(T)$ with $i \sim j$, $j \sim k$, then $f(i) < f(j)$ and the strict convexity of f imply that $f(j) < f(k)$. Let $S \subset V(T)$ be the set of vertices on which f attains its minimum. Let $i, j \in S$ and suppose i and j are not adjacent. Let $i = i_1 - i_2 - \dots - i_k = j$ be an ij -path. By the observation made earlier, $f(i) = f(i_1) < f(i_2)$ implies that $f(i_2) < \dots < f(i_k) = f(j)$ and hence $f(i) < f(j)$, which is a contradiction. Thus any two vertices in S must be adjacent. Since T has no cycles, it follows that S has either 1 or 2 vertices. The second part is proved similarly, using the observation made in the beginning. This completes the proof.

If f is strictly quasiconvex, then again if $i, j, k \in V(T)$ with $i \sim j$, $j \sim k$, and $f(i) < f(j)$, then it follows that $f(j) < f(k)$. The rest of the proof is similar. \square

Theorem 2.2. Let T be a tree with $V(T) = \{1, \dots, n\}$, let $g : [0, +\infty) \rightarrow [0, +\infty)$ be a strictly increasing convex function. Let x_1, \dots, x_n be positive numbers and let $f(i) = \sum_{j=1}^n g(d_{ij})x_j$, $i = 1, \dots, n$. Then f is strictly convex.

Proof. Let i, j, k be vertices of T such that $i \sim j$, $j \sim k$. Let $e_1 = \{i, j\}$, $e_2 = \{j, k\}$ be edges of T . Let T_1, T_2 and T_3 be the components of $T \setminus \{e_1, e_2\}$, containing i, j and k , respectively. We have

$$f(i) = \sum_{\ell \in T_1} g(d_{i\ell})x_\ell + \sum_{\ell \in T_2} g(d_{j\ell} + 1)x_\ell + \sum_{\ell \in T_3} g(d_{k\ell} + 2)x_\ell \tag{2.2}$$

$$f(j) = \sum_{\ell \in T_1} g(d_{i\ell} + 1)x_\ell + \sum_{\ell \in T_2} g(d_{j\ell})x_\ell + \sum_{\ell \in T_3} g(d_{k\ell} + 1)x_\ell \tag{2.3}$$

$$f(k) = \sum_{\ell \in T_1} g(d_{i\ell} + 2)x_\ell + \sum_{\ell \in T_2} g(d_{j\ell} + 1)x_\ell + \sum_{\ell \in T_3} g(d_{k\ell})x_\ell \tag{2.4}$$

Since g is convex, we have

$$g(d_{i\ell}) + g(d_{i\ell} + 2) \geq 2g(d_{i\ell} + 1) \tag{2.5}$$

and

$$g(d_{k\ell} + 2) + g(d_{k\ell}) \geq 2g(d_{k\ell} + 1). \tag{2.6}$$

It follows from (2.2)–(2.6) that

$$f(i) + f(k) \geq \sum_{\ell \in T_1} 2g(d_{i\ell} + 1)x_\ell + \sum_{\ell \in T_2} 2g(d_{j\ell} + 1)x_\ell + \sum_{\ell \in T_3} 2g(d_{k\ell} + 1)x_\ell$$

and since g is strictly increasing we have

$$\begin{aligned} f(i) + f(k) - 2f(j) &\geq \sum_{\ell \in T_2} 2(g(d_{j\ell} + 1) - g(d_{j\ell}))x_\ell \\ &> 0. \end{aligned}$$

That completes the proof. \square

2.1. Sum of the largest p distances

We introduce some notations. Let p be a fixed positive integer, $1 \leq p \leq n$. If $x, y \in R^n$, we define $x \otimes_p y$ to be the sum of the p largest components in $\{x_1y_1, \dots, x_ny_n\}$. If A is an $n \times n$ matrix and if $x \in R^n$, then $A \otimes_p x$ is defined in the natural way, that is, the i -th component of $A \otimes_p x$ is given by $(a_{i1}, \dots, a_{in}) \otimes_p x$, $i = 1, \dots, n$. Note that when $p = n$, $x \otimes y = \sum_{i=1}^n x_iy_i$, whereas when $p = 1$, then $x \otimes y = \max_i x_iy_i$.

If A is an $n \times n$ matrix with $a_{ij} \geq 0$ for all i, j and if $g : [0, +\infty) \rightarrow [0, +\infty)$ is a function, then we define $g(A)$ to be the $n \times n$ matrix with (i, j) -entry $g(a_{ij})$.

Let T be a tree with $V(T) = \{1, \dots, n\}$ and let $y \in R^n$ be a vector with nonnegative coordinates. We say that y is convex (quasiconvex) if $f(i) = y_i$, $i = 1, \dots, n$ is a convex (quasiconvex) function with respect to the tree T . Similar definition is used with respect to strict convexity (quasiconvexity).

Theorem 2.3. *Let T be a tree with $V(T) = \{1, \dots, n\}$, let $g : [0, +\infty) \rightarrow [0, +\infty)$ be a strictly increasing function. Let $x \in R^n$ be a vector with positive coordinates. Let p be a positive integer, $1 \leq p \leq n$. Then $g(D) \otimes_p x$ is strictly quasiconvex.*

Proof. Let i, j, k be vertices of T such that $i \sim j, j \sim k$. We claim that for any $\ell \in V(T)$ we have

$$g(d_{j\ell})x_\ell < \max(g(d_{i\ell})x_\ell, g(d_{k\ell})x_\ell). \tag{2.7}$$

Let $e_1 = \{i, j\}$, $e_2 = \{j, k\}$ be edges of T . Let T_1, T_2 and T_3 be the components of $T \setminus \{e_1, e_2\}$, containing i, j and k , respectively. If $\ell \in T_1$, then $d_{i\ell}x_\ell = (d_{j\ell} - 1)x_\ell$ and $d_{k\ell}x_\ell = (d_{j\ell} + 1)x_\ell$. Hence we have

$$d_{j\ell}x_\ell < \max(d_{i\ell}x_\ell, d_{k\ell}x_\ell). \tag{2.8}$$

Similarly (2.8) holds if $\ell \in T_3$. If $\ell \in T_2$, then $d_{i\ell}x_\ell = (d_{j\ell} + 1)x_\ell$ and $d_{k\ell}x_\ell = (d_{j\ell} + 1)x_\ell$. Hence (2.8) holds as well. Therefore (2.8) holds for any $\ell \in V(T)$. Since g is strictly increasing, (2.7) follows from (2.8).

It follows that the sum of the largest p components in $g(d_{j\ell})x_\ell$, $\ell = 1, \dots, n$ is less than the sum of the largest p components in $g(d_{i\ell})x_\ell$, $\ell = 1, \dots, n$ and the largest p components in $g(d_{k\ell})x_\ell$, $\ell = 1, \dots, n$. Therefore $g(D) \otimes_p x$ is strictly quasiconvex and the proof is complete. \square

We note two special cases of [Theorem 2.3](#). Let T be a tree with $V(T) = \{1, \dots, n\}$. Recall that the *eccentricity* of vertex $\ell \in V(T)$ is defined to be $\text{ecc}(\ell) = \max(d_{\ell 1}, \dots, d_{\ell n})$, while the transmission index of $\ell \in V(T)$ is given by $\text{Tr}(\ell) = \sum_{i=1}^n d(\ell i)$. It follows from [Theorem 2.3](#) that both eccentricity and transmission index are strictly quasiconvex functions and hence satisfy (i) and (ii) of [Lemma 2.1](#). The vertex (or vertices) where the eccentricity is minimized constitute the *center* of the tree, whereas the vertex (or vertices) where the transmission index is minimized, constitute the *median* of the tree.

2.2. Centroid

Let T be a tree with $V(T) = \{1, \dots, n\}$. If $i \in V(T)$, then the weight of i , denoted $\text{wt}(i)$, is defined to be the maximum number of vertices in a component of $T \setminus i$.

Let i, j, k be vertices of T such that $i \sim j, j \sim k$. Let T_1, T_2 and T_3 be the components of $T \setminus \{e_1, e_2\}$, containing i, j and k , respectively. Let $T_\ell^i, \ell = 1, \dots, u$, be the components of $T_1 \setminus i$, let $T_\ell^j, \ell = 1, \dots, v$, be the components of $T_2 \setminus j$, and let $T_\ell^k, \ell = 1, \dots, w$, be the components of $T_3 \setminus k$. We denote the number of vertices in the graph G by $|G|$. We have

$$\text{wt}(i) = \max\{|T_1^i|, \dots, |T_u^i|, |T_1^j| + \dots + |T_v^j| + |T_1^k| + \dots + |T_w^k| + 2\}, \tag{2.9}$$

and

$$\text{wt}(k) = \max\{|T_1^k|, \dots, |T_w^k|, |T_1^j| + \dots + |T_v^j| + |T_1^i| + \dots + |T_u^i| + 2\}. \tag{2.10}$$

It follows from [\(2.9\)](#), [\(2.10\)](#) that

$$\begin{aligned} & \max(\text{wt}(i), \text{wt}(k)) \\ &= \max\{|T_1^j|, \dots, |T_v^j| + |T_1^k| + \dots + |T_w^k| + 2, |T_1^j| + \dots + |T_v^j| + |T_1^i| + \dots + |T_u^i| + 2\} \\ &> \max\{|T_1^i| + \dots + |T_u^i| + 1, |T_1^j| + \dots + |T_v^j|, |T_1^k| + \dots + |T_w^k| + 1\} \\ &= \text{wt}(j). \end{aligned}$$

Thus we have proved the following result.

Theorem 2.4. *Let T be a tree with $V(T) = \{1, \dots, n\}$. Then $\text{wt}(i)$, $i = 1, \dots, n$ is strictly quasiconvex.*

It follows from [Theorem 2.4](#) that the weight function satisfies (i) and (ii) of [Lemma 2.1](#). The vertices with minimum weight constitute the *centroid* of the tree. It can be shown that the centroid and the median of a tree coincide.

3. Perron vector of distance and distance signless Laplacian matrices

The following result is a reformulation of [Theorem 2.2](#). The proof is omitted.

Theorem 3.1. *Let $f : [0, +\infty) \rightarrow [0, +\infty)$ be a strictly increasing convex function and \mathcal{D} be the distance matrix of a tree T . Then $f(\mathcal{D})z$ is strictly convex on T for any positive vector z , where $f(\mathcal{D})$ is the matrix with (i, j) -th entry $f(d_{ij})$.*

The following corollary, which follows immediately from [Theorem 3.1](#), says that the Perron vector of $f(\mathcal{D})$ is strictly convex, which implies that the Perron vector satisfies (i) and (ii) of [Lemma 2.1](#).

Corollary 3.2. *Let \mathcal{D} be the distance matrix of a tree T and let $f : [0, +\infty) \rightarrow [0, +\infty)$ be a strictly increasing convex function. Then the Perron vector of $f(\mathcal{D})$ is strictly convex on T .*

Proof. Let ρ be the spectral radius and \mathbf{x} be the Perron vector of $f(\mathcal{D})$. By the Perron–Frobenius theorem \mathbf{x} is positive. Since $f(\mathcal{D})\mathbf{x} = \rho\mathbf{x}$, then $\mathbf{x} = \frac{1}{\rho}f(\mathcal{D})\mathbf{x}$. By [Theorem 3.1](#) \mathbf{x} is strictly convex. \square

It follows from [Corollary 3.2](#) that the Perron vector of $f(\mathcal{D})$ satisfies (i) and (ii) of [Lemma 2.1](#). The next result says that the Perron vector of the distance signless Laplacian matrix of a tree is strictly quasiconvex and hence satisfies (i) and (ii) of [Lemma 2.1](#).

Theorem 3.3. *Let \mathcal{D}^Q be the distance signless Laplacian of a tree T . Then the Perron vector of \mathcal{D}^Q is strictly quasiconvex on T .*

Proof. Let i, j, k be vertices of T such that $i \sim j, j \sim k$. Let $e_1 = \{i, j\}, e_2 = \{j, k\}$ be edges of T . Let T_1, T_2 and T_3 be the components of $T \setminus \{e_1, e_2\}$, containing i, j and k , respectively. Let \mathbf{x} be the Perron vector of \mathcal{D}^Q . By [Theorem 2.2](#) the transmission index is strictly convex and therefore

$$Tr(i) + Tr(k) - 2Tr(j) > 0.$$

Thus

$$2\{\rho^Q - Tr(j)\} > 2\rho^Q - \{Tr(i) + Tr(k)\}. \tag{3.11}$$

By eigenvector equations of \mathbf{x} at the vertices i, j, k we have

$$\{\rho^{\mathcal{Q}} - Tr(i)\}\mathbf{x}_i = \sum_{\ell \in T_1} d_{i\ell}\mathbf{x}_\ell + \sum_{\ell \in T_2} (d_{j\ell} + 1)\mathbf{x}_\ell + \sum_{\ell \in T_3} (d_{k\ell} + 2)\mathbf{x}_\ell \tag{3.12}$$

$$\{\rho^{\mathcal{Q}} - Tr(j)\}\mathbf{x}_j = \sum_{\ell \in T_1} (d_{i\ell} + 1)\mathbf{x}_\ell + \sum_{\ell \in T_2} d_{j\ell}\mathbf{x}_\ell + \sum_{\ell \in T_3} (d_{k\ell} + 1)\mathbf{x}_\ell \tag{3.13}$$

$$\{\rho^{\mathcal{Q}} - Tr(k)\}\mathbf{x}_k = \sum_{\ell \in T_1} (d_{i\ell} + 2)\mathbf{x}_\ell + \sum_{\ell \in T_2} (d_{j\ell} + 1)\mathbf{x}_\ell + \sum_{\ell \in T_3} d_{k\ell}\mathbf{x}_\ell. \tag{3.14}$$

It follows from (3.12)–(3.14) that

$$\{\rho^{\mathcal{Q}} - Tr(i)\}\mathbf{x}_i + \{\rho^{\mathcal{Q}} - Tr(k)\}\mathbf{x}_k - 2\{\rho^{\mathcal{Q}} - Tr(j)\}\mathbf{x}_j > 0. \tag{3.15}$$

From (3.11) and (3.15), it follows that

$$\{\rho^{\mathcal{Q}} - Tr(i)\}\mathbf{x}_i + \{\rho^{\mathcal{Q}} - Tr(k)\}\mathbf{x}_k > \{2\rho^{\mathcal{Q}} - \{Tr(i) + Tr(k)\}\}\mathbf{x}_j.$$

Thus $\beta\mathbf{x}_i + (1 - \beta)\mathbf{x}_k > \mathbf{x}_j$, where $\beta = \frac{\rho^{\mathcal{Q}} - Tr(i)}{2\rho^{\mathcal{Q}} - \{Tr(i) + Tr(k)\}}$. Therefore $\mathbf{x}_j < \max\{\mathbf{x}_i, \mathbf{x}_k\}$ and hence the Lemma holds. \square

4. Distance spectral radius

For a matrix A we use A_i to denote the i -th column of A . By $\mathbb{1}$ we denote the column vector of all ones and by \mathbb{J} the matrix of all ones of appropriate size. We recall some known results which will be used.

Lemma 4.1 (Weyl’s inequalities). [3] *Let $\lambda_i(M)$ denote the i -th largest eigenvalue of a real symmetric matrix M . If A and B are two real symmetric matrices of order n , then*

$$\lambda_1(A) + \lambda_i(B) \geq \lambda_i(A + B) \geq \lambda_n(A) + \lambda_i(B) \text{ for } i = 1, 2, \dots, n.$$

Lemma 4.2. [3] *If A is a symmetric $n \times n$ matrix with λ_1 as the largest eigenvalue then for any normalized vector $\mathbf{x} \in R^n (\mathbf{x} \neq 0)$,*

$$\mathbf{x}^t A \mathbf{x} \leq \lambda_1.$$

The equality holds if and only if \mathbf{x} is an eigenvector of A corresponding to the eigenvalue λ_1 .

Lemma 4.3. *Let G_v be the graph obtained by identifying a vertex v of the graph G with a fixed vertex u of another graph G' . If i, j, k are consecutive vertices of G such that i and k are on different components of $G - j$ and $\rho_v^{\mathcal{D}}$ denotes the distance spectral radius of G_v , then $\rho_i^{\mathcal{D}} + \rho_k^{\mathcal{D}} - 2\rho_j^{\mathcal{D}} > 0$.*

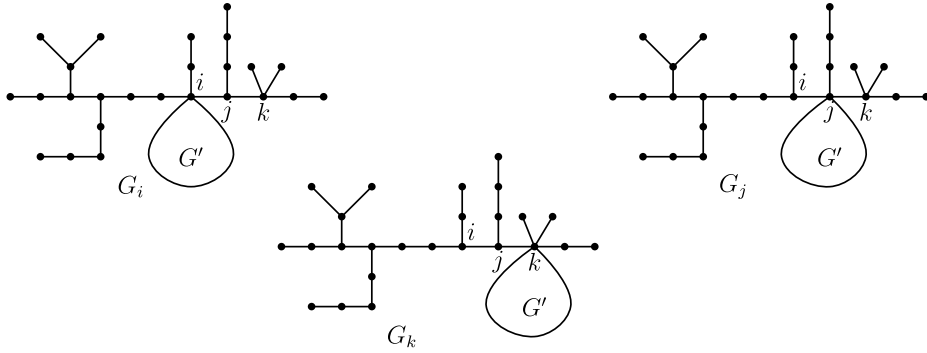


Fig. 1. Graphs G_i , G_j and G_k .

Proof. Let D_j be the j th column of $\mathcal{D}(G)$ and $d_{G'}(u, x)$ be the distance between u and x in G' . By permutation similarity, we have

$$\mathcal{D}(G_j) = \begin{bmatrix} \mathcal{D}(G) & B \\ B^t & \mathcal{D}(G')^u \end{bmatrix},$$

where $B_x = D_j + d_{G'}(u, x)\mathbb{1}$ for $x \in V(G' - u)$ and $\mathcal{D}(G')^u$ is the submatrix of $\mathcal{D}(G')$ after deleting the row and column corresponding to the vertex u .

Thus

$$\mathcal{D}(G_i) + \mathcal{D}(G_k) - 2\mathcal{D}(G_j) = \begin{bmatrix} 0 & (D_i + D_k - 2D_j)\mathbb{1}^t \\ \mathbb{1}(D_i + D_k - 2D_j)^t & 0 \end{bmatrix}.$$

Let \mathbf{x} be the Perron vector of $\mathcal{D}(G_j)$. Since \mathbf{x} is positive, $D_i + D_k - 2D_j \geq 0$ and $D_i + D_k - 2D_j \neq 0$ we see that

$$\mathbf{x}^t \{ \mathcal{D}(G_i) + \mathcal{D}(G_k) - 2\mathcal{D}(G_j) \} \mathbf{x} > 0. \tag{4.16}$$

Also

$$\begin{aligned} \rho_i^{\mathcal{D}} + \rho_k^{\mathcal{D}} - 2\rho_j^{\mathcal{D}} &\geq \rho \{ \mathcal{D}(G_i) + \mathcal{D}(G_k) \} - 2\rho_j^{\mathcal{D}} \text{ [by Lemma 4.1]} \\ &\geq \mathbf{x}^t \{ \mathcal{D}(G_i) + \mathcal{D}(G_k) - 2\mathcal{D}(G_j) \} \mathbf{x} \text{ [by Lemma 4.2]}. \end{aligned} \tag{4.17}$$

From (4.16) and (4.17) we have $\rho_i^{\mathcal{D}} + \rho_k^{\mathcal{D}} - 2\rho_j^{\mathcal{D}} > 0$. (See Fig. 1.) \square

Corollary 4.4. Let T be a tree and G' be a graph. If T_i and $\rho_i^{\mathcal{D}}$ are as defined in Lemma 4.3, then $\rho_i^{\mathcal{D}}$ is strictly convex on T .

Let G be a graph with v as one of the vertices. For $k, \ell \geq 1$, let $G(v, k, \ell)$ be the graph obtained by attaching two paths P_k and P_ℓ at v . We use $\rho_{v,k,\ell}^{\mathcal{D}}$ to denote the distance

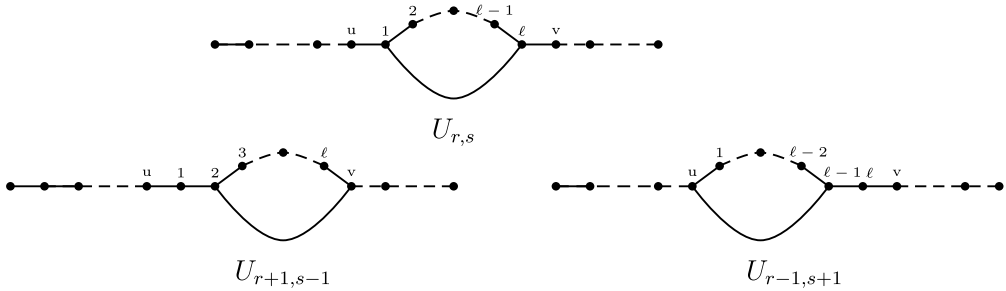


Fig. 2. Graphs $U_{r,s}$, $U_{r+1,s-1}$ and $U_{r-1,s+1}$.

spectral radius of $G(v, k, \ell)$. Stevanović, and Ilić [7] proved that $\rho_{v,k,\ell}^D < \rho_{v,k+1,\ell-1}^D$ for $k \geq \ell \geq 1$, which can now be obtained as an immediate application of Corollary 4.4.

Corollary 4.5. [7, Lemma 3.3] Let G be a graph and v be one of its vertices. If $k \geq \ell \geq 1$, then $\rho_{v,k,\ell}^D < \rho_{v,k+1,\ell-1}^D$.

Example 4.6. The graphs $U_{r,s}$, $U_{r+1,s-1}$ and $U_{r-1,s+1}$ with vertex labelling are as shown in Fig. 2.

The following lemma is crucial in proving our main result of this section.

Lemma 4.7. Let $U_{r,s}$ be a unicyclic graph of girth g with two pendent paths of length r, s ($r, s \geq 1$) respectively such that the intermediate vertices of the shortest path between the roots of the two paths are of degree two. If $\rho_{r,s}$ denotes the distance spectral radius of $U_{r,s}$ then $\rho_{r+1,s-1} + \rho_{r-1,s+1} > 2\rho_{r,s}$.

Proof. Let the vertices 1 and ℓ of the cycle be roots of the paths P_1 and P_2 respectively. Let $P := 1 - 2 - \dots - \ell$ be the shortest path from 1 to ℓ and S be the set of vertices not on these three paths. Partition the vertex set as follows:

$$\begin{aligned}
 V_1 &= \{j \in S : d_{1j} > d_{2j} \text{ and } d_{\ell j} < d_{(\ell-1)j}\} \\
 V_2 &= \{j \in S : d_{1j} < d_{2j} \text{ and } d_{\ell j} > d_{(\ell-1)j}\} \\
 V_3 &= \{j \in S : d_{1j} < d_{2j} \text{ and } d_{\ell j} < d_{(\ell-1)j}\} \\
 V_4 &= \{j \in S : d_{1j} = d_{2j}\} \\
 V_5 &= \{j \in S : d_{\ell j} = d_{(\ell-1)j}\} \\
 V_6 &= V(P_1 \cup P_2) - \{1, \ell\} \\
 V_7 &= V(P) - \{1, \ell\}.
 \end{aligned}$$

For $i \in V_7, j \in \cup_{i=3}^5 V_i$ if the lengths of the two paths in $U_{r,s}$ from i to j differ by k then following changes occur.

- (i) For $k = 0$, d_{ij} decreases by 1 in both $U_{r+1,s-1}$ and $U_{r-1,s+1}$.
- (ii) For $k = 1$, d_{ij} remains unchanged in one of $U_{r+1,s-1}$ and $U_{r-1,s+1}$ and decreases by 1 in the other.
- (iii) For $k \geq 2$, d_{ij} increases by 1 in one of $U_{r+1,s-1}$ and $U_{r-1,s+1}$ and decreases by 1 in the other.

Similarly we can describe the change in distances among other vertices.

For $i \in V_7$, let S_i be the set of vertices j in $\cup_{i=3}^{i=5} V_i$ such the length of the two paths from i to j differ by at most one. Note that if girth g is even then $S_i \subseteq V_3$ for all $i \in V_7$ as $V_4 = V_5 = \emptyset$.

Let $\mathcal{D}_{r,s}$ be the distance matrix of $U_{r,s}$. Then with respect to the ordered partition $V(U_{r,s}) = \{1, \ell\} \cup (V_1 \cup V_2) \cup (V_3 \cup V_4 \cup V_5) \cup V_6 \cup V_7$, we have

$$\mathcal{D}_{r+1,s-1} + \mathcal{D}_{r-1,s+1} - 2\mathcal{D}_{r,s} = \begin{bmatrix} \{1, \ell\} & V_1 \cup V_2 & V_3 \cup V_4 \cup V_5 & V_6 & V_7 \\ \hline 0 & \begin{matrix} 0 & 2\mathbb{1}^t \\ 2\mathbb{1}^t & 0 \end{matrix} & \begin{matrix} 2\mathbb{1}^t & \mathbb{1}^t & 2\mathbb{1}^t \\ 2\mathbb{1}^t & 2\mathbb{1}^t & \mathbb{1}^t \end{matrix} & 0 & 0 \\ \hline \begin{matrix} 0 & 2\mathbb{1} \\ 2\mathbb{1} & 0 \end{matrix} & 0 & 0 & 0 & 0 \\ \hline \begin{matrix} 2\mathbb{1} & 2\mathbb{1} \\ \mathbb{1} & 2\mathbb{1} \\ 2\mathbb{1} & \mathbb{1} \end{matrix} & 0 & 0 & 0 & A^t \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & A & 0 & 0 \end{bmatrix}$$

where $A_{ij} = \begin{cases} -1, & \text{if } j \in S_i \text{ and } g \text{ is odd} \\ -2, & \text{if } j \in S_i \text{ and } g \text{ is even} \\ 0, & \text{otherwise.} \end{cases}$

Let \mathbf{x} be the Perron vector of $\mathcal{D}_{r,s}$, then we have

$$\begin{aligned} & \rho_{r+1,s-1} + \rho_{r-1,s+1} - 2\rho_{r,s} \\ & \geq \rho(\mathcal{D}_{r+1,s-1} + \mathcal{D}_{r-1,s+1}) - 2\rho_{r,s} \quad [\text{by Lemma 4.1}] \\ & \geq \mathbf{x}^t (\mathcal{D}_{r+1,s-1} + \mathcal{D}_{r-1,s+1} - 2\mathcal{D}_{r,s}) \mathbf{x} \quad [\text{by Lemma 4.2}] \\ & = \mathbf{x}_1 \left(\sum_{j \in V_4} \mathbf{x}_j + 2 \sum_{j \in V_2 \cup V_3 \cup V_5} \mathbf{x}_j \right) + \mathbf{x}_\ell \left(\sum_{j \in V_5} \mathbf{x}_j + 2 \sum_{j \in V_1 \cup V_3 \cup V_4} \mathbf{x}_j \right) - \alpha \sum_{j \in S_i} \sum_{i \in V_7} \mathbf{x}_i \mathbf{x}_j \\ & \quad (\alpha = 1 \text{ or } 2 \text{ according as } g \text{ is odd or even}) \\ & \geq \mathbf{x}_1 \left(\sum_{j \in V_4} \mathbf{x}_j + 2 \sum_{j \in V_3 \cup V_5} \mathbf{x}_j \right) + \mathbf{x}_\ell \left(\sum_{j \in V_5} \mathbf{x}_j + 2 \sum_{j \in \cup V_3 \cup V_4} \mathbf{x}_j \right) - \alpha \sum_{j \in S_i} \sum_{i \in V_7} \mathbf{x}_i \mathbf{x}_j \\ & > \alpha \sum_{j \in S_i} \sum_{i \in V_7} \mathbf{x}_j (\mathbf{x}_1 + \mathbf{x}_\ell - \mathbf{x}_i). \end{aligned} \tag{4.18}$$

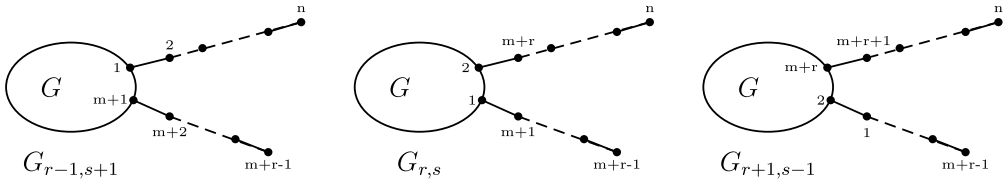


Fig. 3. Graphs $G_{r-1,s+1}$, $G_{r,s}$ and $G_{r+1,s-1}$.

From eigenequations at the vertices $1, \ell, i$ we have

$$\begin{aligned} \rho_{r,s}(\mathbf{x}_1 + \mathbf{x}_\ell - \mathbf{x}_i) &= \sum_{j \in V(U_{r,s})} d_{1j} \mathbf{x}_j + \sum_{j \in V(U_{r,s})} d_{\ell j} \mathbf{x}_j - \sum_{j \in V(U_{r,s})} d_{ij} \mathbf{x}_j \\ &= \sum_{j \in V(U_{r,s})} (d_{1j} + d_{\ell j} - d_{ij}) \mathbf{x}_j. \end{aligned} \tag{4.19}$$

Since $d_{1j} + d_{\ell j} > d_{ij}$ for all $i \in V_7$ and any vertex j , the result follows from (4.18) and (4.19). \square

We use $\mathcal{U}_{n,g}$ to denote the class of unicyclic graph of girth g on n vertices. U_g^n is the unicyclic graph on n vertices obtained by attaching a g -cycle to one of the end vertices of P_{n-g} . The next result is one of our main results of this section which provides the unique graph that maximizes the distance spectral radius among all unicyclic graph with fixed girth.

Theorem 4.8. *The unique graph that maximizes the distance spectral radius in $\mathcal{U}_{n,g}$ is U_g^n .*

Proof. If $g = n$, there is nothing to prove. Let U be a graph that maximizes the distance spectral radius in $\mathcal{U}_{n,g}$. Let C be the cycle in U and T_u be a tree branch attached at a vertex u of C . If T_u is not a path, then T_u contains a vertex of degree at least 3 except u . Let w be a vertex of degree at least 3 which is farthest from u . Then there exist two pendent paths attached at w . Thus $U \cong G(w, k, \ell)$ for some $k, \ell \geq 1$ and unicyclic graph G of girth g . Observe that $G(w, k - 1, \ell + 1), G(w, k + 1, \ell - 1) \in \mathcal{U}_{n,g}$. By Lemma 4.3, the distance spectral radius of $G(w, k + 1, \ell - 1)$ or $G(w, k - 1, \ell + 1)$ is bigger than that of U , which is a contradiction. Hence every tree branch of U attached at a vertex of C is a path. Next we prove that at most one vertex of C has degree 3. To the contrary, assume that u and v are two nearest vertices in C of degree 3. Then $U \cong U_{r,s}$ for some r, s ($r \geq s \geq 1$). By Lemma 4.7, distance spectral radius of $U_{r+1,s-1}$ or $U_{r-1,s+1}$ is bigger than that of U , which is a contradiction. Hence $U \cong U_g^n$. \square

Example 4.9. The graphs $G_{r,s}, G_{r+1,s-1}$ and $G_{r-1,s+1}$ with vertex labelling are as shown above.

Lemma 4.10. *Let $G_{r,s}$ be the graph obtained by attaching two paths of length r and s at two adjacent vertices u and v of a connected graph G , respectively. If $\rho_{r,s}$ denotes the distance spectral radius of $G_{r,s}$ then $\rho_{r+1,s-1} + \rho_{r-1,s+1} > 2\rho_{r,s}$.*

Proof. Let $V(G) = \{1, 2, \dots, m\}$ such that $u = 1, v = 2$. Let $\mathcal{D}_{r,s}$ be the distance matrix of $G_{r,s}$. Observe that $\mathcal{D}_{r+1,s-1} + \mathcal{D}_{r-1,s+1} - 2\mathcal{D}_{r,s} = [\hat{d}_{ij}]$, where

$$\hat{d}_{ij} = \begin{cases} 1 & \text{if } i = 1, 2 \text{ and } d_{1j} = d_{2j}, j = 3, \dots, m \\ 2 & \text{if } d_{1j} < d_{2j}, i = 1, j = 3, \dots, m \text{ or } d_{1j} > d_{2j}, i = 2, j = 3, \dots, m \\ 0 & \text{otherwise.} \end{cases}$$

Thus we see that $\mathbf{x}^t(\mathcal{D}_{r+1,s-1} + \mathcal{D}_{r-1,s+1} - 2\mathcal{D}_{r,s})\mathbf{x} > 0$, if \mathbf{x} is the Perron vector of $\mathcal{D}_{r,s}$. Hence the lemma holds. \square

As an application of Lemma 4.10, we obtain the following result proved by Zhang and Godsil in [9].

Corollary 4.11. [9, Theorem 1.1] *Let $G_{r,s}$ be the graph obtained by attaching two paths of length r and s at two adjacent vertices u and v of a connected graph G , respectively. Let $\rho_{r,s}^{\mathcal{D}}$ be the distance spectral radius of $G_{r,s}$. Then the following holds:*

- (i) $\rho_{r,s}^{\mathcal{D}} < \rho_{r+1,s-1}^{\mathcal{D}}$ if $r > s \geq 1$.
- (ii) $\rho_{r,s}^{\mathcal{D}} < \rho_{r+1,s-1}^{\mathcal{D}}$ or $\rho_{r,s}^{\mathcal{D}} < \rho_{r-1,s+1}^{\mathcal{D}}$ if $r = s \geq 1$.

5. Distance Laplacian spectral radius

The following Lemma is similar to Lemma 4.3 for distance matrix.

Lemma 5.1. *Let G_v be the graph obtained by identifying a vertex v of the graph G with a fixed vertex of another graph G' . If i, j, k are consecutive vertices of G such that i and k are on different components of $G - j$ and ρ_v^L denotes the distance Laplacian spectral radius of G_v , then $\rho_i^L + \rho_k^L - 2\rho_j^L \geq 0$. Furthermore, either $\rho_i^L > \rho_j^L$ or $\rho_k^L > \rho_j^L$.*

Proof. Let G' be of order $m + 1$ with w as a fixed vertex. Assume that G_i, G_j, G_k are obtained by identifying the vertices i, j, k of G with the vertex w of G' , respectively. Let H_i and H_k be the components of $G - j$ containing i and k respectively. Suppose

$$\begin{aligned} V_1 &= \{u \in G : d_{uk} = d_{ui} = d_{uj} + 1\} \\ V_2 &= \{u \in H_k : d_{uk} = d_{jk} \text{ and } d_{ui} = d_{uj} + 1\} \\ V_3 &= \{u \in H_i : d_{ui} = d_{uj} \text{ and } d_{uk} = d_{uj} + 1\} \\ V_4 &= \{u \in H_k : d_{uk} = d_{uj} - 1 \text{ and } d_{ui} = d_{uj} + 1\} \\ V_5 &= \{u \in H_i : d_{ui} = d_{uj} - 1 \text{ and } d_{uk} = d_{uj} + 1\}. \end{aligned}$$

Then $V(G_j) = V(G' - j) \cup (\cup_{i=1}^5 V_i)$ is a partition for $V(G_j)$. If $a = |V_1|$ and $b = |V_2 \cup V_3|$, then by permutation similarity we have

$$\mathcal{D}^L(G_i) + \mathcal{D}^L(G_k) - 2\mathcal{D}^L(G_j) = \begin{bmatrix} V(G' - j) & V_1 & V_2 \cup V_3 & V_4 \cup V_5 \\ (2a + b)I_m & -2\mathbb{J}_{m \times a} & -\mathbb{J}_{m \times b} & 0 \\ -2\mathbb{J}_{a \times m} & 2mI_a & 0 & 0 \\ -\mathbb{J}_{b \times m} & 0 & mI_b & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let \mathbf{x} be an eigenvector corresponding to spectral radius of $\mathcal{D}^L(G_j)$. Observe that

$$\begin{aligned} \rho_i^L + \rho_k^L - 2\rho_j^L &\geq \rho\{\mathcal{D}^L(G_j) + \mathcal{D}^L(G_k)\} - 2\rho_j^L \text{ [by Lemma 4.1]} \\ &\geq \mathbf{x}^t \{\mathcal{D}^L(G_j) + \mathcal{D}^L(G_k) - 2\mathcal{D}^L(G_j)\} \mathbf{x} \text{ [by Lemma 4.2]} \\ &= \sum_{p \in V(G' - j)} \left[2 \sum_{q \in V_1} (\mathbf{x}_p - \mathbf{x}_q)^2 + \sum_{q \in V_2 \cup V_3} (\mathbf{x}_p - \mathbf{x}_q)^2 \right] \geq 0. \end{aligned} \tag{5.20}$$

To the contrary, assume that $\rho_i^L = \rho_j^L = \rho_k^L = \rho$ (say). Then above inequalities hold with equality. Therefore by (5.20), \mathbf{x} is also an eigenvector of $\mathcal{D}^L(G_i)$ and $\mathcal{D}^L(G_k)$ corresponding to ρ and $\mathbf{x}_p = \mathbf{x}_q$ for $p \in V(G' - j)$, $q \in V_1 \cup V_2 \cup V_3$. Thus

$$\{\mathcal{D}^L(G_j) - \mathcal{D}^L(G_i)\} \mathbf{x} = 0 \tag{5.21}$$

$$\text{and } \mathbf{x}_p = \mathbf{x}_w \text{ for all } p \in V(G' - j) \cup V_1 \cup V_2 \cup V_3. \tag{5.22}$$

Let $k = |V_4| - |V_5|$, then we have

$$\mathcal{D}^L(G_j) - \mathcal{D}^L(G_i) = \begin{bmatrix} V(G' - j) & V_1 \cup V_2 \cup V_3 & V_4 & V_5 \\ kI_m & 0 & \mathbb{J} & -\mathbb{J} \\ 0 & 0 & 0 & 0 \\ \mathbb{J} & 0 & -mI & 0 \\ -\mathbb{J} & 0 & 0 & mI \end{bmatrix}.$$

Now for $u \in V_4 \cup V_5$, using (5.22) in the u -th equation of (5.21) we have $\mathbf{x}_u = \mathbf{x}_w$ for all $u \in V_4 \cup V_5$. Thus $\mathbf{x} = \mathbf{x}_w \mathbf{1}$, which implies $\rho = 0$, a contradiction. Hence either $\rho_i^L \neq \rho_j^L$ or $\rho_k^L \neq \rho_j^L$. \square

Corollary 5.2. *Let T be a tree and G' be a graph. If T_i and ρ_i^L are as defined in Lemma 5.1, then ρ_i^L is strictly quasiconvex on T .*

For the distance Laplacian we obtain the following results similar to that of Corollary 4.5 and that of [7, Lemma 3.3] as an immediate corollary to Corollary 5.2.

Corollary 5.3. *Let G be a simple graph and v be one of its vertices. If $k \geq \ell \geq 1$, then $\rho_{v,k,\ell}^L < \rho_{v,k+1,\ell-1}^L$.*

Theorem 5.4. *Let T be a tree with maximal distance Laplacian spectral radius in \mathcal{T}_n^m . Then $T \cong D(n, a, b)$, $a + b = n + 1 - 2m$.*

Proof. If $m = 1$, then the result is trivial. Suppose $m \geq 2$, and let T be a tree in \mathcal{T}_n^m with maximal distance Laplacian spectral radius. Then T has 2 quasipendent vertices. Otherwise, there exists a vertex v in T such that $T - v$ has at least 3 components, and at least two of which are nontrivial. Let T_x, T_y, T_z be three branches at the vertex v of T , and M be an m -matching of T . Then at least two of vx, vy and vz are not in M .

Case 1: Suppose T_x and T_y are non-trivial and vz is not in M . Let x_1, y_1 be quasipendent vertices in T_x, T_y respectively. Then the trees $T' = T - vz + x_1z$ and $T'' = T - vz + y_1z$ are in \mathcal{T}_n^m , and by Corollary 5.2, $\rho^L(T') > \rho^L(T)$ or $\rho^L(T'') > \rho^L(T)$, a contradiction to the maximality of T .

Case 2: Suppose vz is in M and T_z is non-trivial. Then both vx and vy are not in M , and say T_x is non-trivial. Reversing the roles of y and z in Case 1, we have a contradiction.

Case 3: Suppose vz is in M and T_z is trivial. Since T_y is nontrivial therefore there exists a vertex u in T_y which is in a matching. Then both the trees $T' = T - vx + zx$ and $T'' = T - vx + ux$ have matching number m , but by Corollary 5.2 at least one has a larger distance Laplacian spectral radius than T , a contradiction to the maximality of T . \square

Theorem 5.5. *Let T be a tree with maximal distance Laplacian spectral radius in $\mathcal{T}_{n,k}$. Then $T \cong D(n, a, b)$, $a + b = k$.*

Proof. Let $P := i_1 - i_2 - \dots - i_m$ be a path of maximum length in T . If $m = 1, 2, 3$, then there is nothing to prove. Suppose $m \geq 4$. Then degrees of i_3, \dots, i_{m-2} in T are all 2. To the contrary assume that $\text{deg}(i_j) \geq 3$. Let T' be the component of $T - i_{j-1} - i_{j+1}$ containing vertex i_j . Let T_1, T_2 be the new trees obtained by shifting T' to i_{j-1} and i_{j+1} , respectively. By Lemma 5.1, the distance Laplacian spectral radius of T_1 or T_2 is bigger than that of T , a contradiction. Hence the Theorem holds. \square

Let $\mathcal{B}_{n,\Delta}$ denote the class of all trees with n vertices and maximum vertex degree Δ . The broom, denoted by $B_{n,\Delta}$ is the tree obtained from the star $S_{\Delta+1}$ and path $P_{n-\Delta-1}$ by attaching one end of the path to one of the pendent vertex of the star (see Fig. 4).

Stevanović, and Ilić in [7] proved that among all trees with fixed maximum degree Δ , the broom tree $B_{n,\Delta}$ has maximal distance spectral radius. The next result says that broom is the unique tree that maximize the distance Laplacian spectral radius among all trees with fixed maximum degree.

Theorem 5.6. *Among all trees in $\mathcal{B}_{n,\Delta}$ the unique tree that maximizes the distance Laplacian spectral radius is $B_{n,\Delta}$.*



Fig. 4. The broom $B_{n,\Delta}$.

Proof. Let T be tree with maximum distance Laplacian spectral radius among all trees in $\mathcal{B}_{n,\Delta}$. Let v be a vertex of degree Δ and T_1, \dots, T_Δ be the trees attached at v . By Corollary 5.3, it follows that each T_i is a path for $i = 1, \dots, \Delta$. Without loss of generality we assume that the path T_1 has maximum length. By Corollary 5.3, it follows that length of each path T_2, \dots, T_Δ is exactly one. Hence $T \cong B_{n,\Delta}$. \square

Lemma 5.7. Let $G_{r,s}$ be the graph obtained by attaching two paths of length r and s at two adjacent vertices u and v of a graph G , respectively. If $\rho_{r,s}^L$ denotes the distance Laplacian spectral radius of $G_{r,s}$ then $\rho_{r+1,s-1}^L + \rho_{r-1,s+1}^L \geq 2\rho_{r,s}^L$. Furthermore, either $\rho_{r+1,s-1}^L > \rho_{r,s}^L$ or $\rho_{r-1,s+1}^L > \rho_{r,s}^L$.

Proof. Let $V(G) = \{1, 2, \dots, m\}$ such that $u = 1, v = 2$. Let $\mathcal{D}_{r,s}^L$ be the distance Laplacian matrix of $G_{r,s}$. Let $V_1 = \{j \in V(G) - \{1, 2\} | d_{1j} = d_{2j}\}$, $V_2 = \{j \in V(G) - \{1, 2\} | d_{1j} < d_{2j}\}$, $V_3 = \{j \in V(G) - \{1, 2\} | d_{1j} > d_{2j}\}$ and a, b, c be the number of vertices in V_1, V_2 and V_3 , respectively. With respect to the ordered partition $V(G_{r,s}) = \{1, 2\} \cup V_1 \cup V_2 \cup V_3 \cup (V(G_{r,s}) - V(G))$ we have

$$\mathcal{D}_{r+1,s-1}^L + \mathcal{D}_{r-1,s+1}^L - 2\mathcal{D}_{r,s}^L = \begin{bmatrix} 1 & 2 & V_1 & V_2 & V_3 & V(G_{r,s}) - V(G) \\ a + 2b & 0 & -\mathbb{1}^t & -2\mathbb{1}^t & 0 & 0 \\ 0 & a + 2c & -\mathbb{1}^t & 0 & -2\mathbb{1}^t & 0 \\ -\mathbb{1} & -\mathbb{1} & 2I_a & 0 & 0 & 0 \\ -2\mathbb{1} & 0 & 0 & 2I_b & 0 & 0 \\ 0 & -2\mathbb{1} & 0 & 0 & 2I_c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let \mathbf{x} be an eigenvector corresponding to the spectral radius of $\mathcal{D}_{r,s}^L$. Observe that

$$\begin{aligned} \rho_{r+1,s-1}^L + \rho_{r-1,s+1}^L - 2\rho_{r,s}^L &\geq \rho(\mathcal{D}_{r+1,s-1}^L + \mathcal{D}_{r-1,s+1}^L) - 2\rho_{r,s}^L \text{ [by Lemma 4.1]} \\ &\geq \mathbf{x}^t(\mathcal{D}_{r+1,s-1}^L + \mathcal{D}_{r-1,s+1}^L - 2\mathcal{D}_{r,s}^L)\mathbf{x} \text{ [by Lemma 4.2]} \\ &= 2 \left[\sum_{j \in V_2} (\mathbf{x}_1 - \mathbf{x}_j)^2 + \sum_{j \in V_3} (\mathbf{x}_2 - \mathbf{x}_j)^2 \right] \\ &\quad + \sum_{i=1}^2 \sum_{j \in V_1} (\mathbf{x}_i - \mathbf{x}_j)^2 \geq 0. \end{aligned} \tag{5.23}$$

To the contrary, assume that $\rho_{r-1,s+1} = \rho_{r+1,s-1} = \rho_{r,s} = \rho$ (say). Then \mathbf{x} is also an eigenvector of $\mathcal{D}_{r+1,s-1}^L$ and $\mathcal{D}_{r-1,s+1}^L$ corresponding to ρ . Label the vertices of $G_{r,s}$, $G_{r+1,s-1}$ and $G_{r-1,s+1}$ as in Fig. 3. We see that

$$\mathcal{D}_{r+1,s-1}^L - \mathcal{D}_{r,s}^L = \begin{bmatrix} 1 & 2 & \overbrace{3, \dots, m} & \overbrace{m+1, \dots, m+r-1} & \overbrace{m+r, \dots, n} \\ m-2 & 0 & -\mathbb{1}^t & \mathbb{0}^t & \mathbb{0}^t \\ 0 & 0 & z^t & \mathbb{0}^t & \mathbb{0}^t \\ -\mathbb{1} & z & C & -\mathbb{J} & \mathbb{J} \\ 0 & 0 & -\mathbb{J}^t & (m-2)I & 0 \\ 0 & 0 & \mathbb{J}^t & 0 & -(m-2)I \end{bmatrix}.$$

Now we have

$$(\mathcal{D}_{r+1,s-1}^L - \mathcal{D}_{r,s}^L)\mathbf{x} = 0. \tag{5.24}$$

From the first equation in (5.24), we have

$$(m-2)\mathbf{x}_1 = \sum_{j=3}^m \mathbf{x}_j. \tag{5.25}$$

Again from the i -th equation in (5.24), we have

$$(m-2)\mathbf{x}_i = \sum_{j=3}^m \mathbf{x}_j, \text{ for } i = m+1, m+2, \dots, n. \tag{5.26}$$

From the equality in (5.23), we have

$$\mathbf{x}_1 = \mathbf{x}_j \text{ for } d_{2j} \leq d_{1j}, j = 3, \dots, m. \tag{5.27}$$

From equations (5.25), (5.26) and (5.27), we have

$$\mathbf{x}_1 = \mathbf{x}_{m+1} = \mathbf{x}_{m+2} = \dots = \mathbf{x}_n = \mathbf{x}_j \text{ for } d_{2j} \leq d_{1j}, j = 3, \dots, m. \tag{5.28}$$

Similarly, considering the matrix $\mathcal{D}_{r-1,s+1}^L - \mathcal{D}_{r,s}^L$ we can show that

$$\mathbf{x}_2 = \mathbf{x}_{m+1} = \mathbf{x}_{m+2} = \dots = \mathbf{x}_n = \mathbf{x}_j \text{ for } d_{1j} \leq d_{2j}, j = 3, \dots, m. \tag{5.29}$$

By equations (5.28) and (5.29), we see that $\mathbf{x} = \alpha \mathbb{1}$ for some $\alpha \neq 0$, which implies that $\rho = 0$, a contradiction. Hence either $\rho_{r+1,s-1}^L > \rho_{r,s}^L$ or $\rho_{r-1,s+1}^L > \rho_{r,s}^L$. \square

Lemma 5.8. *Let U_0 be a graph obtained from a unicyclic graph by attaching a pendent path P at a vertex p of the cycle. Let $U_1 := U_0 - \{p, r\} + \{q, r\}$ and $U_2 := U_0 - \{p, r\} + \{r, s\}$, where q, r are vertices on the cycle adjacent to p and s on the path adjacent to p . If*

ρ_i^L denotes the Laplacian spectral radius of U_i for $i = 0, 1, 2$ then $\rho_1^L + \rho_2^L - 2\rho_0^L \geq 0$. Furthermore, either $\rho_1^L > \rho_0^L$ or $\rho_2^L > \rho_0^L$.

Proof. Let $V_1 = \{j \in V(U_0 - P) : d_{pj} \leq d_{qj}\}$, and $V_2 = V(U_0) - V_1 - \{p\}$. Let \mathcal{D}_i^L be the distance Laplacian matrix of U_i for $i = 0, 1, 2$. With respect to the ordered partition $\{p\} \cup V_1 \cup V_2$ of $V(U_0)$, we have

$$\mathcal{D}_1^L + \mathcal{D}_2^L - 2\mathcal{D}_0^L = \begin{bmatrix} 2a & -2\mathbb{1}_a^t & 0 \\ -2\mathbb{1}_a & 2I_a & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ where } a = |V_1|.$$

Let \mathbf{x} be an eigenvector corresponding to the spectral radius of \mathcal{D}_0^L . We see that

$$\begin{aligned} \rho_1^L + \rho_2^L - 2\rho_0^L &\geq \rho(\mathcal{D}_1^L + \mathcal{D}_2^L) - 2\rho_0^L \\ &\geq \mathbf{x}^t(\mathcal{D}_1^L + \mathcal{D}_2^L - 2\mathcal{D}_0^L)\mathbf{x} \\ &= 2 \sum_{j \in V_1} (\mathbf{x}_p - \mathbf{x}_j)^2 \geq 0. \end{aligned}$$

Thus $\rho_1^L + \rho_2^L - 2\rho_0^L \geq 0$. If $\rho_1^L = \rho_2^L = \rho_0^L$, then with similar arguments as in the proof of Lemma 5.1 we can prove that $\mathbf{x} = \mathbf{0}$, which is a contradiction. Hence the Lemma holds. \square

Lemma 5.9. Let $C_n = [1, \dots, \lceil \frac{n+1}{2} \rceil, n, n-1, \dots, \lceil \frac{n+1}{2} \rceil + 1]$ be a cycle. Then the following holds.

- (i) The distance Laplacian spectral radius of C_n has multiplicity two.
- (ii) If $n = 2k + 1$, then an eigenvector corresponding to spectral radius of $\mathcal{D}^L(C_n)$ is of the form $\begin{bmatrix} 0 \\ x \\ -x \end{bmatrix}$, where x is positive and strictly decreases along the two paths starting at $\lceil \frac{k+3}{2} \rceil$ and $\mathbf{x}_i > \mathbf{x}_{k-i+3}$ for $i = 2, 3, \dots, \lceil \frac{k+1}{2} \rceil$.
- (iii) If $n = 2k$, then an eigenvector corresponding to spectral radius of $\mathcal{D}^L(C_n)$ is of the form $\begin{bmatrix} x \\ -x \end{bmatrix}$, where x is positive and strictly decreases along the two paths starting at $\lceil \frac{k+1}{2} \rceil$ and $\mathbf{x}_i = \mathbf{x}_{k-i+1}$ for $i = 1, 2, \dots, \lceil \frac{k}{2} \rceil$.

Proof. Let $n = 2k + 1$. Then we observe that

$$\mathcal{D}^L(C_{2k+1}) = \begin{bmatrix} k^2 + k & z^T & z^T \\ z & A & B \\ z & B & A \end{bmatrix},$$

where

$$z^t = [-1 \quad -2 \dots \quad -k], \quad A = \begin{bmatrix} k^2 + k & -1 & -2 & \dots & -(k-1) \\ -1 & k^2 + k & -1 & \dots & -(k-2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -(k-1) & -(k-2) & -(k-3) & \dots & k^2 + k \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} -2 & -3 & -4 & \dots & -(k-1) & -k & -k \\ -3 & -4 & -5 & \dots & -k & -k & -(k-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -k & -k & -(k-1) & \dots & -4 & -3 & -2 \\ -k & -(k-1) & -(k-2) & \dots & -3 & -2 & -1 \end{bmatrix}.$$

Also, we see that for some permutation matrix P ,

$$\mathcal{D}^L(C_{2k+1}) = \left[\begin{array}{c|cc} P^T \begin{pmatrix} A & B \\ B & A \end{pmatrix} P & z_1 & z_2 \\ \hline z_1^T & z_2^T & k^2 + k \end{array} \right],$$

where $z_1^t = [-k \quad -k \quad -(k-1) \quad \dots \quad -2]$, $z_2^t = [-1 \quad -(k-1) \quad -(k-2) \quad \dots \quad -1]$.

Thus it follows that if λ is an eigenvalue of $A - B$ with an eigenvector x , then $\begin{bmatrix} 0 \\ x \\ -x \end{bmatrix}$ and $\begin{bmatrix} Px \\ -Px \\ 0 \end{bmatrix}$ are two linearly independent eigenvectors of $\mathcal{D}^L(C_{2k+1})$ corresponding to the eigenvalue λ . Since $A - B$ has k eigenvalues and 0 is not an eigenvalue of $A - B$, we see that the spectral radius of $A - B$ is the spectral radius of $\mathcal{D}^L(C_{2k+1})$. Moreover, $A - B$ is positive. Thus by Perron–Frobenius theorem x is positive.

For $2 \leq i \leq \lceil \frac{k+1}{2} \rceil$, from eigen-equations we have

$$[\rho^L(C_{2k+1}) - k^2 - k] (\mathbf{x}_{i+1} - \mathbf{x}_i) = 2 \sum_{j=i+1}^{k-i+1} \mathbf{x}_j + \mathbf{x}_{k-i+2} > 0, \tag{5.30}$$

$$[\rho^L(C_{2k+1}) - k^2 - k] (\mathbf{x}_i - \mathbf{x}_{k-i+3}) = \sum_{j=i}^{k-i+2} \mathbf{x}_j > 0 \tag{5.31}$$

$$\text{and } [\rho^L(C_{2k+1}) - k^2 - k] (\mathbf{x}_{k-i+2} - \mathbf{x}_{k-i+3}) = \mathbf{x}_i + 2 \sum_{j=i+1}^{k-i+2} \mathbf{x}_j > 0. \tag{5.32}$$

Since $\rho^L(C_{2k+1}) > k^2 + k$, (ii) follows from (5.30)–(5.32). Similarly we can prove (iii). \square

Lemma 5.10. *Let C_n be a cycle. Then $\rho^L(U_3^n) > \rho^L(C_n)$.*

Proof. Let $C_n = [1, \dots, \lceil \frac{n+1}{2} \rceil, n, \dots, \lceil \frac{n+1}{2} \rceil + 1]$. Remove the edge $\{1, \lceil \frac{n+1}{2} \rceil + 1\}$ and add the edge $\{1, 3\}$ to obtain U_3^n and then observe that distances between 1 and i decreases by 1 for $i = 3, \dots, \lceil \frac{n+1}{2} \rceil$, increases by at least 1 for $i = \lceil \frac{n+1}{2} \rceil, \dots, n$ and distance among

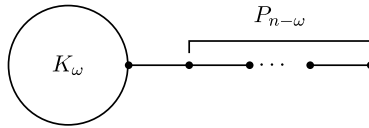


Fig. 5. The graph $PK_{n-\omega, \omega}$.

all other vertices do not decrease. Let \mathbf{x} be an eigenvector of $\mathcal{D}^L(C_n)$ corresponding to the spectral radius. Then we see that

$$\begin{aligned} \rho^L(U_3^n) - \rho^L(C_n) &\geq \mathbf{x}^t(\mathcal{D}^L(U_3^n) - \mathcal{D}^L(C_n))\mathbf{x} \text{ [by Lemma 4.2]} \\ &\geq \sum_{i=\lceil \frac{n+1}{2} \rceil+1}^{n-1} (\mathbf{x}_1 - \mathbf{x}_i)^2 - \sum_{i=3}^{\lceil \frac{n+1}{2} \rceil} (\mathbf{x}_1 - \mathbf{x}_i)^2 \\ &\geq \sum_{i=3}^{\lceil \frac{n+1}{2} \rceil-1} [(\mathbf{x}_1 + \mathbf{x}_i)^2 - (\mathbf{x}_1 - \mathbf{x}_i)^2] \\ &\quad + (\mathbf{x}_1 + \mathbf{x}_2)^2 - (\mathbf{x}_1 - \mathbf{x}_{\lceil \frac{n+1}{2} \rceil})^2, \text{ using Lemma 5.9} \\ &= 4\mathbf{x}_1 \sum_{i=3}^{\lceil \frac{n+1}{2} \rceil-1} \mathbf{x}_i + 2\mathbf{x}_1(\mathbf{x}_2 + \mathbf{x}_{\lceil \frac{n+1}{2} \rceil}) + \mathbf{x}_2^2 - \mathbf{x}_{\lceil \frac{n+1}{2} \rceil}^2 \\ &> 0, \text{ as } \mathbf{x}_2 > \mathbf{x}_{\lceil \frac{n+1}{2} \rceil} \text{ by Lemma 5.9.} \end{aligned}$$

Hence $\rho^L(U_3^n) > \rho^L(C_n)$. \square

Theorem 5.11. *The unique graph that maximizes the distance Laplacian spectral radius among all unicyclic graph on n vertices is U_3^n .*

Proof. Let U be the unicyclic graph that maximizes the distance Laplacian spectral radius among all unicyclic graph of order n . By Lemma 5.1, we see that every tree branch of U attached at a vertex of the cycle is a path and by Lemma 5.7, U has no tree branch at two adjacent vertices of cycle. By repeated application of Lemma 5.8, we see that U is either C_n or U_g^n for some g . By Lemma 5.10, we see that $\rho^L(U_3^n) > \rho^L(C_n)$. Thus $U \not\cong C_n$. If $U \cong U_g^n$, then by Lemma 5.8, we see that either $\rho^L(U_{g-1}^n) > \rho^L(U)$ or $\rho^L(U_{g+1}^n) > \rho^L(U)$. Hence the Theorem holds. \square

A *clique* of a graph is a maximal complete subgraph and *clique number* of a graph is the order of a maximal clique. Let \mathbf{G}_n^ω be the class of all graphs of order n with clique number ω . Here, we find the graph with maximal distance Laplacian spectral radius in \mathbf{G}_n^ω , where $2 \leq \omega \leq n$. A *kite graph*, denoted by $PK_{n-\omega, \omega}$, is a graph obtained by attaching a path $P_{n-\omega}$ to a vertex of a complete graph K_ω [see Fig. 5]. Clearly, $PK_{n-\omega, \omega} \in \mathbf{G}_n^\omega$.

Theorem 5.12. *If $G \in \mathbf{G}_n^\omega$, then $\rho^L(G) \leq \rho^L(PK_{n-\omega,\omega})$. The equality holds if and only if G is isomorphic to $PK_{n-\omega,\omega}$.*

Proof. If $n = \omega$, then $\mathbf{G}_n^\omega = \{K_n\}$. Since $K_n = PK_{0,n}$, so the result holds in this case. Therefore we consider $n > \omega \geq 2$.

(i) If we delete an edge (if any) from G which is not in K but on a cycle, then the resulting graph is still in \mathbf{G}_n^ω . Let G_1 be a spanning subgraph of G obtained by deleting such edges (if any) in an arbitrary order as long as they exist. Thus, G_1 consists of a clique K with a number of rooted trees attached to clique vertices. Since deletion of an edge from a graph does not decrease its distance Laplacian spectral radius, so $\rho^L(G_1) \geq \rho^L(G)$.

(ii) Let T_1, T_2, \dots, T_ℓ be maximal disjoint trees attached at a vertex v of K , in G_1 . We can repeatedly apply the transformation from Lemma 5.1 at any vertex of degree at least three with the largest eccentricity from v in every tree T_i , as long as T_i does not become a path. Each application of this transformation strictly increases the distance Laplacian spectral radius.

Now we can again apply the transformation from Lemma 5.1 at the vertex v as long as there exist at least two paths of length at least one, further increasing the distance Laplacian spectral radius.

(iii) If $\omega = 2$, then $G_2 = P_n = PK_{n-2,2}$, and the result holds in this case. Otherwise, let G_2 consist of clique K and the paths $P_{\ell_1}, P_{\ell_2}, \dots, P_{\ell_m}$ attached to m distinct vertices of K . With the repeated use of Lemma 5.7 to these paths, we finally arrive at the kite graph $PK_{n-\omega,\omega}$. By Lemma 5.7, the distance Laplacian spectral radius is increased at each step.

Therefore, we may conclude that the kite graph $PK_{n-\omega,\omega}$ has larger distance Laplacian spectral radius than any other graph in \mathbf{G}_n^ω . This completes the proof. \square

6. Distance signless Laplacian spectral radius

The following Lemma is similar to Lemma 4.3 and therefore its proof is omitted.

Lemma 6.1. *Let G_v be the graph obtained by identifying a vertex v of the graph G with a fixed vertex u of another graph G' . If i, j, k are consecutive vertices of G such that i and k are on different components of $G - j$ and ρ_v^\circledast denotes the distance signless Laplacian spectral radius of G_v , then $\rho_i^\circledast + \rho_k^\circledast - 2\rho_j^\circledast > 0$.*

Corollary 6.2. *Let T be a tree and G be a graph. If T_i and ρ_i^\circledast are as defined in the above Lemma then ρ_i^\circledast is strictly convex on T .*

The following result similar to that of Corollary 4.5 and Corollary 5.3 follows immediately from Corollary 6.2.

Corollary 6.3. *Let G be a simple graph and v be one of its vertices. If $k \geq \ell \geq 1$, then $\rho_{v,k,\ell}^\circledast < \rho_{v,k+1,\ell-1}^\circledast$.*

Theorem 6.4. *Let T be a tree with maximal distance signless Laplacian spectral radius in \mathcal{T}_n^m . Then $T \cong D(n, a, b)$, $a + b = n + 1 - 2m$.*

Proof. If $m = 1$, then the result is trivial. Suppose $m \geq 2$, and let T be a tree in \mathcal{T}_n^m with maximal distance signless Laplacian spectral radius. Then T has 2 quasipendent vertices. Otherwise, there exists a vertex v in T such that $T - v$ has at least 3 components, and at least two of which are nontrivial. Let T_x, T_y, T_z be three branches at the vertex v of T , and M be an m -matching of T . Then at least two of vx, vy and vz are not in M .

Case 1: Suppose T_x and T_y are non-trivial and vz is not in M . Let x_1, y_1 be quasipendent vertices in T_x, T_y respectively. Then the trees $T' = T - vz + x_1z$ and $T'' = T - vz + y_1z$ are in \mathcal{T}_n^m , and by Corollary 6.2, $\rho^{\mathcal{Q}}(T') > \rho^{\mathcal{Q}}(T)$ or $\rho^{\mathcal{Q}}(T'') > \rho^{\mathcal{Q}}(T)$, a contradiction to the maximality of T .

Case 2: Suppose vz is in M and T_z is non-trivial. Then both vx and vy are not in M , and say T_x is non-trivial. Reversing the roles of y and z in Case 1, we have a contradiction.

Case 3: Suppose vz is in M and T_z is trivial. Since T_y is nontrivial therefore there exists a vertex u in T_y which is in a matching. Then both the trees $T' = T - vx + zx$ and $T'' = T - vx + ux$ have matching number m , but by Corollary 6.2 at least one has a larger distance signless Laplacian spectral radius than T , a contradiction to the maximality of T .

Thus $T \cong D(n, a, b)$, for some a and b . \square

Theorem 6.5. *Let T be a tree with maximal distance signless Laplacian spectral radius in $\mathcal{T}_{n,k}$. Then $T \cong D(n, a, b)$, $a + b = k$.*

Proof. Let $P = i_1 - i_2 - \dots - i_k$ be a path of maximum length in T . If $m = 1, 2, 3$, then there is nothing to prove. Suppose $m \geq 4$. Then degrees of i_3, \dots, i_{m-2} in T are all 2. To the contrary assume that $\text{deg}(i_j) \geq 3$. Let T' be the component of $T - i_{j-1} - i_{j+1}$ containing vertex i_j . Let T_1, T_2 be the new trees obtained by shifting T' to i_{j-1} and i_{j+1} , respectively. By Lemma 6.1, the distance Laplacian spectral radius of T_1 or T_2 is bigger than that of T , a contradiction. Hence the Theorem holds. \square

The next result says that broom is the unique tree that maximizes the distance signless Laplacian spectral radius among all trees with fixed maximum degree.

Theorem 6.6. *Among all trees in $\mathcal{B}_{n,\Delta}$ the unique tree that maximizes the distance signless Laplacian spectral radius is $B_{n,\Delta}$.*

Proof. Let T be a tree with maximum distance Laplacian spectral radius among all trees in $\mathcal{B}_{n,\Delta}$. Let v be a vertex of degree Δ and T_1, \dots, T_Δ be the trees attached at v . By Corollary 6.3, it follows that each T_i is a path for $i = 1, \dots, \Delta$. Without loss of generality we assume that the path T_1 has maximum length. By Corollary 6.3, it follows that length of each path T_2, \dots, T_Δ is exactly one. Hence $T \cong B_{n,\Delta}$. \square

The following Lemma is similar to Lemma 4.10 and Lemma 5.7 whose proof also follows similarly.

Lemma 6.7. *Let $G_{r,s}$ be the graph obtained by attaching two paths of length r and s at two adjacent vertices u and v of a connected graph G , respectively. Let $\rho_{r,s}^{\mathcal{Q}}$ be the distance Laplacian spectral radius of $G_{r,s}$. Then*

$$\rho_{r+1,s-1}^{\mathcal{Q}} + \rho_{r-1,s+1}^{\mathcal{Q}} > 2\rho_{r,s}^{\mathcal{Q}}.$$

Lemma 6.8. *Let U be a unicyclic graph with the cycle $C = [1, 2, \dots, m]$. Let T_1 and T_2 be two tree branches at the vertices 1 and k of C such that $\text{deg}(i) = 2$ for $i = 2, \dots, k - 1$. Let \mathbf{x} be the Perron vector of the distance signless Laplacian of U then $-\mathbf{x}$ induces a quasiconvex function on $\{1, \dots, k\}$.*

Proof. Since $Tr(i) + Tr(i + 2) \leq 2Tr(i + 1)$, we see that

$$2\rho^{\mathcal{Q}} - Tr(i) - Tr(i + 2) \geq 2(\rho^{\mathcal{Q}} - Tr(i + 1)) \text{ for } 1 \leq i \leq k - 2.$$

By eigenequation of \mathbf{x} at $i, i + 1, i + 2$ we have

$$\begin{aligned} &(\rho^{\mathcal{Q}} - Tr(i))\mathbf{x}_i + (\rho^{\mathcal{Q}} - Tr(i + 2))\mathbf{x}_{i+2} - 2(\rho^{\mathcal{Q}} - Tr(i + 1))\mathbf{x}_{i+1} < 0 \\ \Rightarrow &(\rho^{\mathcal{Q}} - Tr(i))\mathbf{x}_i + (\rho^{\mathcal{Q}} - Tr(i + 2))\mathbf{x}_{i+2} - (2\rho^{\mathcal{Q}} - Tr(i) - Tr(i + 2))\mathbf{x}_{i+1} < 0 \\ \Rightarrow &\lambda\mathbf{x}_i + (1 - \lambda)\mathbf{x}_{i+2} < \mathbf{x}_{i+1}, \text{ where } \lambda = \frac{\rho^{\mathcal{Q}} - Tr(i)}{2\rho^{\mathcal{Q}} - Tr(i) - Tr(i + 2)}. \end{aligned}$$

Thus $\mathbf{x}_{i+1} > \min\{\mathbf{x}_i, \mathbf{x}_{i+2}\}$ and the proof is complete. \square

Lemma 6.9. *Let U be a unicyclic graph obtained by attaching a path $P := 1 - 2 - \dots - k$ at the vertex k of the cycle $C := [k, \dots, n]$. Let \mathbf{x} be the Perron vector of the distance signless Laplacian of U . Then \mathbf{x} induces a strictly convex function on $\{1, 2, \dots, k, \dots, k + \lceil \frac{g}{2} \rceil\}$.*

Proof. For our convenience let $t = k + \lceil \frac{g}{2} \rceil$. By eigenequation of \mathbf{x} at t and $t - 1$, we have

$$\rho^{\mathcal{Q}}(\mathbf{x}_t - \mathbf{x}_{t-1}) = \sum_{j=1}^{t-2} \mathbf{x}_j - \sum_{j=t+1}^{t+\lceil \frac{g}{2} \rceil} \mathbf{x}_j + Tr(t)\mathbf{x}_t - Tr(t - 1)\mathbf{x}_{t-1}.$$

Thus we have the following:

$$(\rho^{\mathcal{Q}} - Tr(t) + 1)\mathbf{x}_t - (\rho^{\mathcal{Q}} - Tr(t - 1) - 1)\mathbf{x}_{t-1} > 0.$$

Since $\rho^{\mathcal{Q}} - Tr(t) + 1 < \rho^{\mathcal{Q}} - Tr(t - 1) - 1$, we see that $\mathbf{x}_t > \mathbf{x}_{t-1}$.

By Lemma 6.8, it follows that $-\mathbf{x}_k, \dots, \mathbf{x}_{k+\lceil \frac{n}{2} \rceil}$ is convex. Thus $\mathbf{x}_{k+\lceil \frac{n}{2} \rceil} > \dots > \mathbf{x}_k$. Similarly we can show that $\mathbf{x}_1, \dots, \mathbf{x}_{k+1}$ is strictly convex. Using eigenequation we can show that $\mathbf{x}_1 > \mathbf{x}_2$. Hence the lemma holds. \square

Lemma 6.10. *Let C_n be a cycle. Then $\rho^{\mathcal{Q}}(U_3^n) > \rho^{\mathcal{Q}}(C_n)$.*

Proof. Observe that $\mathbf{1}$ is a Perron vector of $\mathcal{D}^{\mathcal{Q}}(C_n)$ and

$$\mathbf{1}^t \{ \mathcal{D}^{\mathcal{Q}}(U_3^n) - \mathcal{D}^{\mathcal{Q}}(C_n) \} \mathbf{1} = 2 \left\{ \sum_{u \in V(U_3^n)} Tr(u) - \sum_{u \in V(C_n)} Tr(u) \right\} > 0.$$

By Lemma 4.2 we have $\rho^{\mathcal{Q}}(U_3^n) > \rho^{\mathcal{Q}}(C_n)$. \square

The following Lemma is similar to Lemma 5.8 whose proof also follows similarly.

Lemma 6.11. *Let U_0 be a graph obtained from a unicyclic graph by attaching a pendent path P at a vertex p of the cycle. Let $U_1 := U_0 - \{p, r\} + \{q, r\}$ and $U_2 := U_0 - \{p, r\} + \{r, s\}$, where q, r are vertices on the cycle adjacent to p and s on the path adjacent to p . If $\rho_i^{\mathcal{Q}}$ denotes the distance signless Laplacian spectral radius of U_i for $i = 0, 1, 2$ then $\rho_1^{\mathcal{Q}} + \rho_2^{\mathcal{Q}} - 2\rho_0^{\mathcal{Q}} \geq 0$. Furthermore, either $\rho_1^{\mathcal{Q}} > \rho_0^{\mathcal{Q}}$ or $\rho_2^{\mathcal{Q}} > \rho_0^{\mathcal{Q}}$.*

Theorem 6.12. *The unique graph that maximizes the distance signless Laplacian spectral radius among all unicyclic graph is U_3^n .*

Proof. Let U be a unicyclic graph with maximum distance signless Laplacian spectral radius in the class of all unicyclic graph. By Lemma 6.1, we see that every tree branch of U attached at a vertex of the cycle is a path and by Lemma 6.7, U has no path branch at two adjacent vertices of the cycle. By repeated application of Lemma 6.11 and 6.7, we see that U is either C_n or U_3^n . Hence by Lemma 6.10 the Theorem holds. \square

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