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RESEARCH PROJECT REPORT

Compact GCL Preserving Scheme for Navier-Stokes System in Time Varying Curvilinear Meshes

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1 Introduction

The Navier-Stokes (N-S) equations are the base of the mathematical models of important phenomena in many areas of science and technology. They can be used to describe the motion of various fluids under most general settings. Through this formulation accurately represents various flow phenomena, its analytical solution can be found only for some simplified situations hence historically much importance have been given to provide numerical solution of N-S equations in various settings. In this context body-fitted curvilinear co- ordinate system is often required to accurately compute flow around bodies of arbitrary shapes. Additionally for deforming spatial domains a time moving mesh is an absolute necessity. In the literature one can find mainly two different approaches to simulateflow in curvilinear time deforming meshes. The first approach is to use discretized version of nonconservative arbitrary Lagrangian-Eulerian (ALE) description of the flow. The second approach is to resort to the approximation of the conservative form of the N-S system. In this approach efficient discretization strategies are designed to approximate first order derivatives. The second order viscous terms are often tackled by the repeated application of the approximation procedure developed for first order derivatives. As pointed out by Hirt and Nichols [1] and Li *et al.* [2] finite difference approximation of first order advective terms in conservative formulation for variable meshes generally lead to lose of accuracy. But it is important to remember that variable meshes require extra care for efficient implementation, further in many cases, a conservative method is more favorable. Numerical methods based on conservative formulation has the additional advantage that physical identities can be enforced with relative simplicity. For example, discretizing the flow region into contiguous cells varying with time, the finite volume formulation can be used to ensure two additional equations of state. These two geometrical relations, at times ignored for flowing meshes in time-dependent moving boundary problems, ensure that each cell in the flow region is closed and safeguard conservation of volume for a time-varying cell. These principles known as the surface conservation law (SCL) and the volume conservation law (VCL) respectively are together called geometric conservation laws (GCL). As GCL represent geometric identities its differential and integral form are trivial, but to conserve its discretized version additional care is needed. A detailed discussion on conservation laws and their finite volume and finite difference formulation can be found in the work of Vinokur [3].

In this project our main intension is to develop a new numerically scheme to solve the incompressible N-S system and to simulate the flow field in situations involving fluid-structure interaction problems. The scheme thus developed should be easy to implement and preserve basic conservation properties in discretized form. We are interested in compact finite difference approximation of N-S equations. Recently Sen [4, 5] has developed a fourth order compact finite difference scheme for variable coefficient parabolic problems which was found suitable for computing incompressible flow in arbitrary domains. In this project we explore the possibility of extending this formulation to time varying meshes. We are interested in the discretization of non-conservative form of the N-S equations because of its relative ease in implementation. Also in our formulation we emphasize on the proper approximation of the grid metrics so that the underlying geometric identities are correctly satisfied.

2 Mathematical Formulation

2.1 Governing Equation

We begin by describing unsteady two-dimensional (2D) convection-diffusion equation which is often regarded as the simplified version of N-S system. For the unknown transport variable $\phi(x, y, t)$ with spatial variables (x, y) defined over an arbitrary domain $\Omega \subset \mathbb{R}^2$ the equation can be written as

$$\frac{\partial \phi}{\partial t} - a(x,y,t)\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) + c(x,y,t)\frac{\partial \phi}{\partial x} + d(x,y,t)\frac{\partial \phi}{\partial y} = s(x,y,t), \qquad (x,y,t) \in \Omega \times (0,T] \tag{1}$$

with the initial condition

$$\phi(x,y,0) = \phi_0(x,y), \qquad (x,y) \in \Omega \tag{2}$$

and boundary condition

$$b_1(x,y,t)\phi + b_2(x,y,t)\frac{\partial \phi}{\partial n} = g(x,y,t), \qquad (x,y) \in \partial\Omega, \quad t \in (0,T]. \tag{3}$$

Here $a(x, y, t) > 0$ is the diffusion coefficient, $c(x, y, t)$ and $d(x, y, t)$ are convection coefficients, and $s(x, y, t)$ is a forcing function which together with $\phi_0(x, y)$ and $g(x, y, t)$ are assumed to be sufficiently smooth. n is an unit vector in the boundary normal direction and the arbitrary coefficients b_1 and b_2 characterize the boundary condition. For time varying body fitted coordinate system (ξ, η, τ), we introduce the following transformation

$$x = x(\xi, \eta, \tau), \quad y = y(\xi, \eta, \tau), \quad t = \tau \tag{4}$$

from non-dimensionalized Cartesian coordinate system to the curvilinear coordinate system, where Jacobian $J = \dfrac{\partial(x, y)}{\partial(\xi, \eta)}$ is nonsingular. Thus in the computational plane the equation (1) reduces to

$$\begin{aligned}
&\frac{\partial \phi}{\partial \tau} - \frac{a}{J^2}(x_\eta^2 + y_\eta^2)\phi_{\xi\xi} - \frac{a}{J^2}(x_\xi^2 + y_\xi^2)\phi_{\eta\eta} + \frac{2a}{J^2}(x_\xi x_\eta + y_\xi x_\eta)\phi_{\xi\eta} \\
&+ \Bigg[\frac{1}{J}(-J_1 + c y_\eta - d x_\eta) \\
&-\frac{a}{J^3}\Big(J_\eta(x_\xi x_\eta + y_\xi y_\eta) - J_\xi(x_\eta^2 + y_\eta^2) + J(x_\eta x_{\xi\eta} + y_\eta y_{\xi\eta} - x_\xi x_{\eta\eta} - y_\xi y_{\eta\eta}\Big)\Bigg]\phi_\xi \\
&+ \Bigg[\frac{1}{J}(-J_2 - c y_\xi + d x_\xi) \\
&-\frac{a}{J^3}\Big(J_\xi(x_\xi x_\eta + y_\xi y_\eta) - J_\eta(x_\xi^2 + y_\xi^2) + J(x_\xi x_{\xi\eta} + y_\xi y_{\xi\eta} - x_\eta x_{\xi\xi} - y_\eta y_{\xi\xi}\Big)\Bigg]\phi_\eta \\
&= s,
\end{aligned} \tag{5}$$

where a, c, d, s are now defined over (ξ, η, τ) with $J_1 = \dfrac{\partial(x, y)}{\partial(\tau, \eta)}$ and $J_2 = \dfrac{\partial(x, y)}{\partial(\xi, \tau)}$.

2.2 Discretization

The equation (5) which is a generalized parabolic convection-diffusion equation with variable coefficients is perfectly amenable to the fourth order compact discretization procedure newly established by Sen [5]. This implicit finite difference scheme possesses second order temporal accuracy and shows good wave resolution property. Hence we follow the scheme of Sen [5] to approximate equation (5). The coefficients of the above parabolic equation contain grid metrics. Proper estimation of these metrics is imperative for success of any discretization procedure. Visbal and Gaitonde [6, 7] has established that on curvilinear meshes even when analytic metrics are prescribed may lead to intolerable error growth and should be evaded. Further the authors in [6, 7] have commented that in distorted curvilinear meshes identical compact approximation can be used for estimating derivatives of flow variable as well as grid metrics and often lead to relatively less cancelation error. Thus discretization of all first and second order transformation metrics appearing in equation (5) is carried out using the compact procedure delineated in [4, 5].

2.3 Freestream Preservation

Vinokur [3], in his work has emphasized the importance of implicit satisfaction of GCL identities which in differential form for two spatial dimensions can be written as

$$(\xi_x J)_\xi + (\eta_x J)_\eta = 0, \tag{6}$$

$$(\xi_y J)_\xi + (\eta_y J)_\eta = 0, \tag{7}$$

$$(J)_\tau - (J_1)_\xi - (J_2)_\eta = 0. \tag{8}$$

Although GCL identities are always analytically satisfied but their discretized versions are often violated with improper approximation of the spatial and temporal metrics. Further, estimation of Jacobians need careful examination. We thus use the newly developed symmetric-conservative metric evaluation procedure of Abe *et al.* [8, 9] to estimate the Jacobians associated with the transformation. The procedure applicable in case of 2D is as given below,

$$J = [(x_\xi y)_\eta - (x_\eta y)_\xi + (x y_\eta)_\xi - (x y_\xi)_\eta]/2, \tag{9}$$

$$J_1 = [(x_\tau y)_\eta - (x_\eta y)_\tau + (x y_\eta)_\tau - (x y_\tau)_\eta]/2, \tag{10}$$

$$J_2 = [(x_\xi y)_\tau - (x_\tau y)_\xi + (x y_\tau)_\xi - (x y_\xi)_\tau]/2. \tag{11}$$

2.4 Tackling 2D Incompressible Navier-Stokes System

The main goals of this formulation is to design an algorithm to simulate 2D viscous flows for problems involving fluid-structure interaction. In such situations it is imperative to use time varying curvilinear body-fitted coordinate system. Here we choose to work with streamfunction-vorticity ($\psi - \omega$) formulation as it decouples pressure field from velocity calculation. The $\psi - \omega$ transformed using equation (4) takes the form

$$-\widetilde{a}_1\psi_{\xi\xi} - \widetilde{e}_1\psi_{\xi\eta} - \widetilde{b}_1\psi_{\eta\eta} + \widetilde{c}_1\psi_\xi + \widetilde{d}_1\psi_\eta = \widetilde{f}_1, \tag{12}$$

$$\partial_t\omega - \widetilde{a}_2\omega_{\xi\xi} - \widetilde{e}_2\omega_{\xi\eta} - \widetilde{b}_2\omega_{\eta\eta} + \widetilde{c}_2\omega_\xi + \widetilde{d}_2\omega_\eta = 0. \tag{13}$$

In the above system all the coefficients apart from $\widetilde{c}_2$ and $\widetilde{d}_2$ are functions of grid metrics only. $\widetilde{c}_2$ and $\widetilde{d}_2$ are also functions of velocities apart from grid metrics. Note that the velocities are transformed as

$$(u, v) = (\psi_y, -\psi_x) = \frac{1}{J}(-x_\eta\psi_\xi + x_\xi\psi_\eta, -y_\eta\psi_\xi + y_\xi\psi_\eta).$$

The system given by the equations (12) is the steady form of equation (5) whereas (13) is the homogeneous form of equation (5) and hence both the equations (12) and (13) are amenable to the scheme outlined in this work.

3 Numerical Results

3.1 Verification Studies

To numerically verify the proposed formulation we intend to carry out following verification studies:

1. Convection-diffusion of Gaussian pulse on isotropic Cartesian and wavy meshes introduced by Visbal and Gaitonde [6, 7].
2. Convection-diffusion of Gaussian pulse on dynamically deforming 2D mesh [6, 7].
3. Solution of N-S equations amenable to the exact solution in irregular domain such as flow decayed by viscosity problem [10].

3.2 Validation Studies

Finally to validate the scheme developed we propose to solve N-S system in curvilinear time deforming coordinate system. The following problems are proposed to be tackled.

1. Semi circular cavity problem [11].
2. Pitching airfoil problem [7].

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